Numerical ranges of non self-adjoint operators in quantum mechanics

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Motivation

The formulation of conventional quantum mechanics (QM) is based on the theory of self-adjoint operators acting on a Hilbert space $\mathcal{H}$.

These operators describe *observables*,

their eigenvalues are real and provide the possible results of the respective measurements.

Quantum phenomena are adequately described by a wave function $\Psi_t(x)$, where x is the particle coordinate.

In agreement with the principles of QM, the time evolution of the wave function Ψ_t of the system is determined by the time-dependent Schrödinger equation

$$i\frac{\partial \Psi_t}{\partial t} = H\Psi_t,$$

where H is the Hamiltonian operator.

In the second half of last century, energy states of atoms, molecules and atomic nuclei have been described as eigenfunctions of selfadjoint Schrödinger operators.

The eigenvalues E_n are the quantum energy levels.

So, the Hamiltonians eigenfunctions can be taken orthonormal,

or, equivalently, the Hamiltonians can be unitarily diagonalized.

Meanwhile, certain relativistic extensions of quantum mechanics lead to non self-adjoint Hamiltonian operators with a real spectrum

PT-**quantum mechanics**

This motivated an intense research activity, namely on the PT-*quantum mechanics*

where P and T are, respectively, the *parity* and the *time reversal* operators:

$$\mathsf{P}\Psi(x) := \Psi(-x), \quad \mathsf{T}\Psi(x) := \overline{\Psi(x)}.$$

C.M. Bender and S. Boettcher, Real Spectra in Non-Hermitian Hamiltonians Having PT Symmetry , Phys. Rev. Lett., 80 (1998) 5243-5246.

More than 3206 citations.

Nevertheless, the reality of the spectrum by its own is insufficient to draw quantum mechanically relevant conclusions for the system.

Quasi-Hermitian QM

There have been attempts to develop the so called *quasi-Hermitian quantum mechanics* where the Hamiltonian H is represented by a *quasi-self-adjoint* operator,

that is, H satisfies the quasi-self-adjointness relation

$$H^*\Theta = \Theta H, \tag{1}$$

with $\Theta = T^*T$ a positive, bounded and boundedly invertible operator, called a *metric*:

$$\ll \phi, \psi \gg := \langle \Theta\phi, \psi \rangle = \langle T\phi, T\psi \rangle.$$

The concept of quasi-self-adjointness goes back to Dieudonné

is of remarkable interest in the context of quantum mechanics with non-self-adjoint Hamiltonians.

The problem

How to build a consistent quantum theory for these operators.

A necessary condition for developing a consistent quantum theory is the reality of the spectrum, $\sigma(H)$.

But it is far from being sufficient.

A new inner product in the underlying Hilbert space, relatively to which H becomes self-adjoint via the similarity transformation THT^{-1},

$$\widetilde{H} = THT^{-1} = T^{-1}H^{*}T \tag{2}$$

is of interest.

However, it is in general difficult to decide quasi-self-adjointness of an operator.

This motivated an intense research activity, both on the physical and mathematical level

If T is bounded and boundedly invertible, then the spectra of THT^{-1} and H coincide and the eigenfunctions share basis properties.

It is not very common to find in the literature non-self-adjoint models for which such a metric is constructed,

neither the existence of a metric operator is guaranteed.

Several problems arise if T or T^{-1} are unbounded, such as

1) it may happen that the spectrum of H is purely discrete, while THT^{-1} has no eigenvalues.

2) Unbounded transformations may turn an o.n. eigenbasis into a set of functions that do not form any kind of basis.

However, certain relativistic extensions of quantum mechanics lead to the consideration of non self-adjoint Hamiltonian operators with a real spectrum.

The Hilbert space $\mathcal{H}$

Denote by $L^2(\mathbb{R}^2)$ the Hilbert space $\mathcal{H}$ of square integrable functions in two real variables, endowed with the standard inner product

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x,y)\overline{f(x,y)}\mathrm{d}x\mathrm{d}y.$$

Let

$$\mathcal{D} = \left\{ f(x,y)\mathrm{e}^{-(x^2+y^2)} : f(x,y) \text{ is a polynomial in } x,y \right\} \tag{3}$$

which is a dense domain in $L^2(\mathbb{R}^2)$.

Linear operators of interest act as densely defined operators.

The generators of $su(1,1)$

We are particularly concerned with the algebra su(1,1) generated by traceless 2×2 matrices

$$t_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad t_2 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \quad t_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which verify the commutation relations

$$[t_1, t_2] = -2it_3, \quad [t_2, t_3] = 2it_1, \quad [t_3, t_1] = 2it_2.$$

They are J-Hermitian ($J = \text{diag}(1, -1)$), i-e.,

$JH^*J = H$, $J = I_r \oplus -I_{n-r}$.

Bosonic operators

Let $x : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ and $y : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ be the multiplication operators

$$f(x,y) \to xf(x,y), \quad f(x,y) \to yf(x,y),$$

defined in the dense domain $\mathcal{D}$.

Consider the differential operators

$$\frac{\partial}{\partial x} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \quad \text{and} \quad \frac{\partial}{\partial y} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$$

with defined on $\mathcal{D}$,

$$f(x,y) \to \frac{\partial f(x,y)}{\partial x}, \quad f(x,y) \to \frac{\partial f(x,y)}{\partial y},$$

respectively.

Further, let $a, b : \mathcal{D} \to \mathcal{D}$ be the bosonic operators defined by

$$a := x + \frac{1}{2}\frac{\partial}{\partial x}, \quad b := y + \frac{1}{2}\frac{\partial}{\partial y}.$$

Consider also their adjoints $a^*, b^* : \mathcal{D} \to \mathcal{D}$

$$a^* = x - \frac{1}{2}\frac{\partial}{\partial x}, \quad b^* = y - \frac{1}{2}\frac{\partial}{\partial y}.$$

These operators satisfy the commutation rules,

$$[a, a^*] = [b, b^*] = 1,$$

and

$$[a, b^*] = [b, a^*] = [a^*, b^*] = [a, b] = 0,$$

which characterize an algebra of Weil-Heisenberg.

A basis of $\mathcal{H}$

The following holds,

$$a\Phi_0 = b\Phi_0 = 0,$$

for $\Phi_0 = \mathrm{e}^{-(x^2+y^2)} \in \mathcal{D}$, a so-called *vacuum* or *fundamental state*.

The set of the eigenvectors

$$\{\Phi_{m,n} = a^{*m} b^{*n} \Phi_0 : \; m, n \geq 0\},$$

of

$$N := a^* a + b^* b,$$

constitutes a basis of $\mathcal{H}$, that is,

every vector in $L^2(\mathbb{R}^2)$ can be uniquely expressed in terms of this eigensystem,

which is *complete*, since 0 is the only vector orthogonal to all the its elements.

$m + n$ is the eigenvalue of N associated to $\Phi_{m,n}$

Non self-adjoint Hamiltonian

The linear operator $H : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$,

$$H = a^*a + bb^* + \beta(a^*a - b^*b) + \gamma(a^*b^* - ab), \quad \beta, \gamma \in \mathbb{R}, \quad |\gamma| < 1,$$

is non self-adjoint for $\gamma \neq 0$.

It describes a system of two interacting bosons.

The operator H is expressed as a linear combination of the operators

$$a^*a + bb^*, \; ab, \; a^*b^*,$$

which constitute an operator algebra: the commutator of any pair belongs to the space which they generate. The Casimir operator

$$C := a^*a - b^*b,$$

commutes with all of them.

Invariant subspaces of H

Since any $f \in L^2(\mathbb{R}^2)$ may be expanded in the basis $\{\Phi_{mn} : m, n \geq 0\}$, we may identify $L^2(\mathbb{R}^2)$ with $\mathcal{H} = \text{span}\{\Phi_{mn} : m, n \geq 0\}$.

The eigenspaces of the Casimir operator are invariant subspaces of H.

Eigenspaces of the Casimir operator,

$$\mathcal{H}_k := \text{span}\{\Phi_{mn} : m - n = k, m, n \geq 0\}.$$

In $\mathcal{H}_k$, a^*b^* is represented by

$$A_+ = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ \sqrt{|k|+1} & 0 & 0 & 0 & \dots \\ 0 & \sqrt{2(|k|+2)} & 0 & 0 & \dots \\ 0 & 0 & \sqrt{3(|k|+3)} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

while ab is represented by

$$A_- = A_+^*,$$

and $a^*a + bb^*$ is represented by

$$A_0 = \begin{bmatrix} |k|+1 & 0 & 0 & 0 & \dots \\ 0 & |k|+3 & 0 & 0 & \dots \\ 0 & 0 & |k|+5 & 0 & \dots \\ 0 & 0 & 0 & |k|+7 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Raising and lowering operators

The following CR's are satisfied

$$[A_-, A_+] = A_0, \quad [A_0, A_+] = 2A_+, \quad [A_0, A_-] = -2A_-. \qquad (4)$$

The operator A_+ is said to be a raising operator.

That is, if Φ is an eigenvector of A_0, so that

$$A_0\Phi = \Lambda\Phi,$$

then $A_+\Phi$ is an eigenvector associated with an upward shifted eigenvalue,

$$A_0 A_+ \Phi = (\Lambda + 2) A_+ \Phi,$$

$$A_0 A_+^2 \Phi = (\Lambda + 4) A_+^2 \Phi,$$

$$\vdots.$$

The operator A_- is a lowering operator.

$A_-\Phi \neq 0$ is an eigenvector associated with a downward shifted eigenvalue,

$$A_0 A_- \Phi = (\Lambda - 2) A_- \Phi.$$

The spectrum of A_0 is bounded from below and the eigenvector $\Phi_0 = (1, 0, 0, \ldots)^T$ satisfies $A_- \Phi_0 = 0$.

The eigenvector Φ_0 is said to be a lowest weight state.

The eigenvalues of A_0 are positive.

A set of associated eigenvectors is obtained by acting successively with A_+ on Φ_0.

We observe that $A_0 = A_0^*$ and $A_+ = A_-^*$.

Pseudo-Jacobi matrix representation

The compression of the Hamiltonian H to $\mathcal{H}_k$, is represented by a so called pseudo-Jacobi matrix, that is, a Jacobi matrix multiplied by the involution $\mathrm{diag}(1,-1,1,-1,\ldots)$,

$$\begin{bmatrix} \beta k+|k|+1 & -\gamma\sqrt{|k|+1} & 0 & 0 & \ldots \\ \gamma\sqrt{|k|+1} & \beta k+|k|+3 & -\gamma\sqrt{2(|k|+2)} & 0 & \ldots \\ 0 & \gamma\sqrt{2(|k|+2)} & \beta k+|k|+5 & -\gamma\sqrt{3(|k|+3)} & \ldots \\ 0 & 0 & \gamma\sqrt{3(|k|+3)} & \beta k+|k|+7 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

This matrix may be diagonalized by algebraic techniques.

Equations of Motion Method (EMM)

Let $\mathcal{V}$ be the linear space generated by A_0, A_+, A_-.

Let

$$H_0 := H - \beta k I = A_0 + \gamma(A_+ - A_-) \in \mathcal{V}.$$

EMM: Finding a lowering and a raising operator for H_0.

We look for $xA_+ + yA_- + zA_0$ such that

$$[H_0, (xA_+ + yA_- + zA_0)] = \lambda(xA_+ + yA_- + zA_0).$$

We find

$$[H, A_0] = \gamma(-2A_+ - 2A_-),$$

$$[H, A_+] = 2A_+ - \gamma A_0,$$

$$[H, A_-] = -2A_- - \gamma A_0.$$

The eigenvector $xA_+ + yA_- + zA_0$ is determined by the equation

$$\begin{bmatrix} 2 & 0 & -2\gamma \\ 0 & -2 & -2\gamma \\ -\gamma & -\gamma & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The eigenvalues λ and respective eigenvectors are given by

$$2\sqrt{1+\gamma^2}, \quad \left(\frac{1+\sqrt{1+\gamma^2}}{\gamma}, \frac{\gamma}{1+\sqrt{1+\gamma^2}}, -1\right)^T,$$

$$-2\sqrt{1+\gamma^2}, \quad \left(\frac{-1+\sqrt{1+\gamma^2}}{\gamma}, \frac{\gamma}{-1+\sqrt{1+\gamma^2}}, 1\right)^T,$$

and

$$0, \quad (\gamma, -\gamma, 1)^T.$$

Let us consider now the matrices

$$
\begin{aligned}
B_+ &= \frac{\gamma}{2\sqrt{1+\gamma^2}} \left(\frac{1+\sqrt{1+\gamma^2}}{\gamma} A_+ + \frac{\gamma}{1+\sqrt{1+\gamma^2}} A_- - A_0 \right) \\
B_- &= \frac{\gamma}{2\sqrt{1+\gamma^2}} \left(\frac{-1+\sqrt{1+\gamma^2}}{\gamma} A_+ + \frac{\gamma}{-1+\sqrt{1+\gamma^2}} A_- + A_0 \right), \\
B_0 &= \frac{1}{\sqrt{1+\gamma^2}} \Big(A_0 + \gamma (A_+ - A_-) \Big) .
\end{aligned}
$$

where the coefficients have been determined so that the following CR's are satisfied

$$
[B_-, B_+] = B_0, \quad [B_0, B_+] = 2B_+, \quad [B_0, B_-] = -2B_- .
$$

We remark that $B_0 \neq B_0^*$, $B_+ \neq B_-^*$.

The operator B_+ is a raising operator.

That is,

if Ψ is an eigenvector of B_0, so that $B_0\Psi = \Lambda\Psi$, then

$$B_0 B_+ \Psi = (\Lambda + 2) B_+ \Psi$$

.

Similarly, $B_-\Psi \neq 0$ is an eigenvector associated with a downward shifted eigenvalue,

$$B_0 B_- \Psi = (\Lambda - 2) B_- \Psi.$$

The operator B_- is a lowering operator.

Since the spectrum of B_0 is bounded from below, there exist eigenvectors Ψ_0 such that $B_-\Psi_0 = 0$.

The eigenvalues of H

The Casimir operator is

$$C := B_0^2 - 2(B_+B_- + B_-B_+) = B_0^2 - 2B_0 - 4B_+B_- = (k^2 - 1)I.$$

Thus,

$$C\Psi_0 = (B_0^2 - 2B_0)\Psi_0 = (k^2 - 1)\Psi_0$$

which implies that

$$B_0\Psi_0 = (|k| + 1)\Psi_0,$$

so that

$$\sigma(B_0) = \{|k| + 1, |k| + 3, |k| + 5, \ldots\}.$$

Thus

$$\sigma(H_0) = \sqrt{1 + \gamma^2}\{|k| + 1, |k| + 3, |k| + 5, \ldots\}.$$

$$\Psi_0 = \left(1,\ -\alpha(|k|+1)^{1/2},\ \alpha^2 \binom{|k|+2}{2}^{1/2},\ -\alpha^3 \binom{|k|+3}{3}^{1/2},\ \ldots\right)^T$$

$$\alpha = \frac{\gamma}{1+\sqrt{1+\gamma^2}} = \frac{-1+\sqrt{1+\gamma^2}}{\gamma}$$

The operators B_+ and B_- are, respectively, the representations in $\mathcal{H}_k$ of the operators

$$c^{\ddagger}d^{\ddagger}, \quad \text{and} \quad cd,$$

where

$$c = \mathcal{N}((-1+\sqrt{1+\gamma^2})b^* + \gamma a), \quad d = \mathcal{N}((-1+\sqrt{1+\gamma^2})a^* + \gamma b),$$
$$d^{\ddagger} = \mathcal{N}((1+\sqrt{1+\gamma^2})b^* - \gamma a), \quad c^{\ddagger} = \mathcal{N}((1+\sqrt{1+\gamma^2})a^* - \gamma b),$$

with

$$\mathcal{N} = \left(2\gamma\sqrt{1+\gamma^2}\right)^{-1/2}.$$

Notice that $c^{\ddagger} \neq c^*$ and $d^{\ddagger} \neq d^*$.

Eigenfunctions of H

The eigenvalues of H (for $\beta = 0$) are

$$E_{m,n} = (1 + m + n)\sqrt{1 + \gamma^2}, \quad m, n \geq 0.$$

The associated eigenfunctions are expressed as

$$\Psi_{m,n} = \mathrm{e}^{-2\gamma xy}\Phi^{\gamma}_{m,n},$$

where,

$$\Phi^{\gamma}_{m,n} = KH_m((1+\gamma^2)^{1/4}x)\mathrm{e}^{-\sqrt{1+\gamma^2}x^2} \times H_n((1+\gamma^2)^{1/4}y)\mathrm{e}^{-\sqrt{1+\gamma^2}y^2}.$$

Here $K = (2(1+\gamma^2)/\pi)^{1/2}$ and $H_n(t)$ is the nth Hermite polynomial in t.

That is, $\Phi^{\gamma}_{m,n}$ is factorized as follows

$$\Phi^{\gamma}_{m,n}(x, y) = \Phi^{\gamma}_{m}(x)\Phi^{\gamma}_{n}(y),$$

Spectrum of H

The spectrum of an operator on a finite dimensional Hilbert space is exhausted by the eigenvalues, but, in the infinite dimensional setting, there are additional parts to be considered.

The *resolvent set* of H, denoted by $\rho(H)$, is constituted by all the complex numbers for which the *resolvent operator* $\lambda \in \rho(H) \to (H - \lambda)^{-1}$ exists as a bounded operator on $\mathcal{H}$.

The complement

$$\sigma(H) = \mathbb{C} \backslash \rho(H)$$

is the *spectrum* of H.

The set of all eigenvalues of H is the *point spectrum*, denoted by $\sigma_p(H)$, and formed by complex numbers λ for which $H - \lambda : \mathcal{D}(H) \to \mathcal{H}$ is not injective.

Those λ which are not eigenvalues but $H - \lambda$ is not bijective constitute the *continuous* or *residual spectrum*, depending on the range $\mathrm{Ran}(H - \lambda)$ being, respectively, dense or not.

The spectrum $\sigma(H)$ is the union of these three disjoint spectra.

The spectrum of self-adjoint operators is nonempty, real, and the residual spectrum is empty.

Numerical Range of H

It is advantageous for the characterization of the spectrum of H to determine the *numerical range*:

$$W(H) := \{\langle H\psi, \psi\rangle : \psi \in \mathcal{D}(H), \|\psi\| = 1\}.$$

$W(H)$ is convex (Toeplitz-Hausdorff Theorem, 1917-1918), and,

$$\sigma(H) \subset \overline{W(H)}.$$

For simplicity, H denotes the Hamiltonian operator in Cartesian coordinates or its compression to $\mathcal{H}_k$. In the first case, $\mathcal{D}(H) = L^2(\mathbb{R}^2)$. In the second case, $\mathcal{D}(H) = \mathbb{C}^\infty$.

Theorem 1. *The numerical range of H is bounded by the hyperbola branch*

$$y^2 + \gamma^2(1 - x^2) = 0, \quad x \geq 1.$$

Determination of numerical range of the compression of H to $\mathcal{H}_k$, with $\beta = 0$:

The Hermitian part of $\exp(i\theta)H$ is

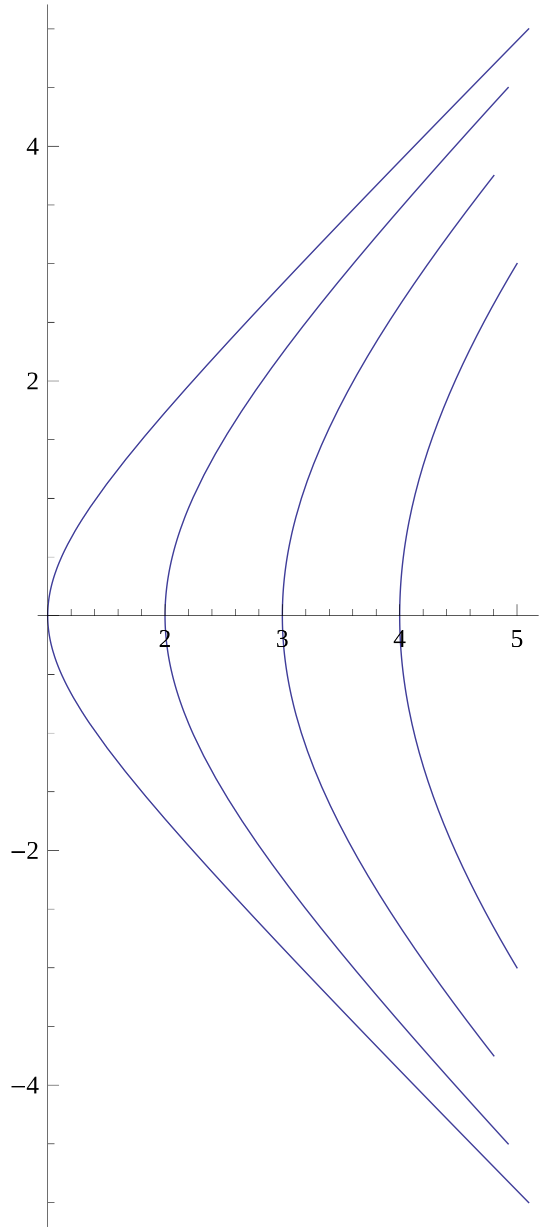
$$H_\theta = A_0 \cos\theta - i\gamma(A_+ - A_-)\sin\theta.$$

The lowest eigenvalue of H_θ, which is equal to the distance to the origin of the supporting line perpendicular to the direction θ, is

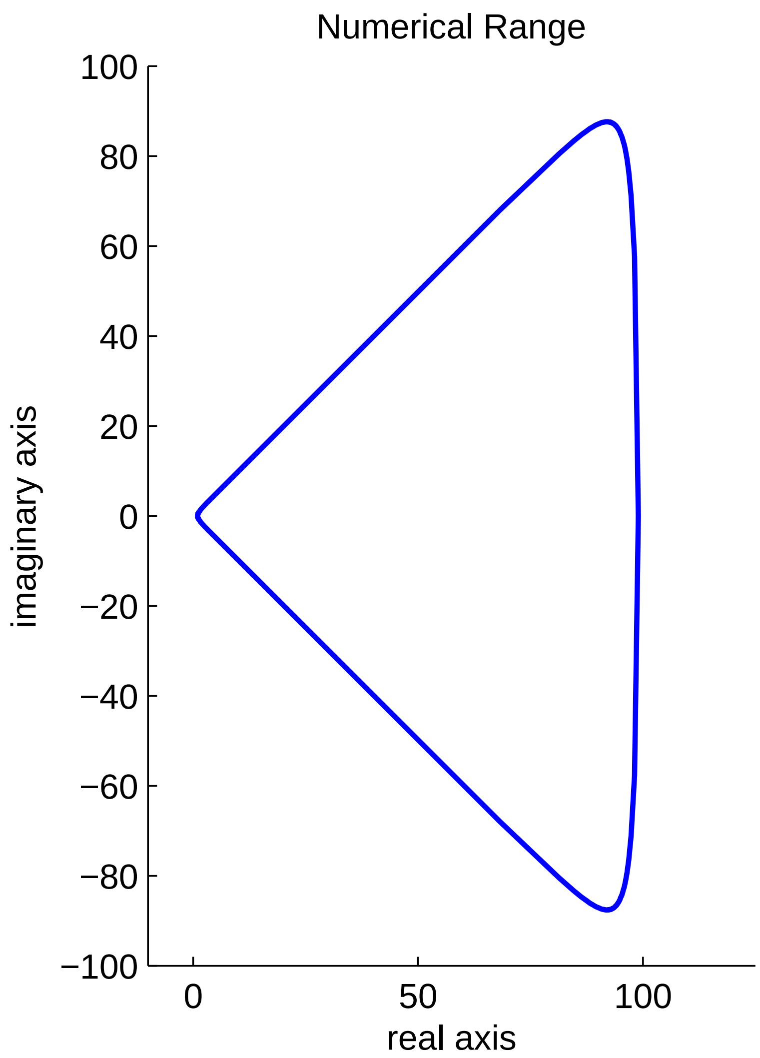
$$2\sqrt{\cos^2\theta - \gamma^2 \sin^2\theta}\ (|k| + 1).$$

The envelope of the supporting lines is the branch $x \geq 1 + |k|$ of the hyperbola

$$x^2 - \frac{y^2}{\gamma^2} = (1 + |k|)^2.$$



Numerical ranges of the compressions of H to the $\mathcal{H}_k$'s, for $\beta = 0,\ \gamma = 1,\ k = 0, 1, 2, 3.$



Numerical range of the compression of H, truncated to the dimension 50.

Sectorial operators are those whose numerical range is the subset of a sector

$$S_{w,\theta} = \{z \in \mathbb{C} : |\arg(z - w)| \leq \theta\}$$

where $w \in \mathbb{R}$ and $0 \leq \theta < \pi/2$, called, respectively, the *vertex* and *semi-angle* of H.

An operator is said to be *accretive* if the vertex can be chosen at the origin, i.e., $W(H) \subset S_{0,\pi/2}$.

An operator H is m-*accretive* iff

1) $\{\lambda \in \mathbb{C} : \Re\lambda < 0\} \subset \rho(H)$

2) the *resolvent bound* for any λ with $\Re\lambda < 0$, holds:

$$\forall\lambda \in \mathbb{C},\ \Re\lambda < 0,\ \|(H - \lambda)^{-1}\| \leq 1/|\Re\lambda|.$$

H is a closed, densely defined and compact operator.

H in $\mathcal{H}$ has a *compact resolvent* if

$$\rho(H) \neq \emptyset$$

and

$(H - \lambda)^{-1}$ is a compact operator, for $\lambda \in \rho(H)$.

The operator H has a compact resolvent, as $\Re(H)$ is an m-accretive operator (since $\Re(H)$ is Hermitian and $W(\Re(H))$ lies on the positive real axis) with compact resolvent and, moreover, V is relatively bounded with respect to $\Re(H)$ with relative bound smaller than 1. Then, $\Re(H) + \lambda V$ has a compact resolvent [Krejcirik, Theorem 5.4.1].

If H has a compact resolvent, then $\sigma(H) = \sigma_p(H)$ [Krejcirik, Theorem IX, 2.3].

For $k = 0$, the eigenvector Ψ_0 in (5) represents the eigenfunction

$$\Psi_{00}(x, y) = \mathrm{e}^{-2\gamma xy - (x^2+y^2)\sqrt{1+\gamma^2}}$$

of H associated with the eigenvalue $E_{00} = \sqrt{1+\gamma^2}$.

$\exp\left(-(x^2+y^2)\sqrt{1+\gamma^2}\right)$ is the vacuum of the bosonic operators

$$g = \frac{1}{2(1+\gamma^2)^{1/4}} \frac{\partial}{\partial x} + (1+\gamma^2)^{1/4} x, \quad h = \frac{1}{2(1+\gamma^2)^{1/4}} \frac{\partial}{\partial y} + (1+\gamma^2)^{1/4} y,$$

$$g^* = -\frac{1}{2(1+\gamma^2)^{1/4}} \frac{\partial}{\partial x} + (1+\gamma^2)^{1/4} x, \quad h^* = -\frac{1}{2(1+\gamma^2)^{1/4}} \frac{\partial}{\partial y} + (1+\gamma^2)^{1/4} y,$$

so that, the following holds

$$g\mathrm{e}^{2\gamma xy}\Psi_{00}(x, y) = h\mathrm{e}^{2\gamma xy}\Psi_{00}(x, y) = 0$$

The physical inner product

The Hamiltonian H acts on functions $\Psi(x,y)$ such that

$$\exp(2\gamma xy)\Psi(x,y) \in L^2(\mathbb{R}^2).$$

This suggests the introduction of the *physical inner product*

$$\ll \Psi_\alpha, \Psi_\beta \gg = \langle e^{4\gamma xy}\Psi_\alpha, \Psi_\beta \rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{4\gamma xy}\Psi_\alpha(x,y)\Psi_\beta(x,y)\,dx\,dy,$$

implying that

$$\ll H\Psi, \Psi \gg = \ll \Psi, H\Psi \gg$$

and

$$\frac{\ll \Psi_{m,n}, \Psi_{p,q} \gg}{\sqrt{\ll \Psi_{m,n}, \Psi_{m,n} \gg \ll \Psi_{p,q}, \Psi_{p,q} \gg}} = \delta_{mp}\delta_{nq}$$

Discussion

We have considered a non self-adjoint Hamiltonian H whose eigenvalues and corresponding eigenfunctions have been explicitly determined.

The investigated Hamiltonian and its adjoint have real eigenvalues and systems of biorthogonal eigenvectors.

They have infinite diagonal matrix representations in the respective eigensystems, which are complete.

Viewing H as the Hamiltonian of a physical model, problems arise from non Hermiticity.

The original inner product defined in $\mathcal{H}$ is not adequate for the physical interpretation of the model.

A new metric, which is appropriate for that purpose is introduced.

Following Mostafazadeh, one can define a subspace of the Hilbert space, and the compression of the Hamiltonian operator to that subspace, so that it has the same spectrum and eigenfunctions as the original one.

The referred subspace remains invariant under the action of H.

Stating that this Hermitian operator represents in a reasonable sense the non-Hermitian operator may be controversial, since relevant information on the Hamiltonian may not be captured in the mentioned subspace.

Although H is non-Hermitian with respect to the inner product $\langle \cdot, \cdot \rangle$, it becomes Hermitian with respect to the physical inner product $\ll \cdot, \cdot \gg = \langle \Theta \cdot, \cdot \rangle$.

Non-Hermitian operators have typically non-trivial pseudospectra.

It is known that the quasi-Hermicity similarity relation holds via a bounded and boundedly invertible positive transformation if and only if the quasi-Hermicity relation holds with a positive bounded and boundedly invertible metric.

Further, if the quasi-Hermicity relation holds with a positive bounded and boundedly invertible metric, then the pseudospectrum of H is trivial.

The concept of pseudospectrum is of great relevance for the description of non-Hermitian operators in the context of QM.

A non trivial pseudospectrum ensures the non existence of a bounded metric.