DETERMINANTAL AND EIGENVALUE INEQUALITIES FOR MATRICES WITH NUMERICAL RANGES IN A SECTOR

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Abstract. Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_n$, where $A_{11} \in M_m$ with $m \le n/2$, be such that the numerical range of A lies in the set $\{e^{i\varphi}z \in \mathbb{C} : |\Im z| \le (\Re z)\tan\alpha\}$, for some $\varphi \in [0, 2\pi)$ and $\alpha \in [0, \pi/2)$. We obtain the optimal containment region for the generalized eigenvalue λ satisfying

$$\lambda \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} x = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} x \quad \text{for some nonzero } x \in \mathbb{C}^n,$$

and the optimal eigenvalue containment region of the matrix $I_m - A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}$ in case A_{11} and A_{22} are invertible. From this result, one can show $|\det(A)| \le \sec^{2m}(\alpha)|\det(A_{11})\det(A_{22})|$. In particular, if A is a accretive-dissipative matrix, then $|\det(A)| \le 2^m|\det(A_{11})\det(A_{22})|$. These affirm some conjectures of Drury and Lin.

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1. Introduction.

Let M_n be the set of $n \times n$ complex matrices. Suppose

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_n \quad \text{with } A_{11} \in M_m, \quad m \le n/2. \tag{1.1}$$

In connection to the study of the growth factor in Gaussian elimination, researchers considered optimal (smallest) $\gamma > 0$ such that

$$|\det(A)| \le \gamma|\det(A_{11})\det(A_{22})|;$$

see [1, 3, 4, 6, 7, 8] and their references. The well-known Fischer inequality asserts that

$$\det(A) \le \det(A_{11})\det(A_{22}) \quad \text{if } A \text{ is positive semi-definite.}$$

In [7], it was shown that if A is accretive and dissipative, i.e., $A + A^*$ and $i(A^* - A)$ are positive semi-definite, then

$$|\det(A)| \le \gamma|\det(A_{11})\det(A_{22})| \qquad \text{with } \gamma = 3^m;$$

in [8], the bound was improved to

$$\gamma = \begin{cases} 2^{3m/2} & \text{if } m \le n/3, \\ 2^{n/2} & \text{if } n/3 < m \le n/2. \end{cases}$$

The author in [8] further proposed the following.

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Conjecture 1 *Suppose A is accretive-dissipative. Then*

$$|\det(A)| \le 2^m |\det(A_{11})\det(A_{22})|. \quad (1.2)$$

The numerical range of a matrix $L \in M_n$ is defined by

$$W(L) = \{x^* L x : x \in \mathbb{C}^n,\ x^* x = 1\}.$$

For any $\alpha \in [0, \pi/2)$, let

$$S_\alpha = \{z \in \mathbb{C} : |\Im z| \le (\Re z)\tan\alpha\}. \quad (1.3)$$

A subset of $\mathbb{C}$ is a sector of half angle α if it is of the form $\{e^{i\varphi}z : z \in S_\alpha\}$ for some $\varphi \in [0, 2\pi)$.

In [1], the author proved that if $W(A)$ is a subset of a sector of half angle $\alpha \in [0, \pi/(2m))$, then

$$|\det(A)| \le \sec^2(m\alpha)|\det(A_{11})\det(A_{22})|.$$

He further proposed the following.

Conjecture 2 *If $W(A)$ is a subset of a sector of half angle $\alpha \in [0, \pi/2)$, then*

$$|\det(A)| \le \sec^{2m}(\alpha)|\det(A_{11})\det(A_{22})|. \quad (1.4)$$

Moreover, if $A_{11} \in M_1$ is nonzero and $A_{22} \in M_{n-1}$ is invertible, then

$$\det(A)/(\det(A_{11})\det(A_{22})) \in \{re^{i2\phi} : 0 \le r \le 2(\cos(2\phi) - \cos(2\alpha))/\sin^2(2\alpha), -\alpha \le \phi \le \alpha\}. \quad (1.5)$$

We will affirm Conjectures 1 and 2 via the study of the following generalized eigenvalue problem

$$\lambda \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} x = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} x \quad \text{for some nonzero } x \in \mathbb{C}^n. \quad (1.6)$$

In Section 2, we prove Theorem 2.4 providing the optimal eigenvalue containment region for those λ satisfying (1.6), and the optimal eigenvalue containment region of the matrix

$$A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}$$

in case A_{11} and A_{22} are invertible. Using the theorem, one can readily verify Conjectures 1 and 2; see Corollaries 2.6 and 2.7.

2. Results, proofs, and remarks. In this section, we will always assume that A has the form in (1.1), and refer to a subset of $\mathbb{C}$ as a sector of half angle α if it is of the form $\{e^{i\varphi}z : z \in S_\alpha\}$ for some $\varphi \in [0, 2\pi)$ and S_α defined in (1.3).

We begin with several lemmas.

Lemma 2.1. *For any $\phi \in [-\pi/2, \pi/2]$, the function $f : (0, \pi/2) \to \mathbb{R}$ defined by*

$$f(\theta) = (\cos(2\phi) - \cos(2\theta))/\sin^2(2\theta)$$

is increasing.

Proof. By direct verification. □

LEMMA 2.2. *Suppose* $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2} \in \mathbb{C}$ *with* $r_1, r_2 \in (0, \infty)$, $\pi/2 > \theta_1 \geq \theta_2 > -\pi/2$, *and* $(z_1 + z_2)/2 = e^{i\psi}$. *Let* $2\theta = \theta_1 - \theta_2$ *and* $2\phi = \theta_1 + \theta_2 - 2\psi$. *Then*

$$r_1 r_2 = 2(\cos(2\phi) - \cos(2\theta))/\sin^2(2\theta).$$

Proof. Consider the triangle T with vertices $0, z_1, z_2$. Because $e^{i\psi}$ is the midpoint of the side joining the vertices z_1 and z_2, T can be divided into two triangles T_1 and T_2 with equal areas, where T_1 has vertices $0, e^{i\psi}, r_1 e^{i\theta_1}$, and T_2 has vertices $0, e^{i\psi}, r_2 e^{i\theta_2}$. Thus,

$$r_1 r_2 \sin(2\theta) = r_1 \sin(\theta_1 - \psi) + r_2 \sin(\psi - \theta_2) \quad \text{and} \quad r_1 \sin(\theta_1 - \psi) = r_2 \sin(\psi - \theta_2).$$

It follows that

$$\begin{aligned}(r_1 r_2 \sin(2\theta))^2 &= (r_1 \sin(\theta_1 - \psi) + r_2 \sin(\psi - \theta_2))^2 \\ &= 4 r_1 r_2 \sin(\theta_1 - \psi) \sin(\psi - \theta_2) \\ &= 2 r_1 r_2 (\cos(2\phi) - \cos(2\theta)).\end{aligned}$$

Hence, $r_1 r_2 = 2(\cos(2\phi) - \cos(2\theta))/\sin^2(2\theta)$. □

We will also use some basic facts about the numerical range; for example, see [5, Chapter 1].

LEMMA 2.3. *Let* $L \in M_n$.

1. *If* $U \in M_n$ *is unitary, then* $W(L) = W(U^* L U)$.
2. *The set* $W(L)$ *is compact and convex.*
3. *If* $\tilde{L}$ *is a principal submatrix of* L, *then* $W(\tilde{L}) \subseteq W(L)$.
4. *If* $L = L_1 \oplus L_2$, *then* $W(L) = \operatorname{conv}(W(L_1) \cup W(L_2))$, *where* $\operatorname{conv}(S)$ *denotes the convex hull of the set* S.
5. *If* L *is normal, then* $W(L)$ *is the convex hull of its eigenvalues.*
6. *If* $x \in \mathbb{C}^n$ *is a unit vector such that* $\mu = x^* L x$ *is a boundary point of* $W(L)$ *with more than one support line, then* $Lx = \mu x$ *and* $L^* x = \overline{\mu} x$.

THEOREM 2.4. *Let* $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_n$ *with* $A_{11} \in M_m$ *be such that* $m \leq n/2$, *and* $W(A)$ *be a subset of a sector of half angle* $\alpha \in [0, \pi/2)$.

(a) *Suppose* $A_{11} \oplus A_{22}$ *is singular, and* $x \in \mathbb{C}^n$ *is a nonzero vector in its kernel. Then* $Ax = 0$ *and* (1.6) *holds for every* $\lambda \in \mathbb{C}$ *with this nonzero vector* x.

(b) *Suppose* A_{11} *and* A_{22} *are invertible. If* $\lambda \in \mathbb{C}$ *satisfies* (1.6), *then*

$$\lambda^2 \in \mathcal{R} = \begin{cases} \left\{1 - re^{i2\phi} : 0 \leq r \leq \frac{2(\cos(2\phi) - \cos(2\alpha))}{\sin^2(2\alpha)}, -\alpha \leq \phi \leq \alpha\right\} & \text{if } \alpha > 0, \\ [0, 1] & \text{if } \alpha = 0. \end{cases}$$

Moreover, for every eigenvalue μ *of the matrix* $A_{11}^{-1} A_{12} A_{22}^{-1} A_{21}$, *there is* $\lambda \in \mathbb{C}$ *satisfying* (1.6) *so that* $\mu = \lambda^2$ *lies in the region* $\mathcal{R}$.

Proof. Without loss of generality, we may assume that $W(A) \subseteq S_\alpha$. Let

$$B_1 = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix}.$$

(a) Suppose B_1 is singular and $x \in \mathbb{C}^n$ is nonzero such that $B_1 x = 0$. By Lemma 2.3, for $y = x/\|x\|$,

$$0 = y^* B_1 y \in \text{conv}\,(W(A_{11} \oplus A_{22})) \subseteq W(A)$$

so that 0 is a boundary point of $W(A)$ with more than one support line. Thus, $Ax = 0$, and $\lambda B_1 x = 0 = (A - B_1)x = B_2 x$ for any $\lambda \in \mathbb{C}$.

(b) Assume A_{11} and A_{22} are invertible. Suppose $\lambda B_1 x = B_2 x$ for some nonzero unit vector $x \in \mathbb{C}^n$. Let $\xi_1 = x^* B_1 x$ and $\xi_2 = x^* B_2 x$. Then $\xi_1 \in W(B_1)$ and $\xi_2 \in W(B_2)$. We see that $\xi_1 + \xi_2 \in W(B_1 + B_2) = W(A)$ and $\lambda \xi_1 = \xi_2$. Note that $A = B_1 + B_2 = Q^*(B_1 - B_2)Q$ with $Q = I_m \oplus (-I_{n-m})$, and hence $W(A) = W(B_1 - B_2)$ contains $x^*(B_1 - B_2)x = \xi_1 - \xi_2$. So, $\xi_1 \pm \xi_2 \in W(A)$, which is a subset of S_α by our assumption.

Observe that $\xi_1 \neq 0$. Otherwise, by Lemma 2.3

$$0 \in W(B_1) = \text{conv}\,(W(A_{11}) \cup W(A_{22})) \subseteq W(A)$$

so that 0 is a boundary point of $W(B_1)$ with more than one support line implying that B_1 is singular, which contradicts our assumption.

Without loss of generality, assume that $\Im(\lambda) = \Im(1 + \xi_2/\xi_1) \geq 0$. Let

$$z_\pm = r_\pm e^{i\theta_\pm} = 1 \pm \xi_2/\xi_1 = 1 \pm \lambda \quad \text{with} \quad r_\pm \geq 0, \quad \theta = \frac{1}{2}(\theta_+ - \theta_-) \quad \text{and} \quad \phi = \frac{1}{2}(\theta_+ + \theta_-).$$

If $\xi_1 = |\xi_1| e^{i\omega}$, then $(\xi_1 \pm \xi_2) = \xi_1(1 \pm \xi_2/\xi_1)$ has arguments $\theta_\pm + \omega \in [-\alpha, \alpha]$ as $\xi_1 \pm \xi_2 \in W(A) \subseteq S_\alpha$. Note also that $(z_1 + z_2)/2 = 1$ has argument 0. It follows that $-\alpha - \omega \leq \theta_- \leq 0 \leq \theta_+ \leq \alpha - \omega$. So $0 \leq \theta \leq \alpha$. Applying Lemma 2.2 with $(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) = (r_+ e^{i\theta_+}, r_- e^{i\theta_-})$ and $\psi = 0$, we have

$$r_+ r_- = \frac{2(\cos(2\phi) - \cos(2\theta))}{\sin^2(2\theta)} \leq \frac{2(\cos(2\phi) - \cos(2\alpha))}{\sin^2(2\alpha)},$$

where the inequality follows from Lemma 2.1. As a result,

$$1 - \lambda^2 = z_+ z_- = r_+ r_- e^{i(\theta_+ + \theta_-)} = r_+ r_- e^{i2\phi}$$

lies in the region

$$\tilde{R} = \left\{ r e^{i2\phi} : 0 \leq r \leq \frac{2(\cos(2\phi) - \cos(2\alpha))}{\sin^2(2\alpha)}, -\alpha \leq \phi \leq \alpha \right\}.$$

Suppose $B = B_1^{-1} B_2$. Then

$$B^2 = \begin{pmatrix} A_{11}^{-1} A_{12} A_{22}^{-1} A_{21} & 0 \\ 0 & A_{22}^{-1} A_{21} A_{11}^{-1} A_{12} \end{pmatrix}.$$

If $A_{11}^{-1} A_{12} A_{22}^{-1} A_{21}$ has eigenvalues $\mu_1, \ldots, \mu_m$, then $A_{22}^{-1} A_{21} A_{11}^{-1} A_{12}$ has eigenvalues $\mu_1, \ldots, \mu_m$ together with $n - m$ zeros. So, we may assume that B has eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $\lambda_j^2 = \lambda_{m+j}^2 = \mu_j$ for $j = 1, \ldots, m$ and $\lambda_\ell = 0$ for $\ell = 2m + 1, \ldots, n$. Note that λ is an eigenvalue of B if and only if λ satisfies (1.6) for some nonzero $x \in \mathbb{C}^n$. The second assertion of (b) follows. □

The containment region in Theorem 2.4 (b) is optimal as shown in the following.

EXAMPLE 2.5. Let $\lambda \in \mathbb{C}$ be such that $\lambda^2 \in \mathcal{R}$ in Theorem 2.4, i.e., $1-\lambda^2 = re^{i2\phi}$ with $r \in \big[0, 2(\cos(2\phi)-\cos(2\alpha))/\sin^2(2\alpha)\big]$ for some $\alpha \in [0, \pi/2)$.

Suppose $r > 0$. By Lemma 2.1, there is $\theta \in (|\phi|, \alpha]$ satisfying $r = 2(\cos(2\phi)-\cos(2\theta))/\sin^2(2\theta)$, here we set $2(\cos(2\phi)-\cos(2\theta))/\sin^2(2\theta) = 1$ if $\theta = 0$ which will imply $\phi = 0$. Let

$$A = \left(I_m \otimes \begin{pmatrix} e^{-i\phi} & a+ib \\ a+ib & e^{-i\phi} \end{pmatrix}\right) \oplus (e^{-i\phi} I_{n-2m})$$

with $a = -\cot\theta \sin\phi$ and $b = \tan\theta\cos\phi$ so that

$$|a| \le |\cot\phi \sin\phi| = \cos\phi, \qquad b \ge |\tan\phi\cos\phi| = |\sin\phi|,$$

$$\frac{-\sin\phi + b}{\cos\phi + a} = \tan\theta, \qquad \text{and} \qquad \frac{-\sin\phi - b}{\cos\phi - a} = -\tan\theta. \tag{2.1}$$

Then A is normal, and by (2.1) the eigenvalues of A has the form $e^{-i\phi} + (a+ib) = r_1 e^{i\theta}$, $r_1 \ge 0$, with multiplicity m, $e^{-i\phi} - (a+ib) = r_2 e^{-i\theta}$, $r_2 \ge 0$, with multiplicity m, and $e^{-i\phi}$ with multiplicity $n-2m$, all in S_α. By Lemma 2.3, $W(A) \subseteq S_\alpha$. Moreover, $\lambda = \pm(a+ib)e^{i\phi}$ satisfy (1.6), and

$$1-\lambda^2 = \det\begin{pmatrix} 1 & (a+ib)e^{i\phi} \\ (a+ib)e^{i\phi} & 1 \end{pmatrix} = e^{2i\phi}\det\begin{pmatrix} e^{-i\phi} & a+ib \\ a+ib & e^{-i\phi} \end{pmatrix} = e^{2i\phi} r_1 e^{i\theta} r_2 e^{-i\theta} = r_1 r_2 e^{2i\phi}.$$

Applying Lemma 2.2 to $r_1e^{i\theta}, r_2e^{-i\theta}$ so that $\theta_1 = -\theta_2 = \theta$ and $\psi = -\phi$, and using the fact that $e^{-i\phi}$ is the midpoint of the line segment joining $r_1e^{i\theta}, r_2e^{-i\theta}$, we have

$$r_1 r_2 = 2(\cos(2\phi)-\cos(2\theta))/\sin^2(2\theta) = r.$$

Suppose $r = 0$. One can verify directly that the matrix

$$A = \left(I_m \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) \oplus I_{n-2m}$$

satisfies $W(A) = [0,2] \subseteq S_\alpha$ and has generalized eigenvalues $\lambda \in \{1,-1\}$ so that $1-\lambda^2 = 0 = r$. □

Using the notation of Theorem 2.4, we see that if A_{11} and A_{22} are invertible, then all eigenvalues of the matrix $C = I_m - A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}$ lie in the set $\left\{re^{i2\phi} : 0 \le r \le \frac{2(\cos(2\phi)-\cos(2\alpha))}{\sin^2(2\alpha)}, |\phi| \le \alpha\right\}$. Thus, the spectral radius of C is bounded by

$$\max_{|\phi|\le\alpha}\left\{\frac{2(\cos(2\phi)-\cos(2\alpha))}{\sin^2(2\alpha)}\right\} = \frac{2(1-\cos(2\alpha))}{\sin^2(2\alpha)} = \sec^2(\alpha),$$

and hence

$$|\det(A)| = |\det(A_{11})\det(A_{22})\det(C)| \le \sec^{2m}(\alpha)|\det(A_{11})\det(A_{22})|.$$

By continuity, one can remove the invertibility assumption on A_{11} and A_{22}. We have the following corollary affirming Conjecture 2.

Corollary 2.6. *Let* $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_n$ *with* $A_{11} \in M_m$ *such that* $m \leq n/2$, *and* $W(A)$ *be a subset of a sector of half angle* $\alpha \in [0, \pi/2)$. *Then*

$$|\det(A)| \leq \sec^{2m}(\alpha) |\det(A_{11}) \det(A_{22})|.$$

If A_{11} *and* A_{22} *are invertible, then the eigenvalues of the matrix* $C = I_m - A_{11}^{-1} A_{12} A_{22}^{-1} A_{21} \in M_m$ *lies in the region*

$$\tilde{\mathcal{R}} = \left\{ re^{i2\phi} : 0 \leq r \leq \frac{2(\cos(2\phi) - \cos(2\alpha))}{\sin^2(2\alpha)}, -\alpha \leq \phi \leq \alpha \right\}.$$

Suppose A is accretive-dissipative. Then $W(e^{-i\pi/4}A) \subseteq S_{\pi/4}$. Applying Corollary 2.6 with $\alpha = \pi/4$, we have the following result affirming Conjecture 1 and verifying a comment in [1] (after Conjecture 0.2).

Corollary 2.7. *Let* $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_n$ *be accretive-dissipative with* $A_{11} \in M_m$ *such that* $m \leq n/2$. *Then*

$$|\det(A)| \leq 2^m |\det(A_{11}) \det(A_{22})|.$$

If A_{11} *and* A_{22} *are invertible, then the eigenvalues of the matrix* $C = I_m - A_{11}^{-1} A_{12} A_{22}^{-1} A_{21} \in M_m$ *lies in the set*

$$\{z \in \mathbb{C} : |z - 1| \leq 1\}.$$

The bound in Corollary 2.6 is best possible by Example 2.5 with $(\phi, \theta) = (0, \alpha)$. In such a case, $(e^{-i\phi}, a + bi) = (1, i \tan \alpha)$, $\det(A) = (1 + \tan^2 \alpha)^m = \sec^{2m}(\alpha)$ and $1 = \det(A_{11}) = \det(A_{22})$. Furthermore, letting $\alpha = \pi/4$, we see that the bound in Corollary 2.7 is also best possible.

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