# Graphs equienergetic with edge-deleted subgraphs

Chi-Kwong Li* Wasin So†

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**Abstract**

The energy of a graph is the sum of the absolute values of the eigenvalues of its adjacency matrix. Two graphs are equienergetic if they have the same energy. We construct infinite families of graphs equienergetic with edge-deleted subgraphs.

**Keywords.** graph energy, equienergetic graph.

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## 1 Introduction

Throughout, $G$ will be a simple graph, i.e., a graph with no loop and no multiple edge. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of $G$ respectively. Also let $A(G)$ denote the adjacency matrix of the graph $G$. The characteristic polynomial and spectrum of a graph are those of its adjacency matrix. If $E$ is a subset of $E(G)$, then $G-E$ will denote the subgraph of $G$ with vertex set $V(G)$ but with edge set $E(G)-E$. A subgraph $H$ of $G$ is an induced subgraph of $G$ if $H$ contains all edges of $G$ that join two vertices of $H$. Clearly $H$ is induced if and only if $A(H)$ is a principal submatrix of $A(G)$. We write $G-H$ for the graph obtained from $G$ by deleting all vertices of an induced subgraph $H$ and all edges incident with $H$. This is also called the complement of $H$ in $G$. Moreover, when no edge of $G$ joins $H$ and its complement $G-H$, we write $G = H \oplus (G-H)$. If $E$ is a set of edges of $G$ such that $G-E$ has more connected components than $G$, then $E$ is called a *cut set* of $G$.

Let $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ be the eigenvalues of an $n \times n$ real symmetric matrix $A$. The *energy* of a graph $G$ is defined as $\mathcal{E}(G) = \sum_{j=1}^{n} |\lambda_j(A(G))|$ [4]. Two graphs are *equienergetic* if they have the same energy. Of course, if two graphs are *cospectral*, i.e., have the same spectrum, then they are equienergetic. However the converse is not true,

*Department of Mathematics, College of William and Mary, Williamsburg, VA 23187-8795. email: ckli@math.wm.edu. Research supported in part by NSF.

†Department of Mathematics, San Jose State University, San Jose, CA 95192-0103. email: so@math.sjsu.edu

see Example 1.2. Recently there is some interest in constructing pairs of graphs which are equienergetic but not co-spectral [1, 2, 5, 6, 7, 8, 9]. In this paper, we are interested in constructing pairs of equienergetic graphs such that one is a subgraph of the other, i.e., the energy of a graph is the same as the energy of a subgraph obtained by deleting some of its edges. Formally we have the following problem.

**Problem 1.1.** Characterize graph $G$ and edge set $E$ such that $\mathcal{E}(G) = \mathcal{E}(G - E)$.

Two examples of small sizes are included.

**Example 1.2.** The disjoint union of two copies of the complete graph on 2 vertices $K_2 \oplus K_2$ is an equienergetic subgraph of the cycle graph on 4 vertices $C_4$. Note that $\mathcal{E}(K_2 \oplus K_2) = 4 = \mathcal{E}(C_4)$, and $K_2 \oplus K_2 = C_4 - E$ where $E$ is a cut set.

**Example 1.3.** Let $G$ be a simple graph obtained by deleting two independent edges from the complete graph on 5 vertices $K_5$. Then the cycle graph on 5 vertices $C_5$ is an equienergetic subgraph of $G$. Note that $\mathcal{E}(G) = 2 + 2\sqrt{5} = \mathcal{E}(C_5)$, and $C_5 = G - E$ where $E$ is not a cut set.

In this paper, we focus on two special cases of Problem 1.1. Section 2 concerns the case when $E$ is a cut set. Two convenient methods of constructing infinitely many graphs equienergetic with disconnected subgraphs are established. Section 3 deals with the case when $E$ is a singleton, i.e., $E$ contains just a single edge. We construct an infinite family of connected graphs equienergetic with subgraphs of one edge fewer. It is worth mentioning that these two cases are mutually exclusive because of the following result from [3].

**Theorem 1.4.** If $e$ is a bridge, i.e., a cut edge of a graph $G$, then $\mathcal{E}(G - \{e\}) < \mathcal{E}(G)$.

# 2 $E$ is a cut set

In this section, we focus on the special case of Problem 1.1 that $E$ is a cut set of $G$. In this case, the adjacency matrix of $G$ is of the form

$$A(G) = \begin{bmatrix} A(H) & X \\ X^T & A(K) \end{bmatrix}$$

where $H$ and $K$ are complementary subgraphs of $G - E$. In [3], it is proved that $\mathcal{E}(G) = \mathcal{E}(G - E)$ if and only if there exist orthogonal matrices $U$ and $V$ such that

$$\begin{bmatrix} UA(H) & UX \\ VX^T & VA(K) \end{bmatrix}$$

is positive semi-definite. However this characterization is not very helpful in finding equienergetic subgraphs. The following example is taken from [3].

**Example 2.1.** For $n \geq 2$, let $G$ be a simple graph on $2n$ vertices with a cut set $E$ such that $A(G) = \begin{bmatrix} J_n - I_n & I_n \\ I_n & J_n - I_n \end{bmatrix}$ and $A(G-E) = \begin{bmatrix} J_n - I_n & 0 \\ 0 & J_n - I_n \end{bmatrix}$ where $J_n$ is the $n \times n$ matrix with all entries equal to 1, and $I_n$ is the $n \times n$ identity matrix. Then $\mathcal{E}(G) = \mathcal{E}(G-E)$.

**Lemma 2.2.** For $n \geq 2$, let $G$ be a simple graph on $2n$ vertices with a cut set $E$ such that $A(G) = \begin{bmatrix} A & I_n \\ I_n & A \end{bmatrix}$ and $A(G-E) = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$. Then $\mathcal{E}(G) = \mathcal{E}(G-E)$ if and only if $|\lambda_i(A)| \geq 1$ for all $i$.

**Proof.** Let the spectrum of $A$ be $\{\lambda_1, \ldots, \lambda_n\}$. Then the spectrum of $A(G)$ is $\{\lambda_1 \pm 1, \ldots, \lambda_n \pm 1\}$. Since $|\lambda_i + 1| + |\lambda_i - 1| \geq 2|\lambda_i|$, we have $\mathcal{E}(G-E) = \mathcal{E}(G)$ if and only if $\sum_i 2|\lambda_i| = \sum_i(|\lambda_i + 1| + |\lambda_i - 1|)$ if and only if $2|\lambda_i| = |\lambda_i + 1| + |\lambda_i - 1|$ for all $i$ if and only if $|\lambda_i| \geq 1$ for all $i$. ■

**Lemma 2.3.** For $n \geq 2$, let $G$ be a simple graph on $2n$ vertices with a cut set $E$ such that $A(G) = \begin{bmatrix} A & A \\ A & A \end{bmatrix}$ and $A(G-E) = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$. Then $\mathcal{E}(G) = \mathcal{E}(G-E)$.

**Proof.** Let the spectrum of $A$ be $\{\lambda_1, \ldots, \lambda_n\}$. Then the spectrum of $A(G)$ is $\{0^{(n)}, 2\lambda_1, \ldots, 2\lambda_n\}$. Hence $\mathcal{E}(G-E) = 2\sum_{i=1}^{n} |\lambda_i| = \mathcal{E}(G)$. ■

**Remark 2.4.** If a graph $G$ has nonzero integer eigenvalues, say any complete graph $K_n$ or the cycle graph $C_6$, then $|\lambda_i(A(G))| \geq 1$ for all $i$. Hence Example 2.1 is a special case of Lemma 2.2. Both Lemmas 2.2 and 2.3 provide convenient constructions of graphs equienergetic with edge-deleted subgraphs. An application of Lemma 2.3 gives the next example.

**Example 2.5.** For $n \geq 2$, let $G$ be a simple graph on $2n$ vertices with a cut set $E$ such that $A(G) = \begin{bmatrix} J_n - I_n & J_n - I_n \\ J_n - I_n & J_n - I_n \end{bmatrix}$ and $A(G-E) = \begin{bmatrix} J_n - I_n & 0 \\ 0 & J_n - I_n \end{bmatrix}$. Then $\mathcal{E}(G) = \mathcal{E}(G-E)$.

**Theorem 2.6.** Suppose that $G$ is a simple graph with a cut set $E$ such that $A(G) = \begin{bmatrix} J_n - I_n & X \\ X^T & J_m - I_m \end{bmatrix}$ and $A(G-E) = \begin{bmatrix} J_n - I_n & 0 \\ 0 & J_m - I_m \end{bmatrix}$. Then $\mathcal{E}(G) = \mathcal{E}(G-E)$ if and only if (i) $n = m$, and (ii) $X = I_n$ or $X = J_n - I_n$

**Proof.** (Sufficiency) By Examples 2.1 and 2.5.

(Necessity) Let $J_n - I_n = P_n \begin{bmatrix} n-1 & 0 \\ 0 & -I_{n-1} \end{bmatrix} P_n^T$ where $P_n$ is an orthogonal matrix with the first column being the vector with all 1's. Since $J_n - I_n$ is nonsingular, there exists a unique orthogonal matrix $U_n$ such that $U_n(J_n - I_n)$ is positive definite, indeed,

$U_n = P_n \begin{bmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{bmatrix} P_n^T$ is also symmetric. By the characterization mentioned in the beginning of this section, if $\mathcal{E}(G) = \mathcal{E}(G-E)$ then

$$\begin{bmatrix} P_n & 0 \\ 0 & P_m \end{bmatrix} \left[\begin{array}{c|c} \begin{bmatrix} n-1 & 0 \\ 0 & I_{n-1} \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{bmatrix} P_n^T X P_m \\ \hline \begin{bmatrix} 1 & 0 \\ 0 & -I_{m-1} \end{bmatrix} P_m^T X^T P_n & \begin{bmatrix} m-1 & 0 \\ 0 & I_{m-1} \end{bmatrix} \end{array}\right] \begin{bmatrix} P_n^T & 0 \\ 0 & P_m^T \end{bmatrix}$$

$$= \left[\begin{array}{c|c} P_n \begin{bmatrix} n-1 & 0 \\ 0 & I_{n-1} \end{bmatrix} P_n^T & P_n \begin{bmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{bmatrix} P_n^T X \\ \hline P_m \begin{bmatrix} 1 & 0 \\ 0 & -I_{m-1} \end{bmatrix} P_m^T X^T & P_m \begin{bmatrix} m-1 & 0 \\ 0 & I_{m-1} \end{bmatrix} P_m^T \end{array}\right]$$

$$= \begin{bmatrix} U_n(J_n - I_n) & U_n X \\ U_m X^T & U_m(J_m - I_m) \end{bmatrix}$$

is positive semi-definite. Hence $\left[\begin{array}{c|c} \begin{bmatrix} n-1 & 0 \\ 0 & I_{n-1} \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{bmatrix} Y \\ \hline \begin{bmatrix} 1 & 0 \\ 0 & -I_{m-1} \end{bmatrix} Y^T & \begin{bmatrix} m-1 & 0 \\ 0 & I_{m-1} \end{bmatrix} \end{array}\right]$ is positive semi-definite where $Y = P_n^T X P_m$ is $n \times m$. By symmetry,

$$\begin{bmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{bmatrix} Y = Y \begin{bmatrix} 1 & 0 \\ 0 & -I_{m-1} \end{bmatrix},$$

and so $P_n^T X P_m = Y = \begin{bmatrix} k & 0 \\ 0 & Z \end{bmatrix}$ and $P_m^T X^T P_n = Y^T = \begin{bmatrix} k & 0 \\ 0 & Z^T \end{bmatrix}$. Recall that both the first column of $P_n$ and $P_m$ are vectors with all 1's, denoted by $\mathbf{1}_n$ and $\mathbf{1}_m$ respectively. It follows that $X\mathbf{1}_m = k\mathbf{1}_n$ and $X^T\mathbf{1}_n = k\mathbf{1}_m$, i.e., $X$ has constant row sums and column sums equal to $k$. Consequently, $n = m$. Now $\left[\begin{array}{c|c} \begin{bmatrix} n-1 & 0 \\ 0 & I_{n-1} \end{bmatrix} & \begin{bmatrix} k & 0 \\ 0 & -Z \end{bmatrix} \\ \hline \begin{bmatrix} k & 0 \\ 0 & -Z^T \end{bmatrix} & \begin{bmatrix} n-1 & 0 \\ 0 & I_{n-1} \end{bmatrix} \end{array}\right]$ is positive semi-definite, and so $|\lambda_i(Z)| \le 1$ for all $i$. Consider the sum of squares of entries for the matrices $X, Y$ and $Z$, we have

$$nk = \text{ tr } X^T X = \text{ tr } Y^T Y = k^2 + \text{ tr } Z^T Z = k^2 + \sum_i |\lambda_i(Z)|^2 \le k^2 + n - 1.$$

Hence $k = 1$ or $k = n-1$, i.e., $X = I_n$ or $X = J_n - I_n$. ■

# 3 $E$ is a singleton

In this section, we consider another special case of Problem 1.1 that $E$ is a singleton. To avoid triviality, the supergraph is required to be connected, but the subgraph does not need

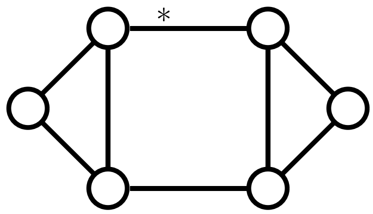


Figure 1: Graph $G_1$.

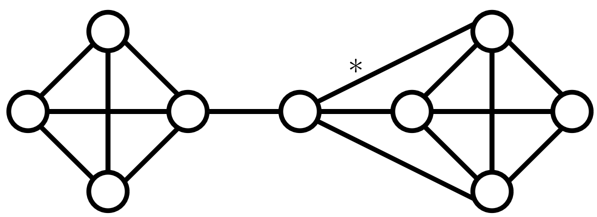


Figure 2: Graph $G_2$.

to be connected. The numbers of connected graphs with 2 to 9 vertices are

$$1,\ \ 2,\ \ 6,\ \ 21,\ \ 112,\ \ 853,\ \ 11117,\ \ 261080,$$

respectively. After an exhaustive search, the numbers of connected graphs equienergetic with subgraphs of one edge fewer are

$$0,\ \ 0,\ \ 0,\ \ 0,\ \ 1,\ \ 0,\ \ 0,\ \ 2,$$

respectively. They are $G_1$ on 6 vertices in Figure 1, $G_2$ on 9 vertices in Figure 2, and $G_3$ on 9 vertices in Figure 3. In these figures, the edge with an asterisk can be deleted without changing the energy. In [3], $G_1$ is shown to be a member of an infinite family of connected graphs equienergetic with subgraphs of one edge fewer. In this section, we construct another infinite family of graphs to which $G_2$ is a member (See Theorem 3.2). It is still unknown whether $G_3$ in Figure 3 is a member of an infinite family of connected graphs equienergetic with subgraphs of one edge fewer.

For $n \geq 2$ and $1 \leq s, r \leq n$, define $KK(n, s, r)$ as a simple connected graph with two copies of complete graph $K_n$ connected via a vertex in the middle. The left complete graph is joined to the middle vertex with $s$ edges and the right complete graph is joined to the middle vertex with $r$ edges. Hence $KK(n, s, r)$ has $2n+1$ vertices and $n^2 - n + s + r$ edges. $G_2$ in Figure 2 is indeed $KK(4, 1, 3)$.

By labeling the two copies of complete graph first and then the middle vertex last, the

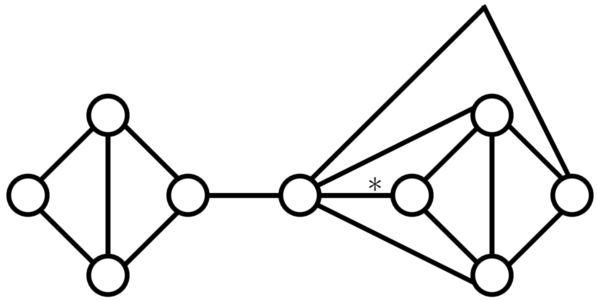


Figure 3: Graph $G_3$.

adjacency matrix of $KK(n,s,r)$ is

$$A = \begin{bmatrix} K & 0 & x_s \\ 0 & K & x_r \\ x_s^T & x_r^T & 0 \end{bmatrix}$$

where $K = J_n - I_n$ is the adjacency matrix of $K_n$, $x_s$ is an $n$-vector with the first $s$ entries equal 1 and the rest equal 0, $x_r$ is an $n$-vector with the first $r$ entries equal 1 and the rest equal 0. In particular, the adjacency matrix for the graph in Figure 2 is

$$A(G_2) = \left[\begin{array}{cccc|cccc|c} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right].$$

Let $\lambda_1 \geq \cdots \geq \lambda_{2n+1}$ be the eigenvalues of $A$. Since $\begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}$ is a principal submatrix of $A$, we have the following interlacing inequalities:

$$\lambda_1 \geq n-1 \geq \lambda_2 \geq n-1 \geq \lambda_3 \geq -1 \geq \lambda_4 \geq -1 \geq \cdots \geq -1 \geq \lambda_{2n} \geq -1 \geq \lambda_{2n+1}.$$

Hence $\lambda_2 = n-1$, $\lambda_4 = \cdots = \lambda_{2n} = -1$, and

$$\lambda_1 \geq n-1 \geq \lambda_3 \geq -1 \geq \lambda_{2n+1}.$$

On the other hand,

$$2trK = trA = \lambda_1 + n - 1 + \lambda_3 + (2n-3)(-1) + \lambda_{2n+1},$$

$$2trKK + 2x_s^T x_s + 2x_r^T x_r = trA^2 = \lambda_1^2 + (n-1)^2 + \lambda_3^2 + (2n-3)(-1)^2 + \lambda_{2n+1}^2,$$
$$2trK^3 + 3x_s^T K x_s + 3x_r^T K x_r = trA^3 = \lambda_1^3 + (n-1)^3 + \lambda_3^3 + (2n-3)(-1)^3 + \lambda_{2n+1}^3.$$

After simplification,

$$\lambda_1 + \lambda_3 + \lambda_{2n+1} = n-2,$$
$$\lambda_1^2 + \lambda_3^2 + \lambda_{2n+1}^2 = 2s + 2r + n^2 - 2n + 2,$$
$$\lambda_1^3 + \lambda_3^3 + \lambda_{2n+1}^3 = 3s(s-1) + 3r(r-1) + n^3 - 3n^2 + 3n.$$

Using Newton's identities, we deduce that $\lambda_1, \lambda_3$, and $\lambda_{2n+1}$ are zeros of the cubic polynomial

$$x^3 - (n-2)x^2 + (1-n-s-r)x - [s^2 + r^2 - (n-1)(s+r)].$$

Therefore we have the following lemma.

**Lemma 3.1.** The characteristic polynomial of $KK(n,s,r)$ is

$$(x-n+1)(x+1)^{2n-3}\left(x^3 - (n-2)x^2 + (1-n-s-r)x - [s^2 + r^2 - (n-1)(s+r)]\right).$$

Let the zeros of

$$p(x) = x^3 - (n-2)x^2 + (1-n-s-r)x - [s^2 + r^2 - (n-1)(s+r)] \tag{1}$$

be $\alpha_1, \alpha_2, \alpha_3$. Then

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= n-2, \\ \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 &= 1-n-s-r, \\ \alpha_1\alpha_2\alpha_3 &= s^2 + r^2 - (n-1)(s+r), \end{aligned}$$

and, by interlacing inequalities, $\alpha_1 \geq n-1 \geq \alpha_2 \geq -1 \geq \alpha_3$.

Similarly, let the zeros of

$$x^3 - (n-2)x^2 + (1-n-s-r+1)x - [s^2 + (r-1)^2 - (n-1)(s+r-1)] \tag{2}$$
$$\text{or} \quad x^3 - (n-2)x^2 + (1-n-s-r)x - [s^2 + r^2 - (n-1)(s+r)] + (x-n+2r) \tag{3}$$

be $\beta_1, \beta_2, \beta_3$. Then

$$\begin{aligned} \beta_1 + \beta_2 + \beta_3 &= n-2, \\ \beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1 &= 2-n-s-r, \\ \beta_1\beta_2\beta_3 &= s^2 + (r-1)^2 - (n-1)(s+r-1), \end{aligned}$$

and, by interlacing inequalities, $\beta_1 \geq n-1 \geq \beta_2 \geq -1 \geq \beta_3$. Moreover, $\alpha_1 > \beta_1$ since $KK(n,s,r-1)$ is a subgraph of the connected graph $KK(n,s,r)$. Also note that if polynomials (1) and (2) have a common zero then it must be $n-2r$.

Now

$$\mathcal{E}(KK(n,s,r)) = 3n-4+|\alpha_1|+|\alpha_2|+|\alpha_3|,$$

and

$$\mathcal{E}(KK(n,s,r-1)) = 3n-4+|\beta_1|+|\beta_2|+|\beta_3|.$$

Consequently, $\mathcal{E}(KK(n,s,r)) = \mathcal{E}(KK(n,s,r-1))$ if and only if $|\alpha_1|+|\alpha_2|+|\alpha_3| = |\beta_1|+|\beta_2|+|\beta_3|$.

**Theorem 3.2.** For $n \geq 2$ and $1 \leq r,s \leq n$, $\mathcal{E}(KK(n,s,r)) = \mathcal{E}(KK(n,s,r-1))$ if and only if

i. $s^2-(2r-1)s+2n^2r-8nr^2+8r^3-n^2+6nr-9r^2-n+3r=0$,

ii. $n<2r$,

iii. $s^2+r^2<(n-1)(s+r)$.

**Proof.** (Sufficiency) Conditions (i) and (ii) imply that $n-2r$ is a common negative root of polynomials (1) and (2) (via equation (3)). Conditions (ii) and (iii) imply that both $\alpha_2$ and $\beta_2$ are positive. Hence $\alpha_3 = \beta_3$. Therefore we have

$$\begin{aligned}
|\alpha_1|+|\alpha_2|+|\alpha_3| &= \alpha_1+\alpha_2-\alpha_3 \\
&= \alpha_1+\alpha_2+\alpha_3-2\alpha_3 \\
&= \beta_1+\beta_2+\beta_3-2\beta_3 \\
&= \beta_1+\beta_2-\beta_3 \\
&= |\beta_1|+|\beta_2|+|\beta_3|
\end{aligned}$$

and so $\mathcal{E}(KK(n,s,r)) = \mathcal{E}(KK(n,s,r-1))$.

(Necessity) Assume that $\mathcal{E}(KK(n,s,r)) = \mathcal{E}(KK(n,s,r-1))$, and so

$$|\alpha_1|+|\alpha_2|+|\alpha_3| = |\beta_1|+|\beta_2|+|\beta_3|.$$

First we claim that $\beta_2 > 0$. Otherwise $\beta_2 \leq 0$, and it leads to a contradiction as follows. If $\alpha_2 \leq 0$ then $\alpha_1 = \beta_1$ is a common zero of polynomials (1) and (2), then $\alpha_1 = \beta_1 = n-2r \geq n-1$, i.e., $1 \geq 2r$, a contradiction. If $\alpha_2 > 0$, then $\alpha_1+\alpha_2 = \beta_1 < \alpha_1$, also a contradiction.

Next we claim that $\alpha_2 > 0$. Otherwise $\alpha_2 \leq 0$, and it leads to a contradiction as follows. Since $\alpha_2 \leq 0$, we have $s^2+r^2-(n-1)(s+r) = \alpha_1\alpha_2\alpha_3 \leq 0$. Therefore we have the following 4 cases to consider.

**Case 1:** $s=r=n-1$.

Then $\alpha_1\alpha_2\alpha_3 = s^2 + r^2 - (n-1)(s+r) = 0$, and so $\alpha_2 = 0$ because $\alpha_1$ and $\alpha_3$ are nonzero. Therefore $\alpha_1 - \alpha_3 = |\alpha_1| + |\alpha_2| + |\alpha_3| = |\beta_1| + |\beta_2| + |\beta_3| = \beta_1 + \beta_2 - \beta_3$. Because $\alpha_1+\alpha_2+\alpha_3 = \beta_1+\beta_2+\beta_3$, it follows that $\alpha_3 = \beta_3$. Hence $\alpha_3 = \beta_3 = n-2r = n-2(n-1) = 2-n$, and $\alpha_1 = n-2-\alpha_3 = n-2-(2-n) = 2n-4$. Consequently, $3-3n = 1-n-s-r = \alpha_1\alpha_3 = (2n-4)(2-n)$ which is impossible because $n$ is an integer.

**Case 2:** $s = r = n$.

Then $\beta_1\beta_2\beta_3 = s^2+(r-1)^2-(n-1)(s+r-1) = n > 0$. However $\beta_2 > 0$ implies that $\beta_1\beta_2\beta_3 < 0$, a contradiction.

**Case 3:** $s = n$ and $r \le n-1$.

Since $0 > \beta_1\beta_2\beta_3 = s^2+(r-1)^2-(n-1)(s+r-1) = r^2-(n+1)r+2n$, it follows that

$$\frac{n+1-\sqrt{n^2-6n+1}}{2} < r < \frac{n+1+\sqrt{n^2-6n+1}}{2} \text{ and } n^2-6n+1 \ge 0.$$

Note that $2 < \frac{n+1-\sqrt{n^2-6n+1}}{2} < 3$ and $n-2 < \frac{n+1+\sqrt{n^2-6n+1}}{2} < n-1$ for $n > 6$. Consequently, $3 \le r \le n-2$ with $n > 6$.

On the other hand, $0 \ge \alpha_1\alpha_2\alpha_3 = s^2+r^2-(n-1)(r+s) = r^2-(n-1)r+n$. It follows that

$$r \le \frac{n-1-\sqrt{n^2-6n+1}}{2} \text{ or } r \ge \frac{n-1+\sqrt{n^2-6n+1}}{2}$$

and $n^2-6n+1 \ge 0$. Note that $1 < \frac{n-1-\sqrt{n^2-6n+1}}{2} < 2$ and $n-3 < \frac{n-1+\sqrt{n^2-6n+1}}{2} < n-2$ for $n > 6$. Consequently $r \le 1$ or $r \ge n-2$ with $n > 6$.

Eventually $r = n-2$ with $n > 6$, so we have

$$\begin{aligned} n-2 = \alpha_1+\alpha_2+\alpha_3 &= \beta_1+\beta_2+\beta_3, & (4)\\ 3-3n = \alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1 &= \beta_1\beta_2+\beta_2\beta_3+\beta_3\beta_1-1, & (5)\\ 2 = \alpha_1\alpha_2\alpha_3 &= \beta_1\beta_2\beta_3+n-4. & (6) \end{aligned}$$

Suppose that $|\alpha_1|+|\alpha_2|+|\alpha_3| = |\beta_1|+|\beta_2|+|\beta_3|$. Since $\alpha_2 \le 0$, by (4), $\alpha_1 = \beta_1+\beta_2$ and $\alpha_2+\alpha_3 = \beta_3$. And by (5), $\alpha_2\alpha_3 = \beta_1\beta_2 - 1$. And by (6), $\frac{2}{\alpha_1} = \frac{6-n}{\beta_3} - 1$ which gives $\beta_3 = (6-n)\frac{\alpha_1}{\alpha_1+2}$. Finally, by (4), $\alpha_1^2+(10-2n)\alpha_1+(4-2n) = 0$ which gives $\alpha_1 = n-5+\sqrt{n^2-8n+21}$. Put it back into (5), we have $3-3n = \alpha_1(\alpha_2+\alpha_3)+\alpha_2\alpha_3 = \alpha_1(n-2-\alpha_1)+\frac{2}{\alpha_1}$ and then $8n^4-146n^3+872n^2-2106n+2164 = 0$ which is impossible because $n$ is an integer.

**Case 4:** $s \le n-1$ and $r = n$.

Since $0 > \beta_1\beta_2\beta_3 = s^2+(r-1)^2-(n-1)(s+r-1) = s^2-(n-1)s$, it follows that $1 \le s \le n-2$.

On the other hand, $0 \geq \alpha_1\alpha_2\alpha_3 = s^2 + r^2 - (n-1)(r+s) = s^2 - (n-1)s + n$. It follows that

$$s \leq \frac{n-1-\sqrt{n^2-6n+1}}{2} \text{ or } s \geq \frac{n-1+\sqrt{n^2-6n+1}}{2}$$

and $n^2 - 6n + 1 \geq 0$. Consequently, $s \leq 1$ or $s \geq n-2$ with $n > 6$.

Eventually, we have two subcases to consider.

**Subsubcase 4.1:** $s = 1$ and $r = n$.

$$\begin{aligned} n-2 = \alpha_1 + \alpha_2 + \alpha_3 &= \beta_1 + \beta_2 + \beta_3 & (7) \\ -2n = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 &= \beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1 - 1 & (8) \\ 2 = \alpha_1\alpha_2\alpha_3 &= \beta_1\beta_2\beta_3 + n & (9) \end{aligned}$$

Suppose that $|\alpha_1| + |\alpha_2| + |\alpha_3| = |\beta_1| + |\beta_2| + |\beta_3|$. Since $\alpha_2 \leq 0$, by (7), it follows that $\alpha_1 = \beta_1 + \beta_2$ and so $\alpha_2 + \alpha_3 = \beta_3$. Consequently, from (8), $\alpha_2\alpha_3 = \beta_1\beta_2 - 1$. From (9), $\frac{2}{\alpha_1} = \frac{2-n}{\beta_3} - 1$ and so $\alpha_2 + \alpha_3 = \beta_3 = (2-n)\frac{\alpha_1}{\alpha_1+2}$. Now, from (7) again, $\alpha_1^2 + (6-2n)\alpha_1 + (4-2n) = 0$. It gives $\alpha_1 = n - 3 + \sqrt{n^2-4n+5}$. Put it back to (8), we obtain $4n^4 - 40n^3 + 136n^2 - 200n + 116 = 0$ which is impossible because $n$ is an integer.

**Subsubcase 4.2:** $s = n-2$ and $r = n$.

$$\begin{aligned} n-2 = \alpha_1 + \alpha_2 + \alpha_3 &= \beta_1 + \beta_2 + \beta_3 & (10) \\ 3-3n = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 &= \beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1 - 1 & (11) \\ 2 = \alpha_1\alpha_2\alpha_3 &= \beta_1\beta_2\beta_3 + n & (12) \end{aligned}$$

Suppose that $|\alpha_1| + |\alpha_2| + |\alpha_3| = |\beta_1| + |\beta_2| + |\beta_3|$. Since $\alpha_2 \leq 0$, by (10), it follows that $\alpha_1 = \beta_1 + \beta_2$ and so $\alpha_2 + \alpha_3 = \beta_3$. Consequently, from (11), $\alpha_2\alpha_3 = \beta_1\beta_2 - 1$. From (12), $\frac{2}{\alpha_1} = \frac{2-n}{\beta_3} - 1$ and so $\alpha_2 + \alpha_3 = \beta_3 = (2-n)\frac{\alpha_1}{\alpha_1+2}$. Now, from (10) again, $\alpha_1^2 + (6-2n)\alpha_1 + (4-2n) = 0$. It gives $\alpha_1 = n - 3 + \sqrt{n^2-4n+5}$. Put it back to (11), we obtain $8n^4 - 90n^3 + 352n^2 - 594n + 380 = 0$ which is impossible because $n$ is an integer.

Finally, from the two claims above, we conclude that $\alpha_3 = \beta_3$ is a common root of polynomials (1) and (2), hence $n - 2r = \alpha_3 = \beta_3 < 0$, which gives condition (ii). It follows that $n - 2r$ is a root of polynomial (1) and it gives condition (i). And $s^2 + r^2 - (n-1)(s+r) = \alpha_1\alpha_2\alpha_3 < 0$, which gives condition (iii). ■

**Remark 3.3.** The conditions (i), (ii), and (iii) in the Theorem 3.2 can be summarized as "$n - 2r$ is the only negative zero of the polynomial $p(x)$ in (1)".

By Theorem 3.2, there are many choices of parameters $n, s, r$ such that $\mathcal{E}(KK(n,s,r)) = \mathcal{E}(KK(n,s,r-1))$. For example, $G_2 = KK(4,1,3)$ in Figure 2 is the example with smallest parameters. Corollary 3.4 below gives explicitly an infinite subclass of these graphs, and hence confirms that the family described by the parameters in Theorem 3.2 is infinite.

**Corollary 3.4.** Let $s = n$. Then $\mathcal{E}(KK(n,s,r)) = \mathcal{E}(KK(n,s,r-1))$ if and only if $n = s = 4k^2 - 9k + 6$ and $r = 2k^2 - 4k + 3$ for $k \geq 3$.

**Proof.** Applying Theorem 3.2 with $s = n$, we see that $\mathcal{E}(KK(n,n,r)) = \mathcal{E}(KK(n,n,r-1))$ if and only if

i. $2n^2r - 8nr^2 + 8r^3 + 4nr - 9r^2 + 3r = 0$,

ii. $n < 2r$,

iii. $r^2 < (n-1)r - n$.

By taking $k = 2r - n$, one can check that (i), (ii), and (iii) hold if and only if $n = s = 4k^2 - 9k + 6$ and $r = 2k^2 - 4k + 3$ for $k \geq 3$. ∎

**Corollary 3.5** Let $s = 1$. Then $\mathcal{E}(KK(n,1,r)) = \mathcal{E}(KK(n,1,r-1))$ if and only if $n = 4$ and $r = 3$.

**Proof.** Applying Theorem 3.2 with $s = 1$, we see that $\mathcal{E}(KK(n,1,r)) = \mathcal{E}(KK(n,1,r-1))$ if and only if

i. $(2r-1)n^2 + (6r - 8r^2 - 1)n + 8r^3 - 9r^2 + r + 2 = 0$,

ii. $n < 2r$,

iii. $1 + r^2 < (n-1)(1+r)$.

One can check that (i), (ii), and (iii) hold if and only if $n = 4$ and $r = 3$. ∎

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