# Operator Radii and Unitary Operators

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ABSTRACT. Let $\rho \geq 1$ and $w_\rho(A)$ be the operator radius of a linear operator $A$. Suppose $m$ is a positive integer. It is shown that for a given invertible linear operator $A$ acting on a Hilbert space, one has $w_\rho(A^{-m}) \geq w_\rho(A)^{-m}$. The equality holds if and only if $A$ is a multiple of a unitary operator.

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## 1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$ which induces the norm $\|\cdot\|$. Denote by $B(\mathcal{H})$ the algebra of bounded linear operators acting on $\mathcal{H}$ with the *operator norm* defined by

$$\|A\| = \sup\{\|Ax\| : x \in \mathcal{H}, \|x\| = 1\} \qquad \text{for } A \in B(\mathcal{H}).$$

It is easy to see that $A \in B(\mathcal{H})$ is *unitary* if and only if it is invertible and

$$\|A\| \leq 1 \quad \text{and} \quad \|A^{-1}\| \leq 1. \tag{1.1}$$

If the requirement (1.1) is weakened as

$$\|A^n\| \leq \rho \quad \text{and} \quad \|A^{-n}\| \leq \rho \quad (n = 1, 2, \dots) \text{ for some } \rho \geq 1, \tag{1.2}$$

then, by a theorem of Sz.-Nagy [8], the operator $A$ is *similar* to a unitary operator, that is,

$$A = S^{-1}US \quad \text{for some invertible } S \text{ and unitary } U,$$

and consequently its *spectrum* $\sigma(A)$ is included in the unit circle of the complex plane.

Recall that the *numerical radius* of $A \in B(\mathcal{H})$ is defined by

$$w(A) = \sup\{|\langle x, Ax\rangle| \ : \ x \in \mathcal{H}, \|x\| \leq 1\}.$$

In [7, Corollary 1] (see also [6]), it was shown that in (1.1) the operator norm $\|\cdot\|$ can be replaced by the numerical radius $w(\cdot)$, namely, that an invertible operator $A$ is unitary if $w(A) \leq 1$ and $w(A^{-1}) \leq 1$. Notice that the map $A \longmapsto w(A)$ is *convex* and (1.2) is guaranteed by the known property of the numerical radius (see [9]), namely, for any $A \in B(\mathcal{H})$

$$w(A) \leq \|A\| \leq 2 \cdot w(A) \quad \text{and} \quad w(A^n) \leq w(A)^n \quad \text{for } n = 1, 2, \dots.$$

Very recently, Choi and Li [2, Theorem 3.9] showed that for a positive integer $m$ and an invertible operator $A \in B(\mathcal{H})$, we have

$$w(A^{-m}) \geq w(A)^{-m}; \tag{1.3}$$

the equality holds if and only if $A$ is a multiple of a unitary operator. Clearly, the same result holds if one replaces the numerical radius by the operator norm. (A short proof of this case is included in Section 3).

In [9], Sz.-Nagy and Foiaş considered the class $\mathcal{C}_\rho$ of operators $T \in B(\mathcal{H})$ which admits a *unitary $\rho$-dilation*, that is, there is a unitary operator $U$ on a superspace $\mathcal{K} \supset \mathcal{H}$ such that

$$T^n = \rho P U^n|_{\mathcal{H}} \quad \text{for } n = 1, 2, \dots,$$

where $P$ is the orthoprojection from $\mathcal{K}$ to $\mathcal{H}$. In connection with this, one can define the *$\rho$-radius* of $A \in B(\mathcal{H})$ by

$$w_\rho(A) = \inf\{\lambda > 0 : \ \lambda^{-1}A \in \mathcal{C}_\rho\}.$$

When $\rho = 1$ and $\rho = 2$, this definition reduces to the operator norm and the numerical radius, respectively. The operator radii have the following properties (see [9], [4] and [5]):

(i) For each $\rho$, the functional $A \longmapsto w_\rho(A)$ is strictly positive, and (non-linear) positive-homogeneous and $w_\rho(A) = w_\rho(A^*)$.

(ii) Let $r(A)$ be the spectral radius of $A \in B(\mathcal{H})$. Then $\lim_{\rho\to\infty} w_\rho(A) = r(A)$, and the function $\rho \longmapsto w_\rho(A)$ is *non-increasing*. Consequently, we have

$$r(A) \le w_\rho(A) \le \|A\|.$$

(iii) For each $A \in B(\mathcal{H})$, we have

$$w_\rho(A) \le \|A\| \le \rho \cdot w_\rho(A) \quad \text{and} \quad w_\rho(A^n) \le w_\rho(A)^n \quad \text{for } n = 1, 2, \ldots.$$

(iv) For $A \in B(\mathcal{H})$, the function $A \longmapsto w_\rho(A)$ is *convex* (only) when $1 \le \rho \le 2$.

In this paper, we show that the inequality (1.3) and the condition for equality are valid if we replace the numerical radius by the $\rho$-radius for any $\rho \ge 1$. Specifically, we have the following.

**Theorem 1.1.** *Let $\rho \ge 1$, and $m$ be a positive integer. If $A \in B(\mathcal{H})$ is invertible, then*

$$w_\rho(A^{-m}) \ge w_\rho(A)^{-m}.$$

*The equality holds if and only if $A$ is a multiple of a unitary operator.*

We will characterize those invertible $A \in B(\mathcal{H})$ satisfying $w_\rho(A) \le 1$ and $w_\rho(A^{-1}) \le 1$ in Section 2, and prove Theorem 1.1 in Section 3. Our proof depends on the following characterization of $A \in B(\mathcal{H})$ satisfying $w_\rho(A) \le 1$ obtained by Ando [1] for the case $\rho = 2$ and by Durszt [3] for the general case (see also [5]).

**Lemma 1.2.** *For an operator $A$ and $\rho > 1$, the condition $w_\rho(A) \le 1$ is valid if and only if there is $0 \le C \le I$ and a contraction $W$, that is, $\|W\| \le 1$, such that*

$$A = \rho(I - C)^{1/2}\{I + \rho(\rho - 2)C\}^{-1/2}WC^{1/2}.$$

*When $A$ is invertible, $0 < C < I$ and $W$ can be chosen as unitary.*

Here, as usual, the order relation $S \le T$ between two selfadjoint operators $S$, $T$ means that $T - S$ is *positive semi-definite*, or equivalently

$$\langle x, Sx\rangle \ \le\ \langle x, Tx\rangle \quad (x \in \mathcal{H}),$$

and $S < T$ means that $T - S$ is invertible in addition.

## 2. Auxiliary results

In this section, we characterize those invertible $A \in B(\mathcal{H})$ such that $w_\rho(A) \le 1$ and $w_\rho(A^{-1}) \le 1$.

We first consider the case when the Hilbert space $\mathcal{H}$ has a *finite dimension*, say $N$. Therefore each $A$ is considered as a matrix, and we can use the *determinant* function.

**Theorem 2.1.** *Suppose $\rho \ge 1$. An invertible matrix $A$ is unitary if and only if $w_\rho(A) \le 1$ and its spectrum $\sigma(A)$ is included in the unit circle.*

*Proof.* The implication ($\Rightarrow$) is clear. We consider the converse. Suppose $\rho = 1$. Then $A$ is unitarily similar to a lower triangular matrix $T$ so that each diagonal entry is an eigenvalue lying in the unit circle. Since $w_1(A) = w_1(T) = \|T\| = 1$, we see that all off diagonal entries of $T$ are zero. Next, assume that $\rho > 1$. Since $w_\rho(A) \le 1$ and $A$ is invertible, by Lemma 1.2 there is $0 < C < I$ and unitary $W$ such that

$$A = \rho(I-C)^{1/2}\{I + \rho(\rho-2)C\}^{-1/2} W C^{1/2}. \tag{2.1}$$

Then since the determinant of a matrix is the product of all its eigenvalues (counting multiplicities) and since $\det(XY) = \det(X)\det(Y)$ for any matrices $X, Y$,

$$\begin{aligned} 1 &= |\det(A)|^2 = \det(A^*A) \\ &= \det\Big(\rho^2 C^{1/2} W^*(I-C)\{I+\rho(\rho-2)C\}^{-1} W C^{1/2}\Big) \\ &= \det\Big(\rho^2 C(I-C)\{I+\rho(\rho-2)C\}^{-1}\Big) \\ &= \prod_{j=1}^{N} \frac{\rho^2\lambda_j(1-\lambda_j)}{1+\rho(\rho-2)\lambda_j}, \end{aligned}$$

where $\lambda_j$ $(j = 1, 2, \ldots, N)$ are the eigenvalues of $C$ with multiplicities counted. It is easy to see that

$$f(t) \equiv f_\rho(t) := \frac{\rho^2 t(1-t)}{1+\rho(\rho-2)t} \le 1 \quad (0 \le t \le 1)$$

and the maximum value 1 is attained only at $t = \rho^{-1}$. We conclude $\lambda_j = \rho^{-1}$ ($j = 1, 2, \ldots, N$), that is, $C = \rho^{-1} I$. Then by (2.1) we have

$$A = \frac{\rho\sqrt{1-\rho^{-1}}}{\sqrt{\rho\{1+\rho(\rho-2)\rho^{-1}\}}} W = W.$$

Therefore $A$ is unitary. □

If $w_\rho(A) \le 1$ and $w_\rho(A^{-1}) \le 1$, then by the property (iii) of the $\rho$-radii and the theorem of Sz.-Nagy [8], the spectrum $\sigma(A)$ is included in the unit circle. Therefore we can conclude from Theorem 2.1 that an invertible matrix is unitary if $w_\rho(A) \le 1$ and $w_\rho(A^{-1}) \le 1$ for any $\rho$.

Now we turn to the infinite dimensional case. The following example shows that the extension of Theorem 2.1 to the infinite dimensional case is not possible even for $\rho = 1$.

**Example 2.2.** *There is a non-unitary contraction which is similar to a unitary operator.*

*Construction.* Let $\mathcal{H} = L^2(-\infty, \infty)$ with respect to the Lebesgue measure on $(-\infty, \infty)$. Let $\varphi(t)$ is a *strictly increasing* continuous function on $(-\infty, \infty)$ such that

$$\lim_{t\to-\infty} \varphi(t) = \alpha > 0 \quad \text{and} \quad \lim_{t\to\infty} \varphi(t) = \beta < \infty.$$

Let $T$ be the *multiplication* operator by the function $\varphi(t)$ and $U$ the *right-shift* operator by unit one, that is,

$$(Tf)(t) = \varphi(t)f(t) \quad \text{and} \quad (Uf)(t) = f(t+1) \quad (-\infty < t < \infty).$$

Let $C$ be the multiplication operator by the function $\frac{\varphi(t-1)}{\varphi(t)}$. Then $T > 0$ and $U$ is unitary while $C$ is a non-unitary contraction because of the strict-increasingness of $\varphi(t)$. Now it is easy to see that $TU = UCT$, which implies that the non-unitary contraction $A = UC$ is similar to the unitary operator $U$. □

The following theorem generalizes a result of Stampfli [7, Corollary 1] (see also [2] and [6]) to general operator radii $w_\rho(\cdot)$.

**Theorem 2.3.** *Suppose $\rho \ge 1$. An invertible operator $A \in B(\mathcal{H})$ is unitary if and only if $w_\rho(A) \le 1$ and $w_\rho(A^{-1}) \le 1$.*

*Proof.* The implication ($\Rightarrow$) is clear. We consider the converse. Suppose $\rho = 1$. If $\|Ax\| < 1$ for any unit vector $x \in \mathcal{H}$, then $\|x\| = \|A^{-1}(Ax)\| < 1$, which is a contradiction.

Thus, $\|Ax\| = 1$ for all unit vector $x \in \mathcal{H}$. Since $A$ is invertible, $A$ is unitary. Next, assume $\rho > 1$. Consider again the function

$$f(t) \equiv f_\rho(t) \ := \ \frac{\rho^2(1-t)}{1+\rho(\rho-2)t} \qquad (0 \le t \le 1). \tag{2.2}$$

Then simple computations will show the following relations:

$$\frac{1}{t} = f(t) + \frac{(1-\rho t)^2}{t\{1+\rho(\rho-2)t\}} \quad \text{and} \quad t = \frac{1}{f(t)} - \frac{(1-\rho t)^2}{\rho^2(1-t)} \quad (0 < t < 1). \tag{2.3}$$

Since by assumption $w_\rho(A) \le 1$ and $w_\rho\Big((A^{-1})^*\Big) = w_\rho(A^{-1}) \le 1$, by Lemma 1.2 there are $0 < X,\ Y < I$ and unitary $U,\ V$ such that

$$A \ = \ f(X)^{1/2}UX^{1/2} \quad \text{and} \quad (A^{-1})^* = f(Y)^{1/2}VY^{1/2}. \tag{2.4}$$

Then it follows from (2.4) that

$$I = A \cdot A^{-1} = f(X)^{1/2}UX^{1/2} \cdot Y^{1/2}V^*f(Y)^{1/2},$$

which implies

$$UX^{1/2}Y^{1/2} = f(X)^{-1/2}f(Y)^{-1/2}V,$$

and hence

$$Y^{1/2}XY^{1/2} = V^*f(Y)^{-1/2}f(X)^{-1}f(Y)^{-1/2}V.$$

This means that $Y^{1/2}XY^{1/2}$ and $f(Y)^{-1/2}f(X)^{-1}f(Y)^{-1/2}$ are *unitarily similar*. Therefore they have the same spectrum

$$\sigma\Big(Y^{1/2}XY^{1/2}\Big) \ = \ \sigma\Big(f(Y)^{-1/2}f(X)^{-1}f(Y)^{-1/2}\Big),$$

which implies obviously

$$\lambda_{\max}\Big(Y^{1/2}XY^{1/2}\Big) \ = \ \lambda_{\max}\Big(f(Y)^{-1/2}f(X)^{-1}f(Y)^{-1/2}\Big) \tag{2.5}$$

and

$$\lambda_{\min}\Big(Y^{1/2}XY^{1/2}\Big) \ = \ \lambda_{\min}\Big(f(Y)^{-1/2}f(X)^{-1}f(Y)^{-1/2}\Big), \tag{2.6}$$

where, for a selfadjoint operator $Z$, the symbols $\lambda_{\max}(Z)$ (resp. $\lambda_{\min}(Z)$) denotes the *maximum* (resp. *minimum*) of the spectrum $\sigma(Z)$.

Now write, according to (2.5),

$$\gamma := \lambda_{\max}(Y^{1/2}XY^{1/2}) = \lambda_{\max}\Big(f(Y)^{-1/2}f(X)^{-1}f(Y)^{-1/2}\Big). \tag{2.7}$$

This $\gamma$ is characterized as a positive number such that $\gamma Y^{-1} - X \geq 0$ and

$$\lim_{n\to\infty} \langle a_n, (\gamma Y^{-1} - X)a_n\rangle = 0 \quad \text{for some } a_n \text{ with } \|a_n\| = 1. \tag{2.8}$$

On the other hand, it follows from (2.3) that

$$\begin{aligned}\gamma Y^{-1} - X &= \gamma f(Y) - f(X)^{-1} + \gamma Y^{-1}\{I + \rho(\rho-2)Y\}^{-1}(I-\rho Y)^2 \\ &\quad + \rho^{-2}(I-X)^{-1}(I-\rho X)^2.\end{aligned}$$

Since $0 < X,\ Y < I$, there is $\epsilon > 0$ such that

$$\gamma Y^{-1} - X \geq \gamma f(Y) - f(X)^{-1} + \epsilon(I-\rho Y)^2 + \epsilon(I-\rho X)^2. \tag{2.9}$$

Since by (2.5) $\gamma Y^{-1} - X \geq 0$ implies $\gamma f(Y) - f(X)^{-1} \geq 0$, it follows from (2.8) and (2.9) that

$$\lim_{n\to\infty} \{\|(I-\rho Y)a_n\|^2 + \|(I-\rho X)a_n\|^2\} = 0. \tag{2.10}$$

Finally we have from (2.8) and (2.10) that

$$0 = \lim_{n\to\infty} \langle a_n, (\gamma Y^{-1} - X)a_n\rangle = \rho\gamma - \rho^{-1}$$

that is,

$$\lambda_{\max}\Big(Y^{1/2}XY^{1/2}\Big) = \gamma = \rho^{-2}. \tag{2.11}$$

Incidentally we have shown that, with $\gamma = \rho^{-2}$,

$$\ker\Big(\rho^{-2}Y^{-1} - X\Big) \ \subset\ \ker(I-\rho X). \tag{2.12}$$

Next write, according to (2.6),

$$\kappa := \lambda_{\min}\Big(f(Y)^{-1/2}f(X)^{-1}f(Y)^{-1/2}\Big) = \lambda_{\min}\Big(Y^{1/2}XY^{1/2}\Big). \tag{2.13}$$

This $\kappa$ is characterized as a positive number such that $f(X)^{-1} - \kappa f(Y) \geq 0$ and

$$\lim_{n\to\infty} \langle b_n, \{f(X)^{-1} - \kappa f(Y)\}b_n\rangle = 0 \quad \text{for some } b_n \text{ with } \|b_n\| = 1. \tag{2.14}$$

Now we have by (2.3)

$$\begin{aligned}f(X)^{-1} - \kappa f(Y) &= X - \kappa Y^{-1} + \rho^{-2}(I-X)^{-1}(I-\rho X)^2 \\ &\quad + \kappa Y^{-1}\{I + \rho(\rho-2)Y\}^{-1}(I-\rho Y)^2.\end{aligned}$$

Since by (2.6) $f(X)^{-1} - \kappa f(Y) \geq 0$ implies $X - \kappa Y^{-1} \geq 0$, as in the foregoing arguments we can conclude that

$$f(X)^{-1} - \kappa f(Y) \ \geq\ X - \kappa Y^{-1} \geq 0 \tag{2.15}$$

and for some $\epsilon > 0$

$$f(X)^{-1} - \kappa f(Y) \ \geq \ \epsilon(I - \rho X)^2 + \epsilon(I - \rho Y)^2. \tag{2.16}$$

Then by (2.14) and (2.16) we have

$$\lim_{n\to\infty} \{\|b_n - \rho X b_n\|^2 + \|b_n - \rho Y b_n\|^2\} = 0,$$

and by (2.14) and (2.15)

$$\lim_{n\to\infty} \langle b_n, (X - \kappa Y^{-1}) b_n\rangle = 0.$$

From the above we can conclude that $\kappa = \rho^{-2}$, hence $\kappa = \gamma$ by (2.11). This means that $Y^{1/2}XY^{1/2} = \rho^{-2}I$, so that $\rho^{-2}Y^{-1} - X = 0$, that is, $\ker(\rho^{-2}Y^{-1} - X) = \mathcal{H}$. Finally by (2.12) this implies $\ker(I - \rho X) = \mathcal{H}$, or equivalently $X = \rho^{-1}I$. Now we can conclude by (2.2) and (2.4)

$$A = f(\rho^{-1})^{1/2}\rho^{-1/2}U = U,$$

that is, $A$ is unitary. This completes the proof. □

## 3. Proof of the main theorem

We use the fact that for any $T \in B(H)$

$$r(T) \leq w_\rho(T) \leq \|T\|,$$

where $r(T)$ is the spectral radius of $T$, and

$$w_\rho(T^k) \leq w_\rho(T)^k, \quad k = 1, 2, \ldots.$$

If $A \in B(\mathcal{H})$ is invertible, then

$$w_\rho(A^{-1}) \geq r(A^{-1}) = 1/\inf\{|\mu| : \mu \in \sigma(A)\} \geq r(A)^{-1} \geq w_\rho(A)^{-1}.$$

Replacing $A$ by $A^m$, we have $w_\rho(A^{-m}) \geq w_\rho(A^m)^{-1}$. Since $w_\rho(A^m) \leq w_\rho(A)^m$, we have

$$w_\rho(A^m)^{-1} \geq w_\rho(A)^{-m}.$$

If $\gamma A$ is unitary for some positive number $\gamma$, then $w_\rho(A^{-m}) = \gamma^{-m} = w_\rho(A)^{-m}$. Conversely, suppose $w_\rho(A^{-m}) = w_\rho(A)^{-m}$. We may replace $A$ by $\gamma A$ for a suitable positive number $\gamma$ and assume that $w_\rho(A^{-m}) = w_\rho(A)^{-m} = 1$. Thus,

$$1 = w_\rho(A^{-m}) \geq w_\rho(A^m)^{-1} \geq w_\rho(A)^{-m} = 1. \tag{3.1}$$

So, $1 = w_\rho(A^m) = w_\rho(A^{-m})$. By Theorem 2.3, $A^m$ is unitary. By (3.1), we also have $w_\rho(A) = 1$. If $\rho = 1$, then for any unit vector $x \in \mathcal{H}$, $\|A\| = 1 = \|A^m x\|$ implies that

$x, Ax, \ldots, A^m x$ are all unit vectors. Thus, $1 = \|Ax\|$ for all $x$. Since $A$ is invertible, $A$ is unitary. Suppose $\rho > 1$. By Lemma 1.2,

$$A = \rho(I - C)^{1/2}\{I + \rho(\rho - 2)C\}^{-1/2} W C^{1/2}$$

for some $0 < C < I$ and a unitary $W$.

Let

$$\tilde{C} = \rho C^{1/2}(I - C)^{1/2}\{I + \rho(\rho - 2)C\}^{-1/2}.$$

As shown in the proof of Theorem 2.1, we have

$$f(t) \equiv f_\rho(t) := \frac{\rho^2 t(1-t)}{1 + \rho(\rho - 2)t} \leq 1 \quad (0 \leq t \leq 1).$$

Thus $\tilde{C}^2 = f(C) \leq I$, and $\tilde{C}$ is a contraction. As a result,

$$A^m = \rho(I - C)^{1/2}\{I + \rho(\rho - 2)C\}^{-1/2} \widetilde{W} C^{1/2}$$

such that

$$\widetilde{W} = W(\tilde{C}W)^{m-1}$$

is a contraction. Since $A^m$ is unitary, we see that

$$\begin{aligned} I &= (A^m)(A^m)^* \\ &= \rho^2 (I - C)^{1/2}\{I + \rho(\rho - 2)C\}^{-1/2} \widetilde{W} C^{1/2} C^{1/2} \widetilde{W}^* \{I + \rho(\rho - 2)C\}^{-1/2}(I - C)^{1/2}. \end{aligned}$$

Thus,

$$\{I + \rho(\rho - 2)C\}(I - C)^{-1} = \rho^2 \widetilde{W} C \widetilde{W}^*. \tag{3.2}$$

When both $\widetilde{W}$ and $C$ are invertible, we know

$$\sigma\Big(\widetilde{W} C \widetilde{W}^*\Big) = \sigma\Big(C^{1/2} \widetilde{W}^* \widetilde{W} C^{1/2}\Big). \tag{3.3}$$

Since in general

$$C^{1/2} \widetilde{W}^* \widetilde{W} C^{1/2} \leq C \quad \text{for } \|\widetilde{W}\| \leq 1,$$

we have

$$\lambda_{\max}(\widetilde{W} C \widetilde{W}^*) \leq \lambda_{\max}(C) \quad \text{and} \quad \lambda_{\min}(\widetilde{W} C \widetilde{W}^*) \leq \lambda_{\min}(C). \tag{3.4}$$

Since the function

$$g(t) := \frac{1 + \rho(\rho - 2)t}{1 - t}$$

is increasing for $0 \leq t < 1$, we have

$$\lambda_{\max}\Big(g(C)\Big) = g\Big(\lambda_{\max}(C)\Big) \quad \text{and} \quad \lambda_{\min}\Big(g(C)\Big) = g\Big(\lambda_{\min}(C)\Big). \tag{3.5}$$

Then it follows from (3.2), (3.4) and (3.5)

$$g(t) \leq \rho^2 t \quad \text{for } t = \lambda_{\max}(C),\ \lambda_{\min}(C). \tag{3.6}$$

Since

$$g(t) - \rho^2 t = \frac{(1-\rho t)^2}{1-t} \geq 0 \quad (0 \leq t < 1),$$

(3.6) is possible only when

$$\lambda_{\max}(C) = \lambda_{\min}(C) = \frac{1}{\rho},$$

and hence $C = \frac{1}{\rho} I$. Consequently

$$A = \rho(I-C)^{1/2}\{I + \rho(\rho-2)C\}^{-1/2} W C^{1/2} = W$$

is unitary as asserted. □

## References


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