# Simultaneous zero inclusion property for spatial numerical ranges

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- $\mathscr{X}$ finite-dimensional complex normed space,
  - usually: $\mathscr{X} = (\mathbb{C}^n, \|\cdot\|)$;
- $\mathscr{X}'$ the dual normed space,
  - $\langle \cdot, \cdot \rangle : \mathscr{X} \times \mathscr{X}' \to \mathbb{C}$ the pairing;
- unit spheres:

$$\begin{array}{l} \mathscr{S}_{\mathscr{X}} = \{x \in \mathscr{X};\ \|x\| = 1\} \\ \mathscr{S}_{\mathscr{X}'} = \{\xi \in \mathscr{X}';\ \|\xi\| = 1\} \end{array};$$

- $\mathcal{L}(\mathscr{X})$ linear operators on $\mathscr{X}$,
  - usually: $\mathcal{L}(\mathscr{X}) = (\mathbb{M}_n, \|\cdot\|)$;
- $I$ the identity operator.

- By the Hahn-Banach theorem:

  $x \in \mathscr{S}_{\mathscr{X}} \quad \Rightarrow \quad \exists\, \xi \in \mathscr{S}_{\mathscr{X}'} : \quad \langle x, \xi \rangle = 1.$

- Denote: $\mathscr{D}(x) = \{\xi \in \mathscr{S}_{\mathscr{X}'}; \quad \langle x, \xi \rangle = 1\}.$
- If $\mathscr{X}$ is strictly convex, then $\mathscr{D}(x)$ is a singleton.
- If $\mathscr{X}$ is a Hilbert space with inner product $(\cdot, \cdot)$, then

$$\mathscr{D}(x) = \{(\cdot, x)\}.$$

- If $\mathscr{X} = \ell_1(n)$ and $e_j = (0, \ldots, 1, \ldots 0)^{\mathsf{T}}$, then

  $\mathscr{D}(e_j) = \{(\gamma_1, \ldots, 1, \ldots, \gamma_n)^{\mathsf{T}}; \quad |\gamma_1| \le 1, \ldots, |\gamma_n| \le 1\} \subset \mathscr{S}_{\ell_\infty(n)}.$

- The spatial numerical range of $A \in \mathcal{L}(\mathscr{X})$ is

$$W_{\mathscr{X}}(A) = \{\langle Ax, \xi \rangle; \quad x \in \mathscr{S}_{\mathscr{X}}, \, \xi \in \mathscr{D}(x)\}.$$

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- Properties:
  - $\sigma(A) \subseteq W_{\mathscr{X}}(A) \subseteq \overline{\mathbb{D}(0, \|A\|)}$;
  - $W_{\mathscr{X}}(A)$ is closed;
  - $W_{\mathscr{X}}(A+B) \subseteq W_{\mathscr{X}}(A) + W_{\mathscr{X}}(B)$;
  - $W_{\mathscr{X}}(\alpha A + \beta I) = \alpha W_{\mathscr{X}}(A) + \beta$;
  - $U$ a linear isometry on $\mathscr{X}$: $W_{\mathscr{X}}(U^{-1}AU) = W_{\mathscr{X}}(A)$.

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  - $U$ a linear isometry on $\mathscr{X}$: $W_{\mathscr{X}}(U^{-1}AU) = W_{\mathscr{X}}(A)$.
- If $\mathscr{X} = \mathscr{H}$ (Hilbert space), then $W_{\mathscr{H}}(A)$ is convex.

  This does not hold for every $\mathscr{X}$.

- **Observation:**

  $\mathscr{X} = \mathscr{H}$ Hilbert space, $A \in \mathcal{L}(\mathscr{H})$ invertible, then

$$0 \in W_{\mathscr{H}}(A) \iff 0 \in W_{\mathscr{H}}(A^{-1}). \tag{1}$$

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Proof.

- $0 \in W_{\mathscr{H}}(A) \quad \Rightarrow \quad \exists\, x \in \mathscr{S}_{\mathscr{H}} : \quad (Ax, x) = 0$
- $y = \frac{1}{\|Ax\|} Ax \in \mathscr{S}_{\mathscr{H}}$ ($Ax \neq 0$ because $A$ is invertible)
- $(A^{-1}y, y) = \frac{1}{\|Ax\|^2}(x, Ax) = 0.$ □

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- Answer: No.

- Example: $\mathscr{X} = \ell_1(2)$.

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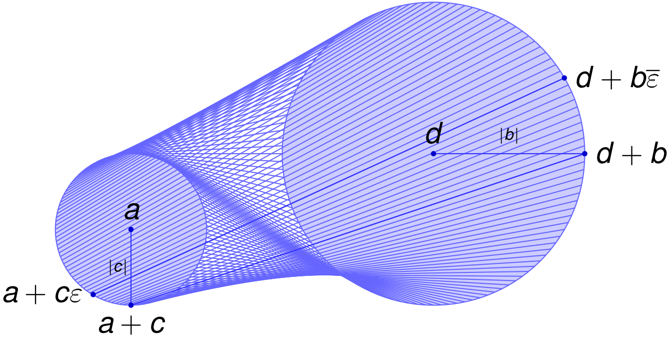
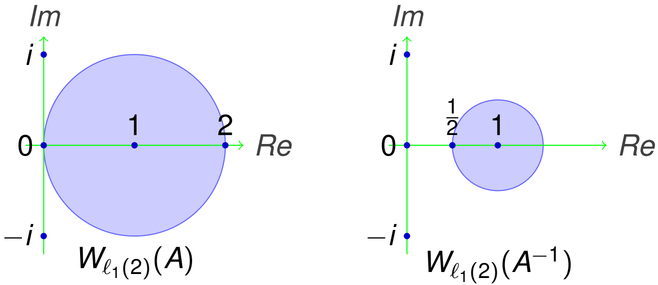
- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_2$;
- $W_{\ell_1(2)}(A)$ is union of:
  - $\overline{\mathbb{D}(a, |c|)}$,
  - $\overline{\mathbb{D}(d, |b|)}$,
  - $[a + c\varepsilon, d + b\overline{\varepsilon}] = \{t(a + c\varepsilon) + (1 - t)(d + b\overline{\varepsilon});\ 0 \le t \le 1\}$,

    line segment for each $\varepsilon \in \mathbb{C},\ |\varepsilon| = 1$.


$d + b\overline{\varepsilon}$
$d$
$|b|$
$d + b$
$a$
$|c|$
$a + c\varepsilon$
$a + c$

- $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{bmatrix};$
- $W_{\ell_1(2)}(A) = \overline{\mathbb{D}(1,1)}$ and $W_{\ell_1(2)}(A^{-1}) = \overline{\mathbb{D}(1,\frac{1}{2})}$;




- $0 \in W_{\ell_1(2)}(A)$ and $0 \notin W_{\ell_1(2)}(A^{-1})$.

- $\ell_1(n), \quad n \geq 2.$

$$A_1 = \begin{bmatrix} 2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

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- $\ell_\infty(n), \quad n \geq 2.$
  - $W_{\ell_\infty(n)}(S) = W_{\ell_1(n)}(S^{\mathsf{T}})$ for every $S \in \mathbb{M}_n$;
  - $0 \in W_{\ell_\infty(n)}(A_1^{\mathsf{T}})$ and $0 \notin W_{\ell_\infty(n)}((A_1^{\mathsf{T}})^{-1}).$

- $\ell_p(n), \quad n \geq 2, \quad 2 < p.$

  There exists $\quad 0 < a_p < 1 \quad$ such that for

$$A_p = \begin{bmatrix} a_p + 1 & 1 & 0 & \cdots & 0 \\ 0 & a_p & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

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- $\ell_q(n), \quad n \geq 2, \quad 1 < q < 2.$
  - $p = \frac{q}{q-1} > 2;$
  - $W_{\ell_q(n)}(S) = W_{\ell_p(n)}(S^{\mathsf{T}})$ for every $S \in \mathbb{M}_n;$
  - $0 \in W_{\ell_q(n)}(A_p^{\mathsf{T}})$ and $0 \notin W_{\ell_q(n)}((A_p^{\mathsf{T}})^{-1})).$

- Definition. Normed space $\mathscr{X}$ has the simultaneous zero inclusion property (S0I) if

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- Hilbert space has S0I.
- $\ell_p(n), \quad 1 \leq p \leq \infty, \ p \neq 2$ does not have S0I.

- $\mathscr{X}$ normed space, $T \in \mathcal{L}(\mathscr{X})$ invertible;
  new norm: $\|x\|_T = \|Tx\| \quad (x \in \mathscr{X})$;
  $\mathscr{X}_T = (\mathscr{X}, \|\cdot\|_T)$.
- Theorem. The following are equivalent:
  - $\mathscr{X}$ has S0I;
  - $\mathscr{X}'$ has S0I;
  - $\mathscr{X}_T$ has S0I.

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  If $0 \in \operatorname{conv}\big(\sigma(A)\big)$, then S0I holds at $A$.

  Proof.
  - $\sigma(A) = \{\lambda_1, \dots, \lambda_k\} \quad \Rightarrow \quad \sigma(A^{-1}) = \{\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_k}\}$;
  - $t_1\lambda_1 + \cdots + t_k\lambda_k = 0, \quad 0 \le t_1, \dots, t_k \le 1, \ t_1 + \cdots + t_k = 1$;
  - for $j = 1, \dots, k$, $s_j = \frac{t_j|\lambda_j|^2}{t_1|\lambda_1|^2 + \cdots + t_k|\lambda_k|^2}$;
  - $0 \le s_1, \dots, s_k \le 1$ and $s_1 + \cdots + s_k = 1$;
  - $s_1\frac{1}{\lambda_1} + \cdots + s_k\frac{1}{\lambda_k} = 0$. □

- Algebra numerical range of $A \in \mathcal{L}(\mathscr{X})$ is

$$V_{\mathscr{X}}(A) \in W_{\mathcal{L}(\mathscr{X})}(L_A) \quad \text{where} \quad L_A : \ T \mapsto AT, \ \forall\, T \in \mathcal{L}(\mathscr{X}).$$

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  - $V_{\mathscr{X}}(A) = \mathrm{conv}\big(W_{\mathscr{X}}(A)\big)$;
  - Williams:

    $$\lambda \in V_{\mathscr{X}}(A) \quad \Longleftrightarrow \quad |z - \lambda| \leq \|zI - A\|, \quad \forall \, z \in \mathbb{C}.$$

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- Theorem. If $A \in \mathcal{L}(\mathscr{X})$ is an invertible contraction and $\|e^{i\varphi} I - A^{-1}\| < 1$, then S0I holds at $A$.

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  Proof.
  - $\|e^{i\varphi} I - A^{-1}\| < 1 \quad \Rightarrow \quad 0 \notin V_{\mathscr{X}}(A^{-1}) \supseteq W_{\mathscr{X}}(A^{-1})$;
  - $\|e^{-i\varphi} I - A\| \le \|e^{-i\varphi} A\| \|e^{i\varphi} I - A^{-1}\| < 1 \quad \Rightarrow$

    $\Rightarrow \quad 0 \notin V_{\mathscr{X}}(A) \supseteq W_{\mathscr{X}}(A)$. □

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- $\mathcal{G}^{out}(\mathscr{X}) = \{A \in \mathcal{L}(\mathscr{X}); \quad A \text{ invertible and } 0 \notin W_{\mathscr{X}}(A) \cup W_{\mathscr{X}}(A^{-1})\},$

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- $\mathscr{X}$ has S0I $\iff$ $\mathcal{G}^{out}(\mathscr{X}) \cup \mathcal{G}^{in}(\mathscr{X})$ all invertible;
- Theorem. $\mathcal{G}^{out}(\mathscr{X})$ is open and $\mathcal{G}^{in}(\mathscr{X})$ is relatively closed subset in the set of all invertible operators.

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  - $\mathscr{X} = \ell_1(n)$: for

$$A_1 = \begin{bmatrix} 2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

$W_{\ell_1(n)}(A_1)$ and $W_{\ell_1(n)}(A_1^{-1})$ are convex $\Rightarrow$

$W_{\ell_1(n)}(A_1) = V_{\ell_1(n)}(A_1)$ and $W_{\ell_1(n)}(A_1^{-1}) = V_{\ell_1(n)}(A_1^{-1})$

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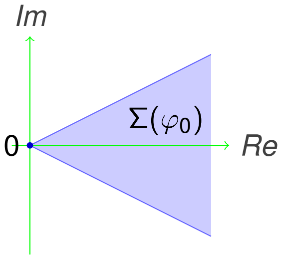
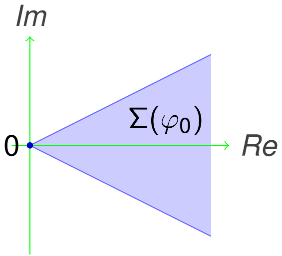
  - $\mathscr{X} = \ell_p(n)$, $1 < p < \infty$, $p \neq 2$: we do not know.

- $\varphi_0 \in [0, \frac{\pi}{2})$: $\Sigma(\varphi_0) = \{re^{i\varphi}; \quad r \geq 0, \ -\varphi_0 \leq \varphi \leq \varphi_0\}$.




- Definition. $\mathscr{X}$ has the simultaneous sector inclusion property (SSI) if, for every invertible $A \in \mathcal{L}(\mathscr{X})$,
$V_{\mathscr{X}}(A) \subseteq \Sigma(\varphi_0)$ for some $\varphi_0$ $\Rightarrow$ $V_{\mathscr{X}}(A^{-1}) \subseteq \Sigma(\varphi_0)$.

- $\varphi_0 \in [0, \frac{\pi}{2})$: $\Sigma(\varphi_0) = \{re^{i\varphi};\quad r \geq 0,\ -\varphi_0 \leq \varphi \leq \varphi_0\}$.

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- Hilbert space has this property;

  $\ell_1(n)$ does not have: $A_1$ is a counter example.
- Is there any relation between SSI and S0I?

Thank you!