Classification of Nonselfadjoint Operators

via Operator System Equivalence

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Operator System Equivalence

Numerical Ranges

Definition (Toeplitz-Hausdorff, 1918)

The *(spatial) numerical range* of a bounded linear operator T on a complex Hilbert space H is the set

$$W_{\mathrm{s}}(T) = \{\langle T\xi, \xi\rangle \,:\, \xi \in \mathsf{H},\ \|\xi\| = 1\}.$$

Definition (Banach Algebra Community, 1950s)

The *numerical range* of an element x in a unital Banach algebra A is the set

$$W(x) = \{\varphi(x) \,:\, \varphi \in \mathsf{A}^*,\ \|\varphi\| = \varphi(1_{\mathsf{A}}) = 1\}.$$

Remarks

- By the Hahn-Banach Extension Theorem, if X is a subspace of a unital Banach algebra A such that $1_A \in X$, then

$$W(x) = \{\varphi(x) \, : \, \varphi \in X^*, \; \|\varphi\| = \varphi(1_A) = 1\},$$

for every $x \in X$.

- Thus, $W(x)$ is determined by certain elements of the unit sphere in the dual space X^* of any subspace of A that contains 1_A and x, such as the 2-dimensional subspace $\mathrm{Span}\,\{1_A, x\}$. Hence,

$$W(x) = \{\lambda \in C \, : \, |\alpha\lambda + \beta| \leq \|\alpha x + \beta 1_A\|, \, \forall \, \alpha, \beta \in C\}.$$

Remarks: numerical range for Hilbert space operators

- In the case of B(H), the spatial numerical range and numerical range of $T \in \mathrm{B(H)}$ are related via:

  $$W(T) = \overline{W_{\mathrm{s}}(T)},$$

  If H has finite dimension, then $W(T) = W_{\mathrm{s}}(T)$ because $W_{\mathrm{s}}(T)$ is already a closed set.

- Because B(H) has an involution $T \mapsto T^*$, it is more natural to consider $W(T)$ in terms of the subspace

  $$\mathrm{S}_T = \mathrm{Span}\,\{1, T, T^*\},$$

  where 1 denotes the identity operator $\xi \mapsto \xi$ on H. That is,

  $$W(T) = \{\varphi(T) \,:\, \varphi \in (\mathrm{S}_T)^*, \, \|\varphi\| = \varphi(1) = 1\}.$$

Remarks: numerical range for Hilbert space operators

- A subspace of the form $\mathsf{S}_T = \mathrm{Span}\{1, T, T^*\}$, for some $T \in \mathsf{B}(\mathsf{H})$, is spanned by its positive elements
- The conditions $\|\varphi\| = \varphi(1) = 1$ are equivalent to $\varphi(1) = 1$ and $\varphi(A) \geq 0$ for every positive $A \in \mathsf{S}_T$.
- If by a *state* on S_T we mean a linear functional φ such that $\varphi(1) = 1$ and $\varphi(A) \geq 0$ for every positive $A \in \mathsf{S}_T$, then

$$W(T) = \{\varphi(T) \,:\, \varphi \text{ is a state on } \mathsf{S}_T\}.$$

- In place of Hahn-Banach Extension Theorem, we have the Krein Extension Theorem: *every state on* S_T *extends to a state on* $\mathsf{B}(\mathsf{H})$. Thus, the definition of $W(T)$ above does not depend on the choice of unital $*$-closed subspace that contains 1, T, and T^*.

Remarks: numerical range for Hilbert space operators

In a subspace of the form $S_T = \mathrm{Span}\{1, T, T^*\}$, elements X are given by

$$X = \alpha 1 + \beta T + \gamma T^*.$$

Thus, $\lambda \in W(T)$ if and only if

$$|\alpha + \beta\lambda + \gamma\bar{\lambda}| \leq \|\alpha 1 + \beta T + \gamma T^*\|,$$

for a scalars α, β, γ.

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Now let us replace the scalars λ, α, β, γ by $n \times n$ complex matrices Λ, A, B, and C

The matricial range for Hilbert space operators

Definition

The *n-th matricial range of* $T \in \mathrm{B(H)}$ is the subset $W_n(T)$ of all $n \times n$ complex matrices Λ such that

$$\|A \otimes 1_n + B \otimes \Lambda + C \otimes \Lambda^*\| \leq \|A \otimes 1 + B \otimes T + C \otimes T^*\|,$$

for all $n \times n$ complex matrices A, B, and C. (Here, 1_n is the $n \times n$ identity matrix and 1 is the identity operator on H.)

Notes:

- 1_n is the $n \times n$ identity matrix and 1 is the identity operator on H
- the norm on each of $\mathrm{M}_n(\mathrm{C}) \otimes \mathrm{M}_n(\mathrm{C})$ and $\mathrm{M}_n(\mathrm{C}) \otimes \mathrm{B(H)}$ is the "operator norm"

A Notion of Equivalence for Operators

Definition

If $S, T \in \mathrm{B(H)}$, then S and T are *operator system equivalent*, denoted by $S \simeq_{\mathrm{os}} T$, if

$$\|A \otimes 1_n + B \otimes S + C \otimes S^*\| = \|A \otimes 1 + B \otimes T + C \otimes T^*\|,$$

for all $n \times n$ complex matrices A, B, and C, and all positive integers n.

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- Sufficient condition for $S \simeq_{\mathrm{os}} T$: $S = U^* TU$, for some unitary U
- Necessary condition for $S \simeq_{\mathrm{os}} T$: $W_n(S) = W_n(T)$, for every n

Examples of Operator System Equivalence

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The following statements are equivalent for operators S and T such that $S^* = S$:

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The following statements are equivalent for operators S and T such that $S^* = S$:

1. $S \simeq_{\mathrm{os}} T$
2. S and T have the same numerical ranges

Note: the numerical range does not determine a hermitian operator up to unitary (or approximate-unitary) equivalence, but it does determine a hermitian operator up to operator-system equivalence

Examples of Operator System Equivalence

Example (2)

If B is the forward shift (a.k.a bilateral shift) operator on $\ell^2(Z)$, and if S is the forward shift (a.k.a unilateral shift) operator on $\ell^2(N)$, then

$$B \simeq_{\mathrm{os}} S.$$

Note: B is unitary and S is not unitary; thus, B and S cannot be unitarily equivalent. Therefore, operator system equivalence is a weaker notion than unitary equivalence.

Arveson's Theorem

Definition

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Theorem (Arveson, 1972)

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The following statements are equivalent for irreducible compact operators:

1. *S and T are operator system equivalent*
2. *S and T are unitarily equivalent*

C^*-Envelope

Theorem (Hamana, 1979)

If $T \in \mathrm{B(H)}$*, then there exists a unital* C^**-algebra, denoted by* $\mathrm{C}^*_{\mathrm{e}}(T)$*, and a unital complete isometry* $\iota_{\mathrm{e}} : \mathrm{S}_T \to \mathrm{C}^*_{\mathrm{e}}(T)$ *such that*

1. $\mathrm{C}^*_{\mathrm{e}}(T)$ *is generated by the range of* ι_{e}*, and*
2. *if* $\phi : \mathrm{S}_T \to \mathrm{A}$ *is a unital complete isometry such that* A *is generated by the range of* ϕ*, then there exists a surjective* $*$*-homomorphism* $\pi : \mathrm{A} \to \mathrm{C}^*_{\mathrm{e}}(T)$ *such that* $\pi \circ \phi = \iota_{\mathrm{e}}$*.*

Definition

If S and T are $*$-closed subspaces of $\mathrm{B(H)}$ in which $1 \in \mathrm{S}$ and $1 \in \mathrm{T}$, then a linear map $\phi : \mathrm{S} \to \mathrm{T}$ is a *unital complete isometry* if

$$\left\| [\phi(S_{ij})]_{i,j=1}^{n} \right\| = \left\| [S_{ij}]_{i,j=1}^{n} \right\|, \forall\, S_{i,j} \in \mathrm{S},\ \forall\, n \in \mathrm{N}$$

Examples of C*-Envelopes

Example (1)

If $T^* = T$, then $\mathrm{C}^*_{\mathrm{e}}(T) = \mathbb{C} \oplus \mathbb{C}$

Example (2)

For each $\lambda \in \mathbb{C}$, let

$$T_\lambda = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{bmatrix}.$$

Then:

$$\mathrm{C}^*_{\mathrm{e}}(T_\lambda) = \left\{ \begin{array}{ll} \mathrm{M}_2(\mathbb{C}) & \text{if } |\lambda| \leq 1/2 \\ \mathrm{M}_2(\mathbb{C}) \oplus \mathbb{C} & \text{if } |\lambda| > 1/2 \end{array} \right\}.$$

Examples of C*-Envelopes

Notation: $J_k(\lambda)$ denotes a $k \times k$ Jordan block with eigenvalue λ

Theorem (Argerami, Farenick)

If $J = \bigoplus_{k=1}^{n} (J_{m_k}(\lambda_k) \otimes 1_{d_k})$, *where each pair* (m_k, λ_k) *unique, and if each* λ_k *is a real number, then the following statements are equivalent:*

1. $\mathrm{C}^*_{\mathrm{e}}(J)$ *is abelian*
2. $m_1 = \cdots = m_k = 1$

C^*-Envelope Equivalence

Definition

If $S, T \in \mathrm{B(H)}$, then S and T are *C^*_{e}-equivalent*, denoted by $S \simeq_{\mathrm{C}^*_{\mathrm{e}}} T$, if there exists a unital $*$-isomorphism $\varrho : \mathrm{C}^*_{\mathrm{e}}(S) \to \mathrm{C}^*_{\mathrm{e}}(T)$ such that $\varrho(S) = T$.

Theorem

The following statements are equivalent for operators S and T:

1. $S \simeq_{\mathrm{os}} T$

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2. $S \simeq_{\mathrm{C}^*_{\mathrm{e}}} T$

C^*-Envelope Equivalence

Definition

If $S, T \in B(H)$, then S and T are *C_e^*-equivalent*, denoted by $S \simeq_{C_e^*} T$, if there exists a unital $*$-isomorphism $\varrho : C_e^*(S) \to C_e^*(T)$ such that $\varrho(S) = T$.

Theorem

The following statements are equivalent for operators S and T:

1. $S \simeq_{\mathrm{os}} T$
2. $S \simeq_{C_e^*} T$

Thus, operator system equivalence amounts to the identification of $C_e^*(T)$ up to $*$-isomorphism.

Irreducible Periodic Weighted Shift Operators

Finite-Dimensional Hilbert Spaces

Definition

If $\xi = (\xi_1, \ldots, \xi_d)$ with $\xi_j \in \mathbb{C} \setminus \{0\}$ for all j, then the *irreducible weighted unilateral shift* with weights $\xi_1, \ldots, \xi_d$ is the operator $W(\xi)$ on $\mathbb{C}^{d+1}$ given by the matrix

$$W(\xi) = \begin{bmatrix} 0 & & & & 0 \\ \xi_1 & 0 & & & \\ & \xi_2 & \ddots & & \\ & & \ddots & 0 & \\ & & & \xi_d & 0 \end{bmatrix}.$$

Finite-Dimensional Hilbert Spaces

Theorem (Farenick, Gerasimova, Shvai, 2011)

The C^*-envelope of an irreducible weighted unilateral shift acting on $\mathbb{C}^{d+1}$ is $M_{d+1}(\mathbb{C})$. Furthermore, for any $\xi, \eta \in \mathbb{C}^d$ with nonzero entries, the following statements are equivalent:

1. $W(\xi) \simeq_{\mathrm{os}} W(\eta)$

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2. $W(\xi) = U^* W(\eta) U$ *for some unitary* U

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1. $W(\xi) \simeq_{\text{os}} W(\eta)$
2. $W(\xi) = U^* W(\eta) U$ *for some unitary* U
3. $|\xi_j| = |\eta_j|$ *for all* j

Infinite-Dimensional Hilbert Spaces

Definition

A *weighted unilateral shift operator of period p* is an operator $W(\omega)$ on $\ell^2(\mathbb{N})$ defined on the standard orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of $\ell^2(\mathbb{N})$ by

$$W(\omega)e_n = \omega_n e_{n+1}\,,\ n \in \mathbb{N},$$

where the *weight sequence* $\omega = \{\omega_n\}_{n\in\mathbb{N}}$ for W consists of complex numbers satisfying

- $\sup_n |\omega_n| < \infty$, and
- $\omega_{n+p} = \omega_n$ for every $n \in \mathbb{N}$, where p is the least positive integer with this property

Remark: If $\omega_n \neq 0$ for all n, the $W(\omega)$ is an irreducible operator.

Recall: the case $p = 1$

An irreducible periodic weighted unilateral shift operator is simply a scalar multiple ω_1 of the (unweighted) unilateral shift. In this case, $S \simeq_{\rm os} U$, for every unitary operator U with $\sigma(U) = \partial \mathbb{D}$.

Thus, it may be more fruitful to determine those operators T for which $\mathrm{C}^*_{\mathrm{e}}(T) \cong \mathrm{C}^*_{\mathrm{e}}(S)$ via a unital $*$-isomorphism ϱ in which $\varrho(T) = S$ — that is, to determine T such that $T \simeq_{\mathrm{C}^*_{\mathrm{e}}} S$.

In the case $p = 1$, this possible because of the Gelfand theory for commutative C*-algebras (even though the C*-algebra generated by S is noncommutative).

Main Result

Definition

If $\omega = \{\omega_1, \ldots, \omega_p\}$ is a sequence of complex numbers, then $G(\omega) : \partial D \to \mathrm{M}_p(\mathrm{C})$ denotes the matrix-valued function given by

$$G(\omega)[z] = \begin{bmatrix} 0 & & & & \omega_p z \\ \omega_1 & 0 & & & \\ & \omega_2 & \ddots & & \\ & & \ddots & 0 & \\ & & & \omega_{p-1} & 0 \end{bmatrix}, \quad \text{for } z \in \partial D.$$

Note: The algebra $\mathrm{M}_p\left(C(\partial D)\right)$ of all continuous functions $\partial D \to \mathrm{M}_p(\mathrm{C})$ is given by the C^*-algebra $C(\partial D) \otimes \mathrm{M}_p(\mathrm{C})$

Main Result

Theorem (Argerami, Farenick)

If $\omega = \{\omega_1, \dots, \omega_p\}$ is a sequence of nonzero complex numbers, and if there is at least one k with $|\omega_k| \notin \{|\omega_j| : j \neq k\}$, then the following statements are equivalent:

1. $W(\omega) \simeq_{\text{os}} G(\omega)$

Main Result

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1. $W(\omega) \simeq_{\mathrm{os}} G(\omega)$
2. $W(\omega) \simeq_{\mathrm{C}^*_\mathrm{e}} G(\omega)$

Moreover, if one of the statements above holds, then

$$\mathrm{C}^*_\mathrm{e}\left(W(\omega)\right) \cong \mathrm{C}^*_\mathrm{e}\left(G(\omega)\right) \cong C(\partial D) \otimes \mathrm{M}_p(\mathrm{C}) \cong \mathrm{M}_p\left(C(\partial D)\right).$$

Main Result: the use of numerical and matricial ranges

For a fixed $\lambda_0 \in \mathbb{C}$ of modulus $|\lambda_0| = 1$, consider the matrix

$$\Omega_{\lambda_0} = G(\omega)[\lambda_0] = \begin{bmatrix} 0 & & & & \omega_p\lambda_0 \\ \omega_1 & 0 & & & \\ & \omega_2 & \ddots & & \\ & & \ddots & 0 & \\ & & & \omega_{p-1} & 0 \end{bmatrix}.$$

- One can think of Ω_{λ_0} as the result of evaluating "point evaluation" $f \mapsto f[\lambda_0]$, for $f \in \mathrm{M}_p(C(\partial\mathbb{D}))$, at $f = G(\omega)$
- In particular, $\Omega_{\lambda_0} \in W_p(G(\omega))$

Main Result: the use of numerical and matricial ranges

- Motivating idea from Gelfand theory: point evaluations on commutative C*-algebras are extremal states
- Matrix case: there is a strong sense in which the "point evaluation matricial state" $\Gamma_{z_0}(f) = f(z_0)$ is extremal, leading to the fact that Ω_{λ_0} is a C*-extreme point of the matricial range $W_p\left(G(\omega)\right)$ of $G(\omega)$
  - *crucial result* of Tsai-Wu (2011): if $r = w(\Omega_{\lambda_0})$ (numerical radius), then

$$W(\Omega_{\lambda_0}) \cap \{z \,:\, |z| = r\} = \{r\zeta^k \lambda_0 \,:\, k = 1, \dots, p\},$$

    where ζ is a primitive p-th root of unity

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    $$W(\Omega_{\lambda_0}) \cap \{z : |z| = r\} = \{r\zeta^k \lambda_0 : k = 1, \dots, p\},$$

    where ζ is a primitive p-th root of unity
  - *personal thought*: the Tsai-Wu result is something that Hausdorff and Toeplitz would have appreciated, 100 years ago

Subspaces Determined by Single-Generated Algebras

A_T

Let $\mathrm{Alg}(T) = \overline{\{f(T) : f \text{ is a complex polynomial}\}}$, which is a unital, abelian Banach algebra

Definition

The *algebra operator system determined by* T is the unital $*$-closed subspace A_T given by

$$\mathrm{A}_T = \{X + Y^* : X, Y \in \mathrm{Alg}(T)\}.$$

Although $\mathrm{A}_T \supseteq \mathrm{S}_T$, the algebra operator system determined by T carries more information about T than S_T does.

Example: Finite Toeplitz Matrices

If $J = J_n(0)$, the $n \times n$ nilpotent Jordan block, then A_J is the unital $*$-closed subspace of all $n \times n$ Toeplitz[1] matrices, which we denoted by $\mathcal{T}_n$.

Theorem (Farenick, Mastnak, Popov)

If $\varphi : \mathcal{T}_n \to M_n(\mathrm{C})$ is a unital linear isometry, then there is a unitary matrix U such that $\varphi(X) = U^* X U$ for every $X \in \mathcal{T}_n$.

Note: φ maps $n \times n$ Toeplitz matrices to $n \times n$ matrices

[1] Yes, the same Toeplitz!

Example: Finite Toeplitz Matrices

Theorem (Farenick, Mastnak, Popov)

If $\varphi : \mathcal{T}_n \to M_m(\mathbb{C})$ *is a unital completely isometric linear map, then* $n \leq m$ *and there are a unitary* $U \in M_m(\mathbb{C})$*, a positive integer* ℓ*, and a unital completely contractive map* $\psi : \mathcal{T}_n \to M_{m-\ell n}(\mathbb{C})$ *such that*

$$\varphi(X) = U^* \left([X \otimes I_\ell] \oplus \psi(X)\right) U,$$

for every $X \in \mathcal{T}_n$*.*

Remark: Allowing $m \neq n$ forces us to move from isometries in the previous theorem to complete isometries in the theorem above

Main Points

1. The numerical range lends itself to be defined in terms of states on a low-dimensional unital subspace
2. Once defined in terms of norms, one can consider "matrix-linear combinations" rather than classical linear combinations to arrive at the notion of a matricial range (there is a spatial approach, also, which I did not mention)
3. Operator system equivalence is a weaker notion than unitary equivalence, and has potential uses in describing various phenomena (e.g., the study of "clean POVMs")
4. The notion of A_T is motivated by Arveson's seminal works in 1969/1972, where he studies nonselfadjoint operator algebras A by way of the analytic features of the subspace $A + A^*$

100 years

Thank you for attending WONRA 2018,

the 100th anniversary of the Toeplitz-Hausdorff Theorem