Convexity and star-shapedness of the joint (p, q)-matricial range

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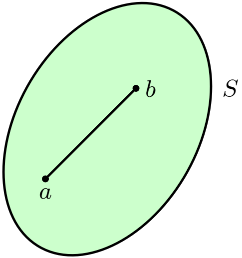
WONRA18, Technical University of Munich, Munich

Convexity and Star-shapedness

A set S is said to be convex if for any $a, b \in S$, the line segment joining a and b also lies in S. That is,

$$ta + (1-t)b \in S \quad \text{for all } t \in [0,1].$$



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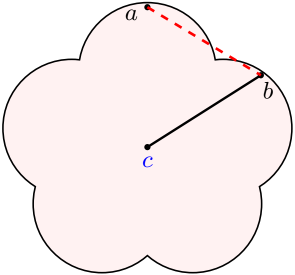

A set S is said to be star-shaped if there is a star-center $c \in S$ such that the line segment joining s and any other point $b \in S$ also lies in S. That is,

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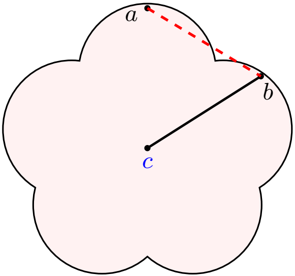

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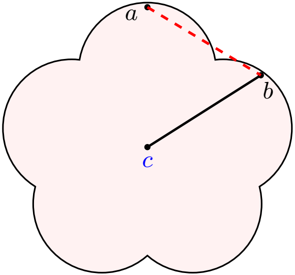

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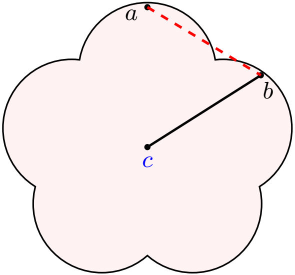

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- The set of all star-centers of a star-shaped set S is convex.

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- $p = q = 1$, $W(A) = W(1 : A) = \Lambda_1(A)$.

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- If $n \geq 3(p-1)+1$, then $\Lambda_p(A)$ is nonempty. [Li, Poon, Sze, LAMA, 2009]

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- How about $W(q : \mathbf{A})$?

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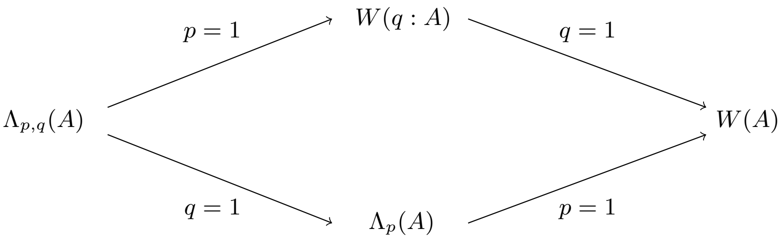

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(c) Every element in

$$\left\{ (a_1 I_q, \ldots, a_m I_q) : (a_1, \ldots, a_m) \in \operatorname{conv} \Lambda_N(\mathbf{A}) \right\}$$

is a star center of $\Lambda_{p,q}(\mathbf{A})$.

Results

Theorem [Lau,Li,Poon,Sze,2018]

Let $\mathbf{A} = (A_1, \ldots, A_m) \in M_n^m$, and $N = p(q^2(2m+1)+1)$. Suppose $n \geq (Np-1)(2m+1)^2$. Then

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- **Open question**: What is smallest dimension n such that $\Lambda_{p,q}(\mathbf{A})$ is always non-empty/star-shaped?

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Theorem [Lau,Li,Poon,Sze,2018]

Let $\mathbf{A} \in \mathcal{B}(\mathcal{H})^m$. Then $\Lambda_{p,q}(\mathbf{A})$ is nonempty and star-shaped.

The joint (p, q)-matricial range

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Let $\mathbf{A} \in \mathcal{B}(\mathcal{H})^m$. Then $\Lambda_{\infty,q}(\mathbf{A})$ is convex. Moreover, if $\mathbf{C} \in \Lambda_{\infty,q}(\mathbf{A})$, then $\mathbf{C}$ is a star-center of $\Lambda_{p,q}(\mathbf{A})$ for all $p \geq 1$.

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Let $\mathbf{A} \in \mathcal{B}(\mathcal{H})^m$. Then $\Lambda_{\infty,q}(\mathbf{A})$ is convex. Moreover, if $\mathbf{C} \in \Lambda_{\infty,q}(\mathbf{A})$, then $\mathbf{C}$ is a star-center of $\Lambda_{p,q}(\mathbf{A})$ for all $p \geq 1$.

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- **Example:** Let $A = \operatorname{diag}(1, 1/2, 1/3, \ldots)$. Then for all positive integer q,

$$\Lambda_{\infty,q}(A) = \emptyset.$$

[Li, Poon, JOT, 2011]

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[Chui, Smith, Smith, Ward, Illinois, 1997], [Müller, Studia Math, 2010]

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Thank you!