Linear maps preserving $\mathcal{AN}$-operators

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Outline

Linear preserving problems

Let H be a separable, inifinite dimensional Hilbert space and $B(H)$ the set of all bounded operators on H.
The following are three typical linear preserving problwms which are pointed out by Chi-Kwong Li-S. Pierce (2001) and Lajos Molnár (2006):
Let M be a subspace of $B(H)$.

Linear preserving problems

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The following are three typical linear preserving problwms which are pointed out by Chi-Kwong Li-S. Pierce (2001) and Lajos Molnár (2006):
Let M be a subspace of $B(H)$.

1. Let F be a (scalar-valued, vector-valued, or set valued) function on M, study the linear preservers of F, i.e., thoses linear operators on M such satisfying $F(\phi(A)) = F(A)$ for all $A \in M$.
2. Let S be a subset of M. Characterize thoses linear preservers ϕ on M which satisfy $\phi(S) \subset S$, or satisfy $A \in S \Leftrightarrow \phi(A) \in S$.
3. Let $\sim$ be a relation on M. Characterize thoses linear preservers ϕ on M which satisfy $A \sim B \Rightarrow \phi(A) \sim \phi(B)$, or satisfy $A \sim B \Leftrightarrow \phi(A) \sim \phi(B)$.

In this talk we consider problem $\boxed{2}$ in the case that S is the set of all $\mathcal{AN}$-operators.

$\mathcal{N}$-operators

Definition

Let H and K be separable infinite dimensional Hilbert spaces. Let $T \in B(H, K)$. T is said to be an $\mathcal{N}$-operator or to satisfy the norm attaining property if there is an element x in the unit sphere of H such that $\|T\| = \|Tx\|$.

We describe $\mathcal{N}(H, K)$ as the set of all $\mathcal{N}$-operators.

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When $T \in B(H)$ and $T = T^*$, T is an $\mathcal{N}$-operator if and only if $\|T\|$ or $-\|T\|$ is an eigenvalue of T.
Indeed, if T is an $\mathcal{N}$-operator, there exists a unit vector $x_0 \in H$ such that $\|Tx_0\| = \|T\|$. We may assume that $\|T\| = 1$. Since $\langle (I - T^2)x_0, x_0\rangle = 0$, $(I - T^2)x_0 = 0$. Hence, $(I + T)(x_0 - Tx_0) = 0$ or $(I - T)(x_0 + Tx_0) = 0$. We know, then, that $\|T\|$ or $-\|T\|$ is an eigenvalue of T.

Properties for $\mathcal{N}$-operators

The following are proved by Carvajal and Neves (2012):

Theorem

Let H and K be separable infinite dimensional Hilbert spaces and $T \in B(H, K)$. The following statements are equivalent.

1. $T \in \mathcal{N}(H, K)$,
2. $T^* \in \mathcal{N}(K, H)$,
3. $|T| \in \mathcal{N}(H, K)$,
4. $|T^*| \in \mathcal{N}(K, H)$,
5. $|T|^2 \in \mathcal{N}(H, K)$,
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When $\|Tx_0\| = \|T\|$, we know $\|T^*(T\frac{1}{\|T\|}x_0)\| = \|T\| = \|T^*\|$.

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When $\|Tx_0\| = \|T\|$, we know $\|T^*(T\frac{1}{\|T\|}x_0)\| = \|T\| = \|T^*\|$.

When $H = K$, we know that $N(H)(= \mathcal{N}(H, H))$ is self-adjoint set.

Examples related to $\mathcal{N}$-operators

1. $D : \ell^2 \to \ell^2$ by

$$D(x_1, x_2, \dots, x_n, \dots) = (x_1, \frac{x_2}{2}, \dots, \frac{x_n}{n}, \dots)$$

   for $x = (x_n) \in \ell^2$. Then $D \geq 0$ and D satisfies the $\mathcal{N}$-property.
   Moreover, any compact operator satisfies the $\mathcal{N}$-property.

2. Let $T = \frac{1}{2}I - D$. Then
   $\|T\| = \sup\{|\frac{1}{2} - \frac{1}{n}| : n \in \mathbf{N}\} = \frac{1}{2} = \|Te_1\|$. Hence $T \in \mathcal{N}(H)$.

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$\|T\| = \sup\{|\frac{1}{2} - \frac{1}{n}| \colon n \in \mathbf{N}\} = \frac{1}{2} = \|Te_1\|$. Hence $T \in \mathcal{N}(H)$.
Let $M = [e_1]^\perp$. Then M is a reducing subspace of T and $\|T|_M\| = \sup\{|\frac{1}{2} - \frac{1}{n}| \colon n \geq 2\} = \frac{1}{2}$. But $T|_M \notin \mathcal{N}(M, \ell^2)$, because that neigher $\frac{1}{2}$ nor $-\frac{1}{2}$ is an eigenvalue of $T|_M$.

Proposition

Let H be a separable, infinite dimensional Hilbert space. Then $\mathcal{N}(H)$ is dense in $B(H)$ with respect to the normed topology.

$\mathcal{AN}$-operators

Definition

Let $T \in B(H,K)$. Then T is said to be an $\mathcal{AN}$ operator or to satisfy the absolutely norm attaining property $\mathcal{AN}$ if for every nonzero closed subspace M of H $T|_M \colon M \to K$ satisfies the norm attaining property.

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When we describe $\mathcal{AN}(H)$ as the set of all $\mathcal{AN}$-operators, then we have $\mathcal{AN}(H) \subset \mathcal{N}(H)$.
Moreover, we know that $\mathcal{AN}(H) \subsetneq \mathcal{N}(H)$.

Necessary conditions for positive $\mathcal{AN}$-operators

Lemma (Pandey-Paulsen 2016)

Let H and K be separable, infinite dimensional Hilbert spaces. For a closed linear space M of H let $V_M \colon M \to H$ be an inclusion map from M to H. Then $T \in \mathcal{AN}(H, K)$ if and only if for every nontrivial closed subspace M of H $TV_M \in \mathcal{N}(M, K)$.

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Let V be an isometry. Then $V \in \mathcal{AN}(H)$.

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Example

Let V be an isometry. Then $V \in \mathcal{AN}(H)$.

Theorem (Pandey-Paulsen 2016)

Let $T \in \mathcal{AN}$ be positive. Then H has an orthonormal basis consisting of eigenvelues of T.

Proposition (Pandey-Paulsen 2016)

Let $T \in \mathcal{AN}(H)$ be positive. Then the spectrum $\sigma(T)$ of T has at most one limit point. Moreover, the uniqune limit point (if exists) can only be the limit of a decreasing sequence in the spectrum.

Proof.

(Scketch) (Step 1) If there exists an increasing ssequence of $\{\lambda_n\}$ such that $\lambda \nearrow \lambda$ with $\lambda_n < \lambda$ $(n \in \mathbf{N})$. Set $M_0 = \mathrm{clos}(\mathrm{spn}\{v_n\})$, where v_n's are the eigenvector corresponding to λ_n. Then $\|T|_{M_0}\| = \sup\{|\lambda_n|\} = \lambda$. However, for every $x = \sum_n \alpha_n v_n \in M_0$ with $\sum_n |\alpha_n|^2 = 1$ so that $\|x\| = 1$,

$$\|T|_{M_0}(x)\| = \sum_n |\alpha_n|^2 |\lambda_n|^2 < \lambda^2 \sum_n |\alpha_n|^2 = \lambda^2,$$

so that $\|T|_{M0}(x)\| < \lambda \leq \|T|_{M0}\|$, contradiction. □

Continuation of the proof

Proposition (Pandey-Paulsen 2016)

Let $T \in \mathcal{AN}(H)$ be positive. Then the spectrum $\sigma(T)$ of T has at most one limit point. Moreover, the uniqune limit point (if exists) can only be the limit of a decreasing sequence in the spectrum.

(Step 2) Suppose that the spectrum of $\sigma(T) = \mathrm{clos}(\{\beta_\alpha\}_{\alpha\in\Lambda})$ has two limits $a < b$. Then from the step 1 there are decreasing sequences $\{a_n\}, \{b_n\} \subset \{\beta_\alpha\}_{\alpha\in\Lambda}$ such that $a_n \searrow a$ and $b_n \searrow b$. May assume that $a_1 < b$. Hence $a_n < b_n$ for $n \in \mathbf{N}$.
Definie $M = \mathrm{clos}(\mathrm{span}\{c_n f_n + \sqrt{1 - c_n^2} g_n : n \in \mathbf{N}\})$, where $c_n \in [0, 1]$, $Tf_n = a_n f_n$ and $Tg_n = b_n g_n$ for each $n \in \mathbf{N}$. Then we know that $T|_M \notin \mathcal{N}(M, H)$. □

Theorem (Pandey-Paulsen 2016)

If $T \in \mathcal{AN}(H)$ is positive, then $T = \sum_{\alpha \in \Lambda} \beta_\alpha v_\alpha \otimes v_\alpha$, where $\{v_\alpha \colon \alpha \in \Lambda\}$ is an orthogonal basis consisiting of eigenvalues of T and for every $\alpha \in \Lambda$ $Tv_\alpha = \beta_\alpha v_\alpha$ with $\beta_\alpha \geq 0$ such that

1. for every finite subset $\Gamma \subset \Lambda$ we have $\sup\{\beta_\alpha \colon \alpha \in \Gamma\} = \max\{\beta_\alpha \colon \alpha \in \Gamma\}$,
2. $\sigma(T)$ has at most one limit point. Moreover, the unique limit point (if it exists) can only the limit of a decreasing sequence in $\sigma(T)$,
3. the set $\{\beta_\alpha\}_{\alpha \in \Lambda}$ eigenvalues of T, without counting multiplicities, is countable and has at most one eigenvalue with infinite multiplicity,
4. if $\sigma(T)$ has both, a limit point β and an eigenvalues $\hat{\beta}$ with infinite multiplicity, then $\beta = \hat{\beta}$.

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Note that if $V \colon \ell^2 \to \ell^2$ be an isometry such that $\dim V(H)^\perp = \infty$, then $V \in \mathcal{AN}(H)$, but $V^* \notin \mathcal{AN}(H)$, and $\mathcal{AN}(H) \subsetneq \mathcal{N}(H)$.

Properties for $\mathcal{AN}(H)$

Theorem (Pandey-Paulsen 2016)

Let H be an arbitrary dimensional Hilbert space and $T \in B(H)$ be a positive. Then $T \in \mathcal{AN}(H)$ if and only if $T = \alpha I + K + F$, where $\alpha \geq 0$, K is positive compact operator and F is self-adjoint finite-rank operator.

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Proposition (Ramesh-Naidu 2017)

Let $T \in B(H, K)$. Then $T \in \mathcal{AN}(H, K)$ if and only if $T^*T \in \mathcal{AN}(H)$.

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Corollary (Ramesh-Naidu 2017)

Let $T \in B(H)$ be normal. Then $T \in \mathcal{AN}(H)$ if and only if $T^* \in \mathcal{AN}(H)$

Maps preserving $\mathcal{AN}$-operators

Proposition (Osaka-Ramesh 2018)

Let $V \in B(H)$ be an isometry and $\phi\colon B(H) \to B(H)$ be defined by $\phi(T) = V^*TV$, for $T \in B(H)$. Then ϕ is a unital C.P. map and $\phi(\mathcal{AN}(H)_+) \subset \mathcal{AN}(H)_+$.
Furthermore, if $V^* \in \mathcal{AN}(H)$, then $\phi(\mathcal{AN}(H)) \subset \mathcal{AN}(H)$.

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Lemma (Carvajal-Neves 2011)

Let $P \in B(H)$ be an orthogonal projection. Then $P \in \mathcal{AN}(H)$ if and only if $\dim\ker(P) < \infty$ or $\dim P(H) < \infty$.

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Corollary (Osaka-Ramesh 2018)

Let $V \in B(H)$ be an isometry and $\phi\colon B(H) \to B(H)$ be defined by $\phi(T) = V^*TV$, for $T \in B(H)$. If $V^* \notin \mathcal{AN}(H)$, then there exists an unitary U such that $\phi(U) \notin \mathcal{AN}(H)$.

Theorem (Osaka-Ramesh 2018)

Let $A \in \mathcal{AN}(H)$. Dfinie $\rho \colon B(H) \to B(H)$ by
$\rho(T) = A^* TA$ for $T \in B(H)$.
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Since $A \in \mathcal{AN}(H)$ if and only if $A^*A \in \mathcal{AN}(H)$, we know that $A^*A = \alpha I + K + F$ for some $\alpha \geq 0$, $K \in K(H)_+$, $F = F^* \in F(H)$. Hence, $\rho(\mathcal{AN}(H)_+) \subset \mathcal{AN}(H)_+$.

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Using this idea we can prove

Corollary

Let $S, T \in \mathcal{AN}(H)$. Then $ST \in \mathcal{AN}(H)$.

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Let $S, T \in \mathcal{AN}(H)$. Then $ST \in \mathcal{AN}(H)$.

Question

How to characterize the map of the form $\rho() = A^*()A$ for some $A \in \mathcal{AN}(H)$?

Let k be a positive integer. We denote by $\mathcal{B}_k(H)$ and $\mathcal{B}_{\leq k}(H)$, the set of all bounded linear operators of rank k and the set of all bounded linear operators of rank at most k, respectively. We say $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is a rank k-preserver if $A \in \mathcal{B}_k(H)$ implies $\phi(A) \in \mathcal{B}_k(H)$. We say ϕ is a rank-k non increasing map if $A \in \mathcal{B}_{\leq k}(H)$ implies $\phi(A) \in \mathcal{B}_{\leq k}(H)$.

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Theorem (M. Gyory, L. Molnar and P. Semrl 1998)

Let k be a positive integer and H be a Hilbert space. Assume that $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a linear rank k non increasing map which is weakly continuous on norm bounded sets. Then either the image of ϕ is a linear space consisting of operators of rank at most k or there exists $A, B \in \mathcal{B}(H)$ such that either $\phi(T) = ATB$ for all $T \in \mathcal{B}(H)$ or $\phi(T) = AT^{tr}B$ for all $T \in \mathcal{B}(H)$. Here T^{tr} denotes the transpose of T relative to any orthonormal basis of H fixed in advance.

Theorem (O-Ramesh 2018)

Let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a linear positive unital, rank k-non increasing map, which is weakly continuous on norm bounded sets. Assume that ϕ preserves the set $\mathcal{AN}(H)_+$. Then there exist an isometry $A \in \mathcal{B}(H)$ such that
either $\phi(T) = A^*TA$ for all $T \in \mathcal{B}(H)$ or $\phi(T) = A^*T^{tr}A$ for all $T \in \mathcal{B}(H)$. In addition if ϕ preserves the set $\mathcal{AN}(H)$, then $A^* \in \mathcal{AN}(H)$.

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either $\phi(T) = A^*TA$ for all $T \in \mathcal{B}(H)$ or $\phi(T) = A^*T^{tr}A$ for all $T \in \mathcal{B}(H)$. In addition if ϕ preserves the set $\mathcal{AN}(H)$, then $A^* \in \mathcal{AN}(H)$.

Theorem (O-Ramesh 2018)

Let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a linear positive, rank k-non increasing map, which is weakly continuous on norm bounded sets. Assume that ϕ preserves the set $\mathcal{AN}(H)_+$ and $\phi(I)$ invertible. Then there exist $A \in \mathcal{AN}(H)$ such that either $\phi(T) = A^*TA$ for all $T \in \mathcal{B}(H)$ or $\phi(T) = A^*T^{tr}A$ for all $T \in \mathcal{B}(H)$.
In addition if ϕ preserves the set $\mathcal{AN}(H)$ and $\phi(I)^{-1} \in \mathcal{AN}(H)$, then $A^* \in \mathcal{AN}(H)$.

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Theorem (O-Ramesh 2018)

Let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a linear positive, rank k-non increasing map, which is weakly continuous on norm bounded sets. Assume that ϕ preserves the set $\mathcal{AN}(H)_+$. Further assume that there exists $S \in \mathcal{B}(H)$ such that rank of $\phi(S)$ is not finite. Then there exist $A \in \mathcal{AN}(H)$ such that either $\phi(T) = A^* TA$ for all $T \in \mathcal{B}(H)$ or $\phi(T) = A^* T^{tr} A$ for all $T \in \mathcal{B}(H)$.

References

1 X. Carvajal and W. Neves, Operators that achieve the norm, Integral Equations Operator Theory **72** (2012), no. 2, 179–195. MR2872473 (2012k:47044)

2 C. K. Li and S. Pierce, *Linear preserver problems*, Amer. Math. Monthly 108 (2001), 591–605. MR1862098

3 L. Molnar, *Selected preserver problems on algebraic structures of linear operators and on function spaces*, Lecture Notes in Mathematics, 1895, Springer-Verlag, Berlin, 2007. MR2267033

4 H. Osaka and G. Ramesh, *Maps preserving $\mathcal{AN}$-operators*, preprint 2018.

5 V. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, 78, Cambridge University Press, Cambridge, 2002. MR1976867

6 Satish Pandey and Vern Paulsen, A Spectral Characterization of $\mathcal{AN}$- Operators, Journal of the Australian Mathematical Society, 2016, doi 10.1017/S1446788716000239

7 G. Ramesh, Structure theorem for $\mathcal{AN}$-operators, J. Aust. Math. Soc. **96** (2014), no. 3, 386–395. MR3217722

8 G. Ramesh and D. Venkunaidu, On Absolutely norm attaining operators, preprint 2016, arXiv:1801.02432.

9 W. F. Stinespring, Positive functions on C^*-algebras, Proc. Amer. Math. Soc. **6** (1955), 211–216. MR0069403