

LINEAR MAPS PRESERVING KY FAN NORMS AND SCHATTEN NORMS OF TENSOR PRODUCTS OF MATRICES

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Abstract. For a positive integer n , let M_n be the set of $n \times n$ complex matrices. Suppose $\|\cdot\|$ is the Ky Fan k -norm with $1 \leq k \leq mn$ or the Schatten p -norm with $1 \leq p \leq \infty$ ($p \neq 2$) on M_{mn} , where $m, n \geq 2$ are positive integers. It is shown that a linear map $\phi : M_{mn} \rightarrow M_{mn}$ satisfying

$$\|A \otimes B\| = \|\phi(A \otimes B)\| \quad \text{for all } A \in M_m \text{ and } B \in M_n$$

if and only if there are unitary $U, V \in M_{mn}$ such that ϕ has the form $A \otimes B \mapsto U(\varphi_1(A) \otimes \varphi_2(B))V$, where $\varphi_s(X)$ is either the identity map $X \mapsto X$ or the transposition map $X \mapsto X^t$. The results are extended to tensor space $M_{n_1} \otimes \cdots \otimes M_{n_m}$ of higher level. The connection of the problem to quantum information science is mentioned.

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1. Introduction and preliminaries. For a positive integer n , let M_n be the set of $n \times n$ complex matrices. Now, suppose that $m, n \geq 2$ are positive integers. Then for $A \in M_m$ and $B \in M_n$, we denote by $A \otimes B \in M_{mn}$ their *tensor product* (a.k.a. the *Kronecker product*). In many applied and pure studies, one considers the tensor product of matrices; for example, see [2, 9, 18, 21]. Most noticeably, the tensor product is often used in quantum information science [19]. In a quantum system, quantum states are represented as density matrices (positive semi-definite matrices with trace one). Suppose $A \in M_m$ and $B \in M_n$ are two quantum states in two quantum systems. Then their tensor product $A \otimes B$ describes the joint state in the bipartite system, in which the general states are density matrices in M_{mn} . More generally, one may consider tensor states and general states in a multipartite system $M_{n_1} \otimes \cdots \otimes M_{n_m}$ identified with M_N where $N = \prod_{i=1}^m n_i$.

In general, it is relatively easy to construct and extract information from matrices in tensor product form. For instance, the eigenvalues (respectively, the singular values) of $A \otimes B$ have the form $a_i b_j$ with $1 \leq i \leq m$ and $1 \leq j \leq n$ if $A \in M_m$ and $B \in M_n$ have eigenvalues (respectively, singular values) a_1, \dots, a_m and b_1, \dots, b_n , respectively. Thus, it is interesting to get information on the tensor space M_{mn} by examining the properties of the small collection of matrices in tensor form $A \otimes B$. In particular, if we consider a linear map $\phi : M_{mn} \rightarrow M_{mn}$ and if one knows the images $\phi(A \otimes B)$ for $A \in M_m$ and $B \in M_n$, then the map ϕ can be completely characterized as every $C \in M_{mn}$ is a linear combination of matrices in tensor form $A \otimes B$. Nevertheless, the challenge is to use the limited information of the linear map ϕ on matrices in tensor form to determine the structure of ϕ . In [5], we considered linear maps preserving the spectrum $\sigma(A \otimes B)$ and spectral

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radius $r(A \otimes B)$ of Hermitian matrices $A \in M_m$ and $B \in M_n$. In [10], the author considered linear maps $\phi : M_{mn} \rightarrow M_{mn}$ satisfying

$$\| \|A \otimes B\| \| = \| \|\phi(A \otimes B)\| \| \quad \text{for all } A \in M_m \text{ and } B \in M_n,$$

where $\| \cdot \|$ is a certain (separability) norm defined in [11]. This family of (separability) norms was shown to be related to the problem of detecting bounded entangled non-positive partial transpose states and characterizing k -positive linear maps in quantum information science.

Suppose $X \in M_N$ has singular values $s_1(X) \geq \dots \geq s_N(X)$. The *Ky Fan k -norm* of X is defined by

$$\|X\|_{(k)} = s_1(X) + \dots + s_k(X).$$

The Ky Fan 1-norm reduces to the *spectral norm* and the Ky Fan N -norm is also called the *trace norm*. For $p \geq 1$, the *Schatten p -norm* of X is defined by

$$\|X\|_p \equiv \left(\sum_{i=1}^N s_i(X)^p \right)^{1/p}.$$

The limiting case $p = \infty$ is just the spectral norm, $\| \cdot \|_1$ is the trace norm, and $\| \cdot \|_2$ is the *Frobenius norm*, i.e., $\|X\|_2 = (\text{tr}(XX^*))^{1/2}$. In bipartite quantum systems, a well known criterion for separability of a state is the computable cross norm (CCNR) criterion [4, 20], which asserts that if a state (density matrix) X in M_{mn} is separable, the trace norm of the realignment of X (see e.g. [4]) is at most 1.

The purpose of this paper is to study linear maps $\phi : M_{mn} \rightarrow M_{mn}$ satisfying

$$\|A \otimes B\| = \|\phi(A \otimes B)\| \quad \text{for all } A \in M_m \text{ and } B \in M_n,$$

where $\| \cdot \|$ denotes the Ky Fan k -norm with $1 \leq k \leq mn$, or the Schatten p -norm with $1 \leq p \leq \infty$.

Note that even if we know that $\phi : M_{mn} \rightarrow M_{mn}$ is linear and satisfies $\|A \otimes B\| = \|\phi(A \otimes B)\|$ for all $A \in M_m$ and $B \in M_n$, it does not ensure that $\|A_1 \otimes B_1 + A_2 \otimes B_2\| = \|\phi(A_1 \otimes B_1) + \phi(A_2 \otimes B_2)\|$ because $A_1 \otimes B_1 + A_2 \otimes B_2$ may not be of the form $A \otimes B$. Thus, the proofs of our main results (Theorems 2.6 and 2.9) are quite delicate as shown in the following discussion (in Section 2). We will also extend the results to multipartite systems $M_{n_1} \otimes \dots \otimes M_{n_m}$ in Section 3.

One may see [1, 3, 7, 13] and their references for some background on linear preserver problems, and the preservers of the Ky Fan k -norms and Schatten p -norms (without the tensor structure). It was shown in these papers that such norm preservers (except for the Schatten 2-norm preservers) $\phi : M_n \rightarrow M_n$ have the form

$$\phi(A) = UAV \quad \text{or} \quad \phi(A) = UA^tV$$

for some unitary matrices $U, V \in M_n$, where A^t is the transpose of A . One can also see [5, 6, 10, 14] and their references for some recent results on linear preserver problems on tensor spaces arising in quantum information science.

In our discussion, we will use X^t and X^* to denote the transpose and the conjugate transpose of a square matrix X , respectively. For any $A \in M_m$ and $B \in M_n$, we denote by $A \oplus B$ their direct sum. The $n \times n$ identity matrix will be denoted by I_n . Denote by E_{ij} the square matrix which the (i, j) -entry is equal to one and all the others are equal to zero, where the size of E_{ij} should be clear in the context.

2. Bipartite systems.

2.1. Spectral norm. In what follows we denote by $\|\cdot\|$ the spectral norm. Recall that the spectral norm is the same as Ky Fan 1-norm. We first present the result for spectral norm.

THEOREM 2.1. *The following are equivalent for a linear map $\phi : M_{mn} \rightarrow M_{mn}$.*

- (a) $\|\phi(A \otimes B)\| = \|A \otimes B\|$ for all $A \in M_m$ and $B \in M_n$.
- (b) *There are unitary matrices $U, V \in M_{mn}$ such that*

$$\phi(A \otimes B) = U(\varphi_1(A) \otimes \varphi_2(B))V \quad \text{for all } A \in M_m \text{ and } B \in M_n,$$

where φ_s is the identity map or the transposition map $X \mapsto X^t$ for $s = 1, 2$.

Proof. The implication (b) \Rightarrow (a) is obvious. Conversely, assume that $\|\phi(A \otimes B)\| = \|A \otimes B\|$ for all $A \in M_m$ and $B \in M_n$. In the following, we first show that ϕ maps the matrix $E_{ii} \otimes E_{jj}$ to $U(E_{ii} \otimes E_{jj})V$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$, where U and V are some unitary matrices.

Let $1 \leq i \leq m$ and $1 \leq j, s \leq n$ with $j \neq s$. Then $\phi(E_{ii} \otimes E_{jj})$ has norm one and the same is true for $\phi(E_{ii} \otimes E_{jj} + \gamma E_{ii} \otimes E_{ss})$ whenever $|\gamma| \leq 1$. Suppose $x_j, y_j \in \mathbb{C}^{mn}$ are the left and right norm attaining unit vectors of $\phi(E_{ii} \otimes E_{jj})$, i.e., $x_j^* \phi(E_{ii} \otimes E_{jj}) y_j = 1$. Then for any unitary matrices X_j and Y_j with x_j and y_j as their first columns, we have $X_j^* \phi(E_{ii} \otimes E_{jj}) Y_j = [1] \oplus G_j$ for some $G_j \in M_{mn-1}$ with $\|G_j\| \leq 1$. Since $\phi(E_{ii} \otimes E_{jj} + \gamma E_{ii} \otimes E_{ss})$ has norm one for all $|\gamma| \leq 1$, the matrix $X_j^* \phi(E_{ii} \otimes E_{ss}) Y_j$ must have the form $[0] \oplus G_s$ for some $G_s \in M_{mn-1}$. It follows that the left and right norm attaining vectors of $\phi(E_{ii} \otimes E_{ss})$, say x_s and y_s , must be orthogonal to x_j and y_j respectively. As j and s are arbitrary, it follows that $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ are orthonormal sets.

Let U_i be an $mn \times mn$ unitary matrix with x_1, \dots, x_n as its first n columns and V_i be an $mn \times mn$ unitary matrix with y_1^*, \dots, y_n^* as its first n rows. From the above discussion, one has

$$\phi(E_{ii} \otimes E_{jj}) = U_i(E_{jj} \oplus P_{ij})V_i \quad \text{for all } j = 1, \dots, n$$

for some $P_{ij} \in M_{mn-n}$ and hence

$$\phi(E_{ii} \otimes D) = U_i(D \oplus P_{i,D})V_i \quad \text{for any diagonal matrix } D \in M_n$$

for some $P_{i,D} \in M_{mn-n}$. Note also that if $1 \leq i, r \leq m$ with $i \neq r$, then $\phi(E_{ii} \otimes D + \gamma E_{rr} \otimes D)$ has norm one for any diagonal unitary matrix $D \in M_n$ and any scalar γ with $|\gamma| \leq 1$. Consequently, we see that the left and right norm attaining vectors of $\phi(E_{ii} \otimes D)$ and those of $\phi(E_{rr} \otimes D)$ are orthogonal. By a similar argument, there are unitary $U, V \in M_{mn}$ such that

$$\phi(E_{ii} \otimes D) = U(E_{ii} \otimes D)V \quad \text{for any } 1 \leq i \leq m \text{ and unitary diagonal } D \in M_n.$$

In particular,

$$\phi(E_{ii} \otimes E_{jj}) = U(E_{ii} \otimes E_{jj})V \quad \text{for } 1 \leq i \leq m \text{ and } 1 \leq j \leq n.$$

For the sake of the simplicity, we assume that U and V are identity matrices. Next, we show that $\phi(E_{ii} \otimes B) = E_{ii} \otimes \varphi_i(B)$ for all $B \in M_n$, where φ_i is a linear map on M_n . For any unitary $Y \in M_n$ and $1 \leq i \leq m$, $1 \leq j \leq n$ we can apply the argument in the preceding paragraphs to conclude that $\phi(E_{ii} \otimes YE_{jj}Y^*)$ has rank one. Since

$$\|\phi(E_{ii} \otimes I_n) + \gamma\phi(E_{ii} \otimes YE_{jj}Y^*)\| = \|\phi(E_{ii} \otimes I_n + \gamma E_{ii} \otimes YE_{jj}Y^*)\| = 1 + \gamma$$

for any positive scalar γ , the left and right norm attaining vectors of $\phi(E_{ii} \otimes YE_{jj}Y^*)$ are also left and right norm attaining vectors of $\phi(E_{ii} \otimes I_n) = E_{ii} \otimes I_n$. Thus, $\phi(E_{ii} \otimes YE_{jj}Y^*) = E_{ii} \otimes Z$ for some rank one $Z \in M_n$. Since this is true for any unitary $Y \in M_n$, by linearity of ϕ , we conclude that there exists a linear map $\varphi_i : M_n \rightarrow M_n$ such that

$$\phi(E_{ii} \otimes B) = E_{ii} \otimes \varphi_i(B) \quad \text{for all } B \in M_n.$$

Clearly, φ_i preserves the spectral norm. Therefore, φ_i has the form $X \mapsto W_i X \tilde{W}_i$ or $X \mapsto W_i X^t \tilde{W}_i$ for some unitary $W_i, \tilde{W}_i \in M_n$ (e.g., see [3] and its references). For simplicity, we may assume that $W_i = \tilde{W}_i = I_n$ for all $i = 1, \dots, m$.

Let $X \in M_m$ be any unitary matrix. Repeating the same argument as above, one can show that

$$\phi(XE_{ii}X^* \otimes B) = U_X(E_{ii} \otimes \varphi_{i,X}(B))V_X$$

for $1 \leq i \leq m$ and $B \in M_n$, where $U_X, V_X \in M_{mn}$ are unitary matrices depending on X and $\varphi_{i,X} : M_n \rightarrow M_n$ is either the identity map or the transposition map depending on i and X . Moreover, since $\phi(I_{mn}) = I_{mn}$, we have $V_X = U_X^*$.

Now we show that all the maps $\varphi_{i,X}$ are the same. For any real symmetric $S \in M_n$ and any unitary $X \in M_m$ we have

$$\phi(I_m \otimes S) = \phi\left(\sum_{i=1}^m XE_{ii}X^* \otimes S\right) = U_X\left(\sum_{i=1}^m XE_{ii}X^* \otimes S\right)U_X^* = U_X(I_m \otimes S)U_X^*.$$

In particular, when $X = I_m$, we have $\phi(I_m \otimes S) = I_m \otimes S$. Thus, $U_X(I_m \otimes S)U_X^* = I_m \otimes S$ and this yields that U_X commutes with $I_m \otimes S$ for all real symmetric S . Hence, U_X has the form $W_X \otimes I_n$ for some unitary $W_X \in M_m$ and

$$\phi(XE_{ii}X^* \otimes B) = (W_X E_{ii} W_X^*) \otimes \varphi_{i,X}(B) \quad \text{for } 1 \leq i \leq m \text{ and } B \in M_n.$$

Now, consider the linear maps $\text{tr}_1 : M_{mn} \rightarrow M_n$ and $\text{Tr}_1 : M_{mn} \rightarrow M_n$ defined by

$$\text{tr}_1(A \otimes B) = (\text{tr } A)B \quad \text{and} \quad \text{Tr}_1(A \otimes B) = \text{tr}_1(\phi(A \otimes B))$$

for all $A \in M_m$ and $B \in M_n$. Notice that the map tr_1 is known as the partial trace function in quantum information science context. Then

$$\text{Tr}_1(\phi(XE_{ii}X^* \otimes B)) = \varphi_{i,X}(B).$$

So, Tr_1 induces a map $XE_{ii}X^* \mapsto \varphi_{i,X}$, where $\varphi_{i,X}$ is either the identity map or the transpose map. Note that Tr_1 is linear and therefore continuous, and the set

$$\{XE_{ii}X^* : 1 \leq i \leq m, X^*X = I_m\} = \{xx^* \in M_m : x^*x = 1\}$$

is connected. So, all the maps $\varphi_{i,X}$ have to be the same. Replacing ϕ by the map $A \otimes B \mapsto \phi(A \otimes B^t)$, if necessary, we may assume that this common map is the identity map. Next, using the linearity of ϕ , one can conclude that for every $A \in M_m$ and $B \in M_n$ we have

$$\phi(A \otimes B) = \varphi_1(A) \otimes B,$$

where $\varphi_1(A) \in M_m$ depends on A only. Recall that $\varphi_1 : M_m \rightarrow M_m$ is a linear map and $\|\varphi_1(A)\| = \|A\|$ for all $A \in M_m$. Hence, φ_1 has the form $A \mapsto UAV$ or $A \mapsto UA^tV$ for some unitary $U, V \in M_m$. This completes the proof. \square

2.2. Ky Fan k -norms. We now turn to Ky Fan k -norms. Two matrices $A, B \in M_n$ are called *orthogonal* if $AB^* = A^*B = 0$ (see [16]). We write $A \perp B$ to indicate that A and B are orthogonal. It is shown in [16] that $A \perp B$ if and only if there are unitary matrices $U, V \in M_n$ such that $UAV = \text{diag}(a_1, \dots, a_n)$ and $UBV = \text{diag}(b_1, \dots, b_n)$ with $a_i, b_i \geq 0$ and $a_i b_i = 0$ for $i = 1, \dots, n$. The matrices A_1, \dots, A_t are said to be *pairwise orthogonal* if $A_i^* A_j = A_i A_j^* = 0$ for any distinct $i, j \in \{1, \dots, t\}$. In this case, there are unitary matrices $U, V \in M_n$ such that $UA_iV = D_i$ for $i = 1, \dots, t$ with each D_i being nonnegative diagonal matrix and $D_i D_j = 0$ for any distinct $i, j \in \{1, \dots, t\}$.

We have the following lemmas relating to orthogonality, which are useful in the proof of Ky Fan k -norm results (Theorems 2.6 and 3.1).

LEMMA 2.2. [16] *Let $A, B \in M_n$ be nonzero matrices. Then*

$$\|\alpha A + \beta B\|_{(k)} = |\alpha| \|A\|_{(k)} + |\beta| \|B\|_{(k)}$$

for every pair of complex numbers α and β if and only if $A \perp B$ and $\text{rank } A + \text{rank } B \leq k$.

Denote by $\sigma(A)$ the spectrum of a matrix $A \in M_n$. Using the same arguments as in the proof of Lemma 2 in [16], we have the following result. (One can also see [8, p.468, Problem 3] for part (a) of Lemma 2.3.)

LEMMA 2.3. *Let $A \in M_n$ be positive semidefinite and let $B \in M_n$ be Hermitian.*

- (a) $\sigma(AB) \subseteq \mathbb{R}$.
- (b) *If $\sigma(AB) = \{0\}$, then there exists a unitary $U \in M_n$ such that*

$$UAU^* = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad UBU^* = \begin{bmatrix} 0 & X \\ X^* & B_1 \end{bmatrix},$$

where $A_1 \in M_s$ is invertible and $B_1 \in M_{n-s}$ with $0 \leq s \leq n$.

LEMMA 2.4. *Let $1 \leq k \leq n$ and $A, B \in M_n$ with spectral norm at most 1. Suppose*

$$\text{rank } A \leq k, \quad \|A + \alpha B\|_{(k)} = k \quad \text{and} \quad \|2A + \alpha B\|_{(k)} = \|A\|_{(k)} + \|A + \alpha B\|_{(k)}$$

for any unit complex number α . Then $A \perp B$ and $\sigma(A^*A) \subseteq \{0, 1\}$.

Proof. First, let $\alpha = 1$ and suppose $2A + B$ has singular value decomposition $2A + B = U_1 D V_1$ with $D = \text{diag}(d_1, \dots, d_n)$ and $d_1 \geq \dots \geq d_n \geq 0$. Then

$$D = U_1^*(2A + B)V_1^* = U_1^* A V_1^* + U_1^*(A + B)V_1^*.$$

Denote the diagonal entries of $U_1^* A V_1^*$ and $U_1^*(A + B)V_1^*$ by a_1, \dots, a_n and b_1, \dots, b_n , respectively. Then we have

$$\sum_{i=1}^k |a_i| \leq \|A\|_{(k)}, \quad \sum_{i=1}^k |a_i + b_i| \leq \|A + B\|_{(k)}, \quad \sum_{i=1}^k (2a_i + b_i) = \sum_{i=1}^k d_i = \|2A + B\|_{(k)}.$$

It follows that

$$\sum_{i=1}^k (a_i + (a_i + b_i)) \leq \sum_{i=1}^k (|a_i| + |a_i + b_i|) \leq \|A\|_{(k)} + \|A + B\|_{(k)} = \|2A + B\|_{(k)} = \sum_{i=1}^k (2a_i + b_i),$$

and so all the above inequalities are indeed equalities, which ensure that $a_i = |a_i|$ and $a_i + b_i = |a_i + b_i|$ are nonnegative real numbers with

$$\sum_{i=1}^k a_i = \sum_{i=1}^k |a_i| = \|A\|_{(k)} \quad \text{and} \quad \sum_{i=1}^k (a_i + b_i) = \sum_{i=1}^k |a_i + b_i| = \|A + B\|_{(k)}.$$

By Theorem 3.1 of [12], we have

$$U_1^* A V_1^* = A_1 \oplus A_2 \quad \text{and} \quad U_1^*(A + B)V_1^* = (A_1 + B_1) \oplus (A_2 + B_2),$$

where A_1 and $A_1 + B_1$ are $k \times k$ positive semidefinite matrices and

$$\|A\|_{(k)} = \|A_1\|_{(k)} = \text{tr } A_1, \quad \|A + B\|_{(k)} = \|A_1 + B_1\|_{(k)} = \text{tr}(A_1 + B_1).$$

Since $\text{rank } A \leq k$, it follows that $A_2 = 0$ and we may assume that $B_2 = \text{diag}(d_{k+1}, \dots, d_n)$. Without loss of generality, we also assume that $U_1 = V_1 = I_n$ and

$$A = A_1 \oplus 0, \quad B = B_1 \oplus \text{diag}(d_{k+1}, \dots, d_n).$$

Now take a unit $\alpha_1 \neq \pm 1$. Suppose the singular value decomposition of $2A_1 + \alpha_1 B_1$ is $2A_1 + \alpha_1 B_1 = U_2 D_1 V_2$ with $D_1 = \text{diag}(\beta_1, \dots, \beta_k)$ and $\beta_1 \geq \dots \geq \beta_k \geq 0$. Let $U_3 = U_2^* \oplus I_{n-k}$ and $V_3 = V_2^* \oplus \alpha_1^{-1} I_{n-k}$. Set $A_3 = U_2^* A_1 V_2^*$ and $B_3 = \alpha_1 U_2^* B_1 V_2^*$. Then

$$\begin{aligned} \text{diag}(\beta_1, \dots, \beta_k, d_{k+1}, \dots, d_n) &= U_3(2A + \alpha_1 B)V_3 \\ &= U_2^* A_1 V_2^* \oplus 0_{n-k} + U_2^*(A_1 + \alpha_1 B_1)V_2^* \oplus \text{diag}(d_{k+1}, \dots, d_n) \\ &= A_3 \oplus 0_{n-k} + (A_3 + B_3) \oplus \text{diag}(d_{k+1}, \dots, d_n). \end{aligned}$$

First, assume that $\|2A + \alpha_1 B\|_{(k)} = \sum_{i=1}^k \beta_i$. Using the same argument as above, we see that A_3 and $A_3 + B_3$ are $k \times k$ positive semidefinite matrices with $\|A\|_{(k)} = \text{tr } A_3$ and $\|A + \alpha_1 B\|_{(k)} = \text{tr}(A_3 + B_3)$. By Lemma 2.3(a), $\sigma(A_1 B_1) \subseteq \mathbb{R}$ and $\sigma(A_3 B_3) \subseteq \mathbb{R}$. Also

$$\sigma(\alpha_1 A_3 B_3) = \sigma(\alpha_1 A_3 B_3^*) = \sigma(\alpha_1 (U_2^* A_1 V_2^*) (\alpha_1 U_2^* B_1 V_2^*)^*) = \sigma(U_2^* A_1 B_1 U_2) = \sigma(A_1 B_1) \subseteq \mathbb{R}.$$

This implies $\sigma(A_1B_1) = \sigma(A_3B_3) = \{0\}$. According to Lemma 2.3(b), we know that there exists a unitary U_4 such that

$$U_4A_3U_4^* = A_4 \oplus 0, \quad U_4B_3U_4^* = \begin{bmatrix} 0 & X \\ X^* & B_4 \end{bmatrix},$$

where $A_4 \in M_s$ is positive definite for some $0 \leq s \leq k$, $B_4 \in M_{k-s}$, and $\|A_3 + B_3\|_{(k)} = \text{tr } A_4 + \text{tr } B_4$. Since both A and B have spectral norm at most 1, the diagonal entries of A_4 and B_4 are less than or equal to one. So $\text{tr}(A_3 + B_3) = \|A_3 + B_3\|_{(k)} = \|A + \alpha_1 B\|_{(k)} = k$ ensures that all the diagonal entries of A_4 and B_4 are equal to one and that the singular values of $A_3 + B_3$ are all equal to one. It follows that $X = 0$, $A_4 = I_s$, and $B_4 = I_{k-s}$, which implies $A \perp B$ and $\sigma(A^*A) \subseteq \{0, 1\}$.

Now, assume that $\|2A + \alpha_1 B\|_{(k)} \neq \sum_{i=1}^k \beta_i$ and suppose that the largest k singular values of $2A + \alpha_1 B$ are $\beta_1, \dots, \beta_r, d_{k+1}, \dots, d_{k+s}$ with $r + s = k$ and $r < k$. Denote by $\tilde{A} = A_3 \oplus 0$ and $\tilde{B} = B_3 \oplus \text{diag}(d_{k+1}, \dots, d_n)$. Applying Theorem 3.1 of [12] again, it follows that

$$\tilde{A} = \tilde{A}[1, \dots, r] \oplus 0_{n-r} \geq 0 \quad \text{and} \quad \tilde{B} = \tilde{B}[1, \dots, r] \oplus \text{diag}(\beta_{r+1}, \dots, \beta_k, d_{k+1}, \dots, d_n) \geq 0,$$

where $\tilde{A}[1, \dots, r]$ and $\tilde{B}[1, \dots, r]$ are the principal submatrices of \tilde{A} and \tilde{B} indexed by $1, \dots, r$. Without loss of generality, we can assume

$$A = \tilde{A}[1, \dots, r] \oplus 0_{n-r} \quad \text{and} \quad B = \tilde{B}[1, \dots, r] \oplus \text{diag}(\beta_{r+1}, \dots, \beta_k, d_{k+1}, \dots, d_n).$$

Choosing a unit $\alpha_2 \neq \{\pm 1, \pm \alpha_1\}$ and repeating the above process at most $n - 2$ times, we can show that $A \perp B$ and $\sigma(A^*A) \subseteq \{0, 1\}$. \square

LEMMA 2.5. *Let $1 \leq k \leq n$ and $A, B \in M_n$ such that A has rank at least k and $\sigma(A^*A) \subseteq \{0, 1\}$. If*

$$\|A + \alpha B\|_{(k)} = k$$

for all unit complex numbers α , then $A \perp B$.

Proof. Without loss of generality, we may assume that $A = I_s \oplus 0$ with $s \geq k$. Denote $A = [a_{ij}]$ and $B = [b_{ij}]$. We claim that for any arbitrary and fixed $1 \leq p_1 < \dots < p_k \leq s$, $b_{ij} = 0$ whenever $\{i, j\} \cap \{p_1, \dots, p_k\} \neq \emptyset$. If the claim holds, then B has the form $0_s \oplus B_2$ and therefore $A \perp B$. To prove the claim, it suffices to show the case when $(p_1, \dots, p_k) = (1, \dots, k)$. Notice that

$$k = \|A + \alpha B\|_{(k)} \geq \sum_{j=1}^k |a_{jj} + \alpha b_{jj}| \geq \left| \sum_{j=1}^k (a_{jj} + \alpha b_{jj}) \right| = \left| k + \alpha \sum_{j=1}^k b_{jj} \right|.$$

The two equalities holds for all complex unit α if and only if $b_{jj} = 0$ for all $1 \leq j \leq k$. Thus, we have $\|A + \alpha B\|_{(k)} = k = \sum_{j=1}^k |a_{jj} + \alpha b_{jj}|$ for all complex unit α . Now by [12, Theorem 3.1],

$$A + \alpha B = C_\alpha \oplus D_\alpha$$

where C_α is positive semidefinite and $\|C_\alpha\|_{(k)} = k$. It follows that $B = B_1 \oplus B_2$ with $B_1 \in M_k$. Notice that $C_\alpha = I_k + \alpha B_1$ is positive semidefinite for all complex unit α . Then B_1 has to be the zero matrix, i.e., $B = 0_k \oplus B_2$. Therefore, the claim holds and the result follows. \square

We now present of the result for the Ky Fan k -norm.

THEOREM 2.6. *Let $m, n \geq 2$ and $2 \leq k \leq mn$ and $\phi : M_{mn} \rightarrow M_{mn}$ be a linear map. The following are equivalent.*

- (a) $\|\phi(A \otimes B)\|_{(k)} = \|A \otimes B\|_{(k)}$ for all $A \in M_m$ and $B \in M_n$.
- (b) *There are unitary matrices $U, V \in M_{mn}$ such that*

$$\phi(A \otimes B) = U(\varphi_1(A) \otimes \varphi_2(B))V \quad \text{for all } A \in M_m \text{ and } B \in M_n,$$

where φ_s is the identity map or the transposition map $X \mapsto X^t$ for $s = 1, 2$.

Proof. The implication (b) \Rightarrow (a) is obvious. Conversely, assume that $\|\phi(A \otimes B)\|_{(k)} = \|A \otimes B\|_{(k)}$ for all $A \in M_m$ and $B \in M_n$. We assert that there exist unitary $U, V \in M_{mn}$ such that

$$\phi(E_{ii} \otimes E_{jj}) = U(E_{ii} \otimes E_{jj})V \quad \text{for } 1 \leq i \leq m \text{ and } 1 \leq j \leq n. \quad (2.1)$$

It suffices to show that

$$\phi(E_{ii} \otimes E_{jj}) \perp \phi(E_{rr} \otimes E_{ss}) \quad (2.2)$$

for any distinct pairs (i, j) and (r, s) with $1 \leq i, r \leq m$ and $1 \leq j, s \leq n$. We distinguish three cases.

Case 1. Suppose $i = r$ or $j = s$. We have

$$\|\alpha\phi(E_{ii} \otimes E_{jj}) + \beta\phi(E_{rr} \otimes E_{ss})\|_{(k)} = |\alpha| \|\phi(E_{ii} \otimes E_{jj})\|_{(k)} + |\beta| \|\phi(E_{rr} \otimes E_{ss})\|_{(k)}$$

for all complex numbers α and β . Applying Lemma 2.2, we have (2.2) and

$$\text{rank } \phi(E_{ii} \otimes E_{jj}) + \text{rank } \phi(E_{rr} \otimes E_{ss}) \leq k. \quad (2.3)$$

Case 2. Suppose $i \neq r$ and $j \neq s$. Let

$$G = \phi(E_{ii} \otimes (E_{jj} + E_{ss})) \quad \text{and} \quad H = \phi(E_{rr} \otimes (E_{jj} + E_{ss})).$$

Subcase 2.a. Assume first that $k \leq 3$. By Case 1, $\phi(E_{ii} \otimes E_{jj})$ and $\phi(E_{ii} \otimes E_{ss})$ are orthogonal with $\|\phi(E_{ii} \otimes E_{jj})\|_{(k)} = \|\phi(E_{ii} \otimes E_{ss})\|_{(k)} = 1$. So, $\|G\| \leq 1$ and $\text{rank } G \leq k$. Similarly, $\|H\| \leq 1$ and $\text{rank } H \leq k$. Furthermore, $\|G + \gamma H\|_{(k)} = k$, and $\|2G + \gamma H\|_{(k)} = \|G\|_{(k)} + \|G + \gamma H\|_{(k)}$ for all complex units γ . Applying Lemma 2.4, we get $G \perp H$. Because $\phi(E_{ii} \otimes E_{jj}) \perp \phi(E_{ii} \otimes E_{ss})$ and $\phi(E_{rr} \otimes E_{jj}) \perp \phi(E_{rr} \otimes E_{ss})$, we have (2.2).

Subcase 2.b. Now, suppose that $k > 3$. We have

$$\|\alpha G + \beta H\|_{(k)} = |\alpha| \|G\|_{(k)} + |\beta| \|H\|_{(k)}$$

for all complex numbers α, β . Applying Lemma 2.2 again, we get $G \perp H$ and, hence, (2.2) follows.

From above, we showed that (2.1) holds. For the sake of the simplicity, we assume that U and V are identity matrices. Now, for any unitary $Y \in M_n$ and $1 \leq i \leq m$, $1 \leq j \leq n$ we can apply the argument in the preceding paragraphs to conclude that $\phi(E_{ii} \otimes YE_{jj}Y^*)$ has rank one and it is orthogonal to $\phi(E_{rr} \otimes YE_{ss}Y^*)$ for any distinct pairs (i, j) and (r, s) . It follows that

$$\|\phi(E_{ii} \otimes I_n) + \gamma\phi(E_{ii} \otimes YE_{jj}Y^*)\| = \|\phi(E_{ii} \otimes I_n + \gamma E_{ii} \otimes YE_{jj}Y^*)\| = 1 + \gamma$$

for any positive scalar γ . Thus, $\phi(E_{ii} \otimes Y E_{jj} Y^*) = E_{ii} \otimes Z$ for some rank one $Z \in M_n$. Since this is true for any $1 \leq i \leq m$ and unitary $Y \in M_n$, we conclude that there exists a linear map $\varphi_i : M_n \rightarrow M_n$ such that

$$\phi(E_{ii} \otimes B) = E_{ii} \otimes \varphi_i(B)$$

for all complex matrices $B \in M_n$. Clearly, φ_i preserves the Ky Fan k -norm of all $B \in M_n$ and $\varphi_i(E_{jj}) = E_{jj}$ for all $1 \leq j \leq n$. (Here the Ky Fan k -norm reduces to the trace norm if $k \geq n$.) Hence, φ_i has the form $X \mapsto W_i X \tilde{W}_i$ or $X \mapsto W_i X^t \tilde{W}_i$ for some unitary $W_i, \tilde{W}_i \in M_n$. Now, we can adapt the arguments in the last two paragraphs in the proof of Theorem 2.1 to obtain our conclusion. \square

2.3. Schatten p -norms. We now study the linear preserver for Schatten p -norms. We first present a key lemma of the result.

LEMMA 2.7. [17] *Let $T, S \in M_n$. Then*

$$(i) \quad 2^{p-1}(\|T\|_p^p + \|S\|_p^p) \leq \|T + S\|_p^p + \|T - S\|_p^p \leq 2(\|T\|_p^p + \|S\|_p^p) \text{ if } 1 \leq p \leq 2,$$

$$(ii) \quad 2(\|T\|_p^p + \|S\|_p^p) \leq \|T + S\|_p^p + \|T - S\|_p^p \leq 2^{p-1}(\|T\|_p^p + \|S\|_p^p) \text{ if } 2 \leq p < \infty.$$

If $p = 2$, all equalities always hold; if $p \neq 2$, any equality holds if and only if $T^ T S^* S = S^* S T^* T = 0$.*

REMARK 2.8. Suppose $p \neq 2$ and the singular values of T and S are $\{t_1, \dots, t_n\}$ and $\{s_1, \dots, s_n\}$, respectively. If any of equality in Lemma 2.7 holds, then $T^* T S^* S = S^* S T^* T = 0$ implies $T^* T \perp S^* S$. So there exists unitary $U \in M_n$ such that $U^* T^* T U = \text{diag}(t_1^2, \dots, t_n^2)$ and $U^* S^* S U = \text{diag}(s_1^2, \dots, s_n^2)$ with $s_i t_i = 0$ for $i = 1, \dots, n$. Replacing T and S with T^* and S^* , we get $T T^* S S^* = 0$ and there exists unitary $V \in M_n$ such that $V^* T T^* V = \text{diag}(t_1^2, \dots, t_n^2)$ and $V^* S S^* V = \text{diag}(s_1^2, \dots, s_n^2)$. It follows that $V^* T U = \text{diag}(t_1, \dots, t_n)$ and $V^* S U = \text{diag}(s_1, \dots, s_n)$, which implies $T \perp S$.

THEOREM 2.9. *Let $1 \leq p < \infty$ and $p \neq 2$ and $\phi : M_{mn} \rightarrow M_{mn}$ be a linear map. Then the following are equivalent.*

$$(a) \quad \|\phi(A \otimes B)\|_p = \|A \otimes B\|_p \text{ for all } A \in M_m \text{ and } B \in M_n.$$

(b) *There are unitary matrices $U, V \in M_{mn}$ such that*

$$\phi(A \otimes B) = U(\varphi_1(A) \otimes \varphi_2(B))V \quad \text{for all } A \in M_m \text{ and } B \in M_n,$$

where φ_s is the identity map or the transposition map $X \mapsto X^t$ for $s = 1, 2$.

Proof. The implication (b) \Rightarrow (a) is obvious. Conversely, assume that $\|\phi(A \otimes B)\|_p = \|A \otimes B\|_p$ for all $A \otimes B \in M_{mn}$. We first conclude there exist unitary $U, V \in M_{mn}$ such that

$$\phi(E_{ii} \otimes E_{jj}) = U(E_{ii} \otimes E_{jj})V \quad \text{for } 1 \leq i \leq m \text{ and } 1 \leq j \leq n. \quad (2.4)$$

Then we can adapt the arguments in the last three paragraphs in the proof of Theorem 2.1 to verify that ϕ has the form claimed in (b).

As in the proof of Theorem 2.6, to prove (2.4), it suffices to show that

$$\phi(E_{ii} \otimes E_{jj}) \perp \phi(E_{rr} \otimes E_{ss}) \quad (2.5)$$

for any distinct pairs (i, j) and (r, s) with $1 \leq i, r \leq m$ and $1 \leq j, s \leq n$.

If $i = r$ or $j = s$, we have

$$\|\phi(E_{ii} \otimes E_{jj}) + \phi(E_{rr} \otimes E_{ss})\|_p^p + \|\phi(E_{ii} \otimes E_{jj}) - \phi(E_{rr} \otimes E_{ss})\|_p^p = 2 (\|\phi(E_{ii} \otimes E_{jj})\|_p^p + \|\phi(E_{rr} \otimes E_{ss})\|_p^p).$$

Applying Lemma 2.7 and Remark 2.8, we get (2.5). If $i \neq r$ and $j \neq s$, we have

$$\begin{aligned} & \|\phi(E_{ii} \otimes (E_{jj} + E_{ss})) + \phi(E_{rr} \otimes (E_{jj} + E_{ss}))\|_p^p + \|\phi(E_{ii} \otimes (E_{jj} + E_{ss})) - \phi(E_{rr} \otimes (E_{jj} + E_{ss}))\|_p^p \\ &= 2 (\|\phi(E_{ii} \otimes (E_{jj} + E_{ss}))\|_p^p + \|\phi(E_{rr} \otimes (E_{jj} + E_{ss}))\|_p^p). \end{aligned}$$

By Lemma 2.7, we have $\phi(E_{ii} \otimes (E_{jj} + E_{ss})) \perp \phi(E_{rr} \otimes (E_{jj} + E_{ss}))$. With the fact that $\phi(E_{ii} \otimes E_{jj}) \perp \phi(E_{ii} \otimes E_{ss})$ and $\phi(E_{rr} \otimes E_{jj}) \perp \phi(E_{rr} \otimes E_{ss})$, we conclude that (2.5) holds. This completes the proof. \square

REMARK 2.10. For $p = 2$, i.e., the Frobenius norm case, the statements (a) and (b) in Theorem 2.9 are not equivalent. One can consider the linear map $\psi(A) = [b_{ij}]$ for $A = [a_{ij}] \in M_{mn}$ such that $b_{1,mn} = a_{mn,1}$, $b_{mn,1} = a_{1,mn}$ and $b_{ij} = a_{ij}$ for $(i, j) \notin \{(1, mn), (mn, 1)\}$. In fact, any linear map on M_{mn} preserving the inner product $(A, B) = \text{tr}(AB^*)$ on M_{mn} will preserve the Frobenius norm.

REMARK 2.11. Note that the maps in Theorems 2.1, 2.6, and 2.9 may not satisfy $\|\phi(C)\| = \|C\|$ for all $C \in M_{mn}$. For instance, if $\phi(A \otimes B) = A \otimes B^t$ and if

$$C_r = r^2 E_{11} \otimes E_{11} + r(E_{12} \otimes E_{12} + E_{21} \otimes E_{21}) + E_{22} \otimes E_{22} \quad \text{for } r \geq 0,$$

then C_r has singular values $r^2 + 1, 0, \dots, 0$, and $\phi(C_r)$ has singular values $r^2, r, r, 1, 0, \dots, 0$. Then for any positive number $r \neq 1$, $\|\phi(C_r)\| \neq \|C_r\|$ unless $\|\cdot\|$ is the Frobenius norm. However, if we assume that ϕ satisfies $\|\phi(C)\| = \|C\|$ for C of the tensor form $A \otimes B$, and also for $C = C_r$ for some positive number $r \neq 1$, then one easily deduces that both φ_i mentioned in the theorems have to be of the same type, i.e., both are identity map, or both are the transposition map. It follows that there are unitary $U, V \in M_{mn}$ such that ϕ has the form

$$X \mapsto UXV \quad \text{or} \quad X \mapsto UX^tV$$

and hence, $\|\phi(X)\| = \|X\|$ for all $X \in M_{mn}$.

3. Multipartite systems. In this section we extend the previous results to multipartite systems $M_{n_1} \otimes \dots \otimes M_{n_m}$, $m \geq 2$. Let $A_i \in M_{n_i}$, $i = 1, \dots, m$. We denote $\bigotimes_{i=1}^m A_i = A_1 \otimes A_2 \otimes \dots \otimes A_m$.

THEOREM 3.1. *Let $1 \leq k \leq \prod_{i=1}^m n_i$ and $\phi : M_{n_1 \dots n_m} \rightarrow M_{n_1 \dots n_m}$ be a linear map. The following are equivalent.*

- (a) $\|\phi(A_1 \otimes \dots \otimes A_m)\|_{(k)} = \|A_1 \otimes \dots \otimes A_m\|_{(k)}$ for all $A_i \in M_{n_i}$, $i = 1, \dots, m$.
- (b) *There are unitary matrices $U, V \in M_{n_1 \dots n_m}$ such that*

$$\phi(A_1 \otimes \dots \otimes A_m) = U(\varphi_1(A_1) \otimes \dots \otimes \varphi_m(A_m))V \quad \text{for all } A_i \in M_{n_i}, i = 1, \dots, m, \quad (3.1)$$

where φ_s is the identity map or the transposition map $X \mapsto X^t$ for $s = 1, \dots, m$.

Proof. The sufficiency part is clear. To prove the necessity part, we use induction on m . By Theorem 2.1 and Theorem 2.6, we already know that the statement of Theorem 3.1 is true for bipartite systems. So, assume that $m \geq 3$ and that the result holds for all $(m-1)$ -partite systems. We need to prove that the same is true for m -partite systems.

Denote $N = \prod_{i=1}^m n_i$. First we claim that there exist unitary $U, V \in M_N$ such that

$$\phi(E_{j_1 j_1} \otimes \cdots \otimes E_{j_m j_m}) = U(E_{j_1 j_1} \otimes \cdots \otimes E_{j_m j_m})V \quad \text{for all } 1 \leq j_i \leq n_i. \quad (3.2)$$

When $k = 1$, as in the proof of Theorem 2.1, we can successively consider

$$\begin{aligned} & \phi(E_{i_1 i_1} \otimes \cdots \otimes E_{i_{m-2}, i_{m-2}} \otimes E_{i_{m-1}, i_{m-1}} \otimes (E_{i_m i_m} + \gamma E_{j_m j_m})), \\ & \phi(E_{i_1 i_1} \otimes \cdots \otimes E_{i_{m-2}, i_{m-2}} \otimes (E_{i_{m-1} i_{m-1}} + \gamma E_{j_{m-1} j_{m-1}}) \otimes D_m), \quad \dots, \\ & \phi((E_{i_1 i_1} + \gamma E_{j_1 j_1}) \otimes D_2 \otimes \cdots \otimes D_{m-2} \otimes D_{m-1} \otimes D_m) \end{aligned}$$

to obtain (3.2), where D_2, \dots, D_m are arbitrary diagonal matrices.

When $k \geq 2$, as in the proof of Theorem 2.6, it suffices to show

$$\phi(E_{i_1 i_1} \otimes \cdots \otimes E_{i_m i_m}) \perp \phi(E_{j_1 j_1} \otimes \cdots \otimes E_{j_m j_m}) \quad \text{for any } (i_1, \dots, i_m) \neq (j_1, \dots, j_m). \quad (3.3)$$

To confirm (3.3), it suffices to verify that for any $1 \leq r \leq m$,

$$\phi \left(\bigotimes_{u=1}^{r-1} (E_{i_u i_u} + E_{j_u j_u}) \otimes E_{i_r i_r} \otimes \bigotimes_{u=r+1}^m E_{i_u i_u} \right) \perp \phi \left(\bigotimes_{u=1}^{r-1} (E_{i_u i_u} + E_{j_u j_u}) \otimes E_{j_r j_r} \otimes \bigotimes_{u=r+1}^m E_{i_u i_u} \right) \quad (3.4)$$

for any distinct $\mathbf{i} = (i_1, \dots, i_m)$ and $\mathbf{j} = (j_1, \dots, j_m)$ with $i_u \neq j_u$, $1 \leq u \leq r$. Denote by $G_r = G_r(\mathbf{i}, \mathbf{j})$ and $\hat{G}_r = \hat{G}_r(\mathbf{i}, \mathbf{j})$ the two matrices in (3.4) accordingly. We consider two cases.

Case 1. For $r \leq \log_2 k$, as $\|\alpha G_r + \beta \hat{G}_r\|_{(k)} = |\alpha| \|G_r\|_{(k)} + |\beta| \|\hat{G}_r\|_{(k)}$ for all complex α and β . Applying Lemma 2.2, we get

$$G_r \perp \hat{G}_r \quad \text{and} \quad \text{rank } G_r + \text{rank } \hat{G}_r \leq k \quad \text{for all } r \leq \log_2 k.$$

Now as $G_{s+1} = G_s + \hat{G}_s$ and $G_s \perp \hat{G}_s$ for all $s \leq \log_2 k$,

$$\|G_{s+1}\| \leq \max\{\|G_s\|, \|\hat{G}_s\|\} \leq \max\{\|G_s(\mathbf{i}, \mathbf{j})\| : i_u \neq j_u, 1 \leq u \leq s\},$$

where $\|\cdot\|$ is the spectral norm. Hence,

$$\max\{\|G_{s+1}(\mathbf{i}, \mathbf{j})\| : i_u \neq j_u, 1 \leq u \leq s+1\} \leq \max\{\|G_s(\mathbf{i}, \mathbf{j})\| : i_u \neq j_u, 1 \leq u \leq s\}.$$

As the inequality holds for all $s \leq \log_2 k$, it follows that

$$\begin{aligned} \|G_r\| & \leq \max\{\|G_1(\mathbf{i}, \mathbf{j})\| : i_1 \neq j_1\} = \max\{\|\phi(E_{i_1 i_1} \otimes \cdots \otimes E_{i_m i_m})\| : (i_1, \dots, i_m)\} \\ & \leq \max\{\|\phi(E_{i_1 i_1} \otimes \cdots \otimes E_{i_m i_m})\|_{(k)} : (i_1, \dots, i_m)\} = 1. \end{aligned}$$

Similarly, one conclude that $\|\hat{G}_r\| \leq 1$.

Case 2. For $r > \log_2 k$, we claim that G_r and \hat{G}_r are orthogonal and both of them have singular values 0 and 1 only. We prove the claim by induction on r .

Suppose $\log_2 k < r \leq 1 + \log_2 k$. Notice that $G_r = G_{r-1} + \hat{G}_{r-1}$. By Case 1,

$$G_{r-1} \perp \hat{G}_{r-1}, \quad \|G_{r-1}\| \leq 1, \quad \|\hat{G}_{r-1}\| \leq 1, \quad \text{and} \quad \text{rank } G_{r-1} + \text{rank } \hat{G}_{r-1} \leq k.$$

Therefore, $\|G_r\| \leq 1$ and $\text{rank } G_r \leq k$. Since

$$\|G_r + \alpha \hat{G}_r\|_{(k)} = k \quad \text{and} \quad \|2G_r + \beta \hat{G}_r\|_{(k)} = \|G_r\|_{(k)} + \|G_r + \beta \hat{G}_r\|_{(k)},$$

for any complex unit α , by Lemma 2.4, we obtain $G_r \perp \hat{G}_r$. Moreover, G_r has singular values 0 and 1 only. Similarly, one can conclude that \hat{G}_r has singular values 0 and 1 only. Now assume that the claim holds for some $r > \log_2 k$. We will show that the claim also holds for $r + 1$. By induction assumption and the fact that $G_{r+1} = G_r + \hat{G}_r$, we conclude that G_{r+1} has singular values 0 and 1 only. The same conclusion holds for \hat{G}_{r+1} . By Lemma 2.5 and the fact that $\|G_{r+1} + \alpha \hat{G}_{r+1}\| = k$ for all complex unit α , we get $G_{r+1} \perp \hat{G}_{r+1}$. Therefore the claim holds.

Combining the above two cases, we see that (3.4) holds and hence the statement (3.3) follows. Therefore, the claim (3.2) holds for all k . Without loss of generality, we may assume $U = V = I_N$ in (3.2). Following a similar argument as in Theorems 2.1 and 2.6, one can conclude that

$$\phi \left(E_{j_1 j_1} \otimes \cdots \otimes E_{j_{m-1} j_{m-1}} \otimes B \right) = E_{j_1 j_1} \otimes \cdots \otimes E_{j_{m-1} j_{m-1}} \otimes \varphi_{j_1, \dots, j_{m-1}}(B)$$

for all $1 \leq j_i \leq n_i$ with $1 \leq i \leq m-1$ and $B \in M_{n_m}$, where $\varphi_{j_1, \dots, j_{m-1}}$ can be assumed to be either the identity map or the transposition map. Following the same argument, we can further conclude that for any $X = X_1 \otimes \cdots \otimes X_{m-1}$ with $X_i \in M_{n_i}$ being unitary for $1 \leq i \leq m-1$, there are unitary U_X and V_X such that

$$\phi \left(\left(\bigotimes_{i=1}^{m-1} X_i E_{j_i j_i} X_i^* \right) \otimes B \right) = U_X \left(\left(\bigotimes_{i=1}^{m-1} X_i E_{j_i j_i} X_i^* \right) \otimes \varphi_{j_1, \dots, j_{m-1}, X}(B) \right) V_X$$

for all $1 \leq j_i \leq n_i$ with $1 \leq i \leq m-1$ and $B \in M_{n_m}$, where $\varphi_{j_1, \dots, j_{m-1}, X}$ can be assumed to be either the identity map or the transposition map, depending on j_1, \dots, j_{m-1} and X . By the fact that $\phi(I_N) = I_N$, we have $V_X^* = U_X$.

Again, considering all symmetric $S \in M_{n_m}$ as in the proof of Theorem 2.1, we can show that there exists $W_X \in M_{n_1 \dots n_{m-1}}$ such that

$$\phi \left(\left(\bigotimes_{i=1}^{m-1} X_i E_{j_i j_i} X_i^* \right) \otimes B \right) = W_X \left(\bigotimes_{i=1}^{m-1} X_i E_{j_i j_i} X_i^* \right) W_X^* \otimes \varphi_{j_1, \dots, j_{m-1}, X}(B)$$

for all $1 \leq j_i \leq n_i$ with $1 \leq i \leq m-1$ and $B \in M_{n_m}$. Consider the linear map (known as the partial trace function in quantum information science context) on M_N defined by $X \otimes Y \mapsto (\text{tr } X)Y$ for $X \in M_{n_1 \dots n_{m-1}}$ and $Y \in M_{n_m}$ and apply to the above equation, we see that every choice of $\bigotimes_{j=1}^{m-1} X_j E_{j_j j_j} X_j^*$ gives rise to a linear map $\varphi_{j_1, \dots, j_m, X}$, which is either the identity map or the transposition map. Evidently, the map

$$\bigotimes_{j=1}^{m-1} X_j E_{j_j j_j} X_j^* \mapsto \varphi_{j_1, \dots, j_m, X}$$

is linear and hence continuous; the set

$$\left\{ \bigotimes_{j=1}^{m-1} X_j E_{j_i j_i} X_j^* : 1 \leq j_i \leq n_i \text{ and } X_j^* X_j = I_{n_j}, \text{ for } j = 1, \dots, m-1 \right\}$$

$$= \{x_1 x_1^* \otimes \cdots \otimes x_{m-1} x_{m-1}^* : x_i \in \mathbb{C}^{n_i} \text{ with } 1 \leq i \leq m-1\}$$

is connected. Thus, all the maps $\varphi_{j_1, \dots, j_{m-1}, X}$ have to be the same. Assume that this common map is φ_m , which is either the identity map or the transposition map. By linearity, one can conclude that for any $A_1 \otimes \cdots \otimes A_m \in M_{n_1} \otimes \cdots \otimes M_{n_m}$, we have

$$\phi(A_1 \otimes \cdots \otimes A_m) = \psi(A_1 \otimes \cdots \otimes A_{m-1}) \otimes \varphi_m(A_m)$$

for some $\psi(A_1 \otimes \cdots \otimes A_{m-1}) \in M_{n_1 \cdots n_{m-1}}$. Note that $\psi : M_{n_1 \cdots n_{m-1}} \rightarrow M_{n_1 \cdots n_{m-1}}$ preserves the Ky Fan k -norm of all matrices in $M_{n_1} \otimes \cdots \otimes M_{n_{m-1}}$. By the induction hypothesis, we know there exist unitary \tilde{U}, \tilde{V} such that

$$\psi(A_1 \otimes \cdots \otimes A_{m-1}) = \tilde{U}(\varphi_1(A_1) \otimes \cdots \otimes \varphi_{m-1}(A_{m-1}))\tilde{V}$$

with each φ_j being either the identity map or the transposition map. Hence, ϕ has the desired form and the proof is completed. \square

Using a similar argument and applying Lemma 2.7, we can extend Theorem 2.9 to multipartite systems as follows.

THEOREM 3.2. *Let $1 \leq p < \infty$ and $p \neq 2$ and $\phi : M_{n_1 \cdots n_m} \rightarrow M_{n_1 \cdots n_m}$ be a linear map. The following are equivalent.*

- (a) $\|\phi(A_1 \otimes \cdots \otimes A_m)\|_p = \|A_1 \otimes \cdots \otimes A_m\|_p$ for all $A_i \in M_{n_i}$, $i = 1, \dots, m$.
- (b) *There are unitary matrices $U, V \in M_{n_1 \cdots n_m}$ such that*

$$\phi(A_1 \otimes \cdots \otimes A_m) = U(\varphi_1(A_1) \otimes \cdots \otimes \varphi_m(A_m))V, \quad \text{for all } A_i \in M_{n_i}, i = 1, \dots, m,$$

where φ_s is the identity map or the transposition map $X \mapsto X^t$ for $s = 1, \dots, m$.

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