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Linear Preserver Problems: A Brief Introduction and Some Special Techniques

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ABSTRACT

Linear preserver problems concern the characterization of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant. The earliest papers on linear preserver problems date back to 1897, and a great deal of effort has been devoted to the study of this type of question since then. We present a brief picture of the subject, aiming at giving a gentle introduction to the reader. Then we describe some techniques used in our recent papers on this type of problem.

1. INTRODUCTION

One of the most active and fertile subjects in matrix theory during the past one hundred years is the *linear preserver problem* (LPP), which concerns the characterization of linear operators on matrix spaces that leave certain func-

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tions, subsets, relations, etc., invariant. The earliest papers such as [17] and [24] on LPPs date back to 1897. Since then, a great deal of effort has been devoted to the study of this type of question (for example, see the excellent surveys [39, 40, 18] by Marcus and Grone). In this paper, we present a brief picture of the subject and describe several techniques used in our recent papers on this type of problem. As the LPP is a vast topic, we make no attempt at a detailed survey of it in this short paper. In fact, coordinated by Steve Pierce, a group of people (including the authors) are working on a monograph on LPPs.

In Section 2 we describe some general types of LPPs. Then we discuss some motivations for the study in Section 3. A brief list of some active research topics is given in Section 4. Section 5 is a discussion on the diversification and unification of the LPP. Finally, some special techniques are described in Section 6. The content of these sections are based on our experience with the subject. Other authors may have different emphasis or opinions.

In the following we shall always assume ϕ to be a linear operator on the matrix space $\mathcal{M}$, which may be any one of the following:

$\mathbb{F}^{m\times n}$: the set of all $m \times n$ matrices over the field $\mathbb{F}$, where $\mathbb{F}$ is usually $\mathbb{R}$ or $\mathbb{C}$,

$S_n(\mathbb{F})$: the set of all $n \times n$ symmetric matrices over $\mathbb{F}$,

$K_n(\mathbb{F})$: The set of all $n \times n$ skew-symmetric matrices over $\mathbb{F}$,

H_n: the set of all hermitian matrices.

2. GENERAL TYPES OF LINEAR PRESERVER PROBLEMS

In this section we shall describe four general types of LPPs and give some examples. Once again, we stress that no detailed survey of the results will be given here. The objective is just to give a brief overview.

The first type of general question is concerned with the study of those *linear operators preserving certain functions.*

PROBLEM I. Let F be a (scalar-valued, vector-valued, or set-valued) function on $\mathcal{M}$. Characterize those linear operators ϕ on $\mathcal{M}$ that satisfy

$$F(\phi(A)) = F(A) \qquad \text{for all} \quad A \in \mathcal{M}.$$

Probably the first problem of this kind was considered by Frobenius [17], who proved that if $\mathcal{M} = \mathbb{C}^{n\times n}$ and $F(A) = \det A$, then ϕ must be of the form

$$\phi(A) = MAN \qquad \text{for all} \quad A \in \mathcal{M} \tag{1}$$

or

$$\phi(A) = MA^tN \qquad \text{for all} \quad A \in \mathcal{M} \tag{2}$$

for some nonsingular matrices $M, N \in \mathbb{C}^{n\times n}$ with $\det MN = 1$. In fact, for the function $F(A) = \det A$, he also considered the case when (i) $\mathcal{M}$ is the space of real symmetric (or odd order skew-symmetric) matrices, and (ii) $\mathcal{M} = \{A \in \mathbb{C}^{n\times n} : \operatorname{tr} A = 0\}$. It turns out that for both problems ϕ is also of the form described in (1) or (2) with some additional assumptions on the matrices M and N: in problem (i) $N = \mu M^t$ for some constant μ so that $\det MN = 1$; in problem (ii) $N = \mu M^{-1}$ for some constant μ so that $\det MN = 1$. As can be seen in other examples in the sequel and the discussion in Section 4, it is very common for people to extend or consider the same LPP on different matrix spaces after certain initial results are obtained. Although the results on different matrix spaces look very similar, the techniques involved for the proof or the degree of difficulty of the problem may be very different. It is worth mentioning that many linear preservers (in other LPPs) have the "usual form" described in (1) or (2) with different conditions on the matrices M and N. In some particular situations, it is interesting to find linear preservers which are not of these usual forms.

A variation of Problem I is to consider ϕ on $\mathcal{M}$ satisfying

$$G(\phi(A)) = F(A) \qquad \text{for all} \quad A \in \mathcal{M}$$

for different functions G and F. Of course, for this question one has to answer the existence question first. For example, it is shown in [41] that there does not exist a linear operator on $\mathcal{M} = \mathbb{R}^{n\times n}$ that satisfies per $\phi(A) = \det A$ for all $A \in \mathcal{M}$, where per stands for the permanent.

A second type of general problem concerns those *linear operators preserving certain subsets.*

PROBLEM II. Let $\mathcal{S} \subset \mathcal{M}$. Characterize those linear operators ϕ on $\mathcal{M}$ that satisfy

$$\phi(\mathcal{S}) \subset \mathcal{S} \quad \text{or} \quad \phi(\mathcal{S}) = \mathcal{S}.$$

Let $\mathcal{M} = \mathbb{C}^{n\times n}$, and $\mathcal{S}$ be the unitary group $\mathcal{U}_n$. In [38] Marcus showed that a linear operator ϕ on $\mathcal{M}$ satisfies $\phi(\mathcal{S}) \subset \mathcal{S}$ if and only if (1) or (2) holds with $M, N \in \mathcal{U}_n$. This result was extended to rectangular matrices by Grone [18]. It was shown that a linear operator ϕ on $\mathcal{M} = \mathbb{C}^{m\times n}$ with $m \leqslant n$ satisfies $\phi(\mathcal{S}) \subset \mathcal{S}$, where $\mathcal{S}$ is the set of all matrices $A \in \mathcal{M}$ satisfying

$AA^* = I_m$, if and only if (1) or (2) holds with $M \in \mathcal{U}_m$ and $N \in \mathcal{U}_n$. Let $\mathbb{R}^m_\downarrow$ denote the collection of all vectors in $\mathbb{R}^m$ with nonnegative entries arranged in descending order. For $c = (c_1, \ldots, c_m) \in \mathbb{R}^m_\downarrow$, let $\mathcal{S}(c)$ denote the set of matrices in $\mathbb{C}^{m\times n}$ (assume w.l.o.g. that $m \leqslant n$) with singular values $c_1, \ldots, c_m$. Then the set $\mathcal{S}$ considered by Marcus and Grone can be regarded as $\mathcal{S}(1, \ldots, 1)$. It is natural to consider the structure of those linear operators satisfying $\phi(\mathcal{S}(c)) \subset \mathcal{S}(c)$ for a certain fixed nonzero $c \in \mathbb{R}^m_\downarrow$. Very recently, the authors [35] extended the result to any nonzero $c \in \mathbb{R}^m_\downarrow$. Again, after obtaining the results for complex matrices, people try to work on matrices over other fields. In fact, the same problem over real matrices has also been considered (see [35] and its references).

A variation of Problem II is to consider ϕ on $\mathcal{M}$ satisfying $\phi(\mathcal{S}_1) \subset \mathcal{S}_2$ or $\phi(\mathcal{S}_1) = \mathcal{S}_2$ for different subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathcal{M}$. For example, in [30, 34, 35], we have considered those linear operators ϕ on various matrix spaces that map $\mathcal{S}(c)$ into or onto $\mathcal{S}(d)$ for fixed vectors c and d in $\mathbb{R}^m_\downarrow$.

Another type of general question is the study of those *linear operators preserving certain relations.*

PROBLEM III. Let $\sim$ be a relation or an equivalence relation on $\mathcal{M}$. Characterize those linear operators ϕ on $\mathcal{M}$ that satisfy

$$\phi(A) \sim \phi(B) \qquad \text{whenever} \quad A \sim B$$

or

$$\phi(A) \sim \phi(B) \quad \text{if and only if} \quad A \sim B.$$

Let $\mathcal{M} = \mathbb{F}^{n\times n}$, where $\mathbb{F}$ is any field, and let $\sim$ be defined by $A \sim B$ if $AB = BA$. It was shown (e.g. see [44]) that for $n \geqslant 3$, a nonsingular linear operator ϕ on $\mathcal{M}$ satisfies $\phi(A)\phi(B) = \phi(B)\phi(A)$ whenever $AB = BA$ if and only if

$$\phi(A) = \alpha X^{-1} A X + f(A) I \qquad \text{for all} \quad A \in \mathcal{M} \tag{3}$$

or

$$\phi(A) = \alpha X^{-1} A^t X + f(A) I \qquad \text{for all} \quad A \in \mathcal{M}, \tag{4}$$

for some nonsingular matrix $X \in \mathcal{M}$, $\alpha \in \mathbb{R}$, and linear functional f on $\mathcal{M}$. As with most other LPPs, people have also considered the same problem in other matrix spaces such as $S_n(\mathbb{R})$ or H_n (see [10]). It was shown that ϕ has the same

structure except that X in (3) and (4) has to be real orthogonal or complex unitary according as $\mathcal{M} = S_n(\mathbb{R})$ or $\mathcal{M} = H_n$. In all cases, it has been shown that if $n = 2$, then there are counterexamples of nonsingular linear preservers of commutativity, which are not of the form (3) or (4). In [14], linear preservers of commutativity without the nonsingularity assumption were studied. In many LPPs, after a result on linear preservers with an *a priori* nonsingularity assumption is obtained, removing the nonsingularity assumption poses a new LPP. It is often more difficult to characterize singular linear preservers or to prove that they cannot exist.

Notice that Problems I, II, and III are related in certain ways. First, given a function F, one could define a subset $\mathcal{S}$ as $F^{-1}(\mathcal{T})$ for some subset $\mathcal{T}$ in the range of F and study the linear operators preserving $\mathcal{S}$. For example, if $F(A) = \det A$, then $\mathcal{S} = F^{-1}(0)$ is the set of all singular matrices and $\mathcal{S} = F^{-1}(\mathbb{F} \setminus \{0\})$ is the set of all nonsingular matrices. In fact, in both cases the corresponding set preserver problems have been studied (see, e.g., [15] and [2]). On the other hand, if a set $\mathcal{S}$ is given, one may consider a suitable function F so that $\mathcal{S}$ can be regarded as the inverse image $F^{-1}(\mathcal{T})$ of a certain subset $\mathcal{T}$. In this way linear preservers of F (i.e., Problem I type preservers) can be related to linear preservers of $\mathcal{S}$ (i.e., Problem II type preservers). Also, given a partition of $\mathcal{M}$ into subsets $\mathcal{S}$, one could define an equivalence relation based on the disjoint subsets $\mathcal{S}$. This relates Problem II with Problem III.

After our talk at the Conference was presented, G.H. Chan pointed out that the study of those *linear operators commuting with certain transformations on* $\mathcal{M}$ is also commonly regarded as a LPP:

PROBLEM IV. Given a transformation $F: \mathcal{M} \to \mathcal{M}$, characterize those linear operators ϕ on $\mathcal{M}$ that satisfy

$$F(\phi(A)) = \phi(F(A)) \qquad \text{for all} \quad A \in \mathcal{M}. \tag{5}$$

Notice that if ϕ is nonsingular, then (5) can be rewritten as

$$\phi^{-1} \circ F \circ \phi = F,$$

which means that the transformation F is *preserved by a linear change of basis* ϕ. The condition (5) might be considered as a generalization of this concept. It turns out that the techniques used in solving this kind of problem and the results obtained are very similar to those of the previous three problem types.

Let $\mathcal{M} = \mathbb{C}^{n\times n}$ and $F(A) = \text{adj } A$, the adjoint of the matrix A. In [47] Sinkhorn proved that a linear operator ϕ commutes with F if and only if

(i) ϕ is of the form (1) or (2) with $MN = \mu I$ where $\mu^{n-2} = 1$; or

(ii) $n = 2$ and $\phi(A)$ is a linear combination of maps of the form $PA(F(P))$ and $QA^t(F(Q))$ for some $P, Q \in \mathcal{M}$.

This result is extended in [13] to $\mathbb{F}^{n\times n}$, $S_n(\mathbb{F})$, and $K_n(\mathbb{F})$ for arbitrary infinite fields $\mathbb{F}$. In the same paper, characterizations are also given for linear maps ϕ on $\mathcal{M} = \mathbb{F}^{n\times n}$ or $S_n(\mathbb{F})$ with $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ that satisfy $\phi(e^A) = e^{\phi(A)}$. It is shown that such ϕ must be of the form (1) or (2) with $MN = I$ if $\mathcal{M} = \mathbb{F}^{n\times n}$, and $M = N^t = N^{-1}$ if $\mathcal{M} = S_n(\mathbb{F})$. Very recently, Chan and Lim [12] have considered this type of problem when $\mathcal{M} = \mathbb{F}^{n\times n}$ and $F(A) = A^k$ for some positive integer $k > 1$, and they have shown that a linear operator ϕ on $\mathcal{M}$ commutes with F if and only if it is of the form (1) or (2) with $MN = \mu I$ where $\mu^{k-1} = 1$.

3. SOME MOTIVATIONS

Many subjects in matrix analysis can be broadly classified into categories such as functions on matrices, subsets of matrices, relations on matrices, and transformations (not necessarily linear) on matrix spaces, etc. On the other hand, since matrix spaces are linear spaces, the linear transformation is the most natural among all transformations on them. These two observations together give rise naturally to the LPP: as soon as the word "preserve" is defined (and, in most cases, the definition of "preserve" is clear from the context), one may ask what kind of linear operators would preserve a certain function, a certain subset, a certain relation, or a certain transformation on a matrix space. This suggests that the LPP is of fundamental theoretical interest in matrix theory.

Apart from this, there are also other motivations for the study of the LPPs. First, linear preserver problems arise naturally when one considers the converse problems of some basic results in matrix theory. For example, suppose a linear operator ϕ on $\mathbb{C}^{n\times n}$ is defined by $\phi(A) = MAN$ or $\phi(A) = MA^tN$ for some nonsingular matrices $M, N \in \mathbb{C}^{n\times n}$. Then it is clear that ϕ will preserve the rank of a matrix. It is somewhat surprising that such ϕ's are the only linear operators on $\mathbb{C}^{n\times n}$ that preserve rank (see, e.g., [42]). For another example, consider a linear operator ϕ on $\mathbb{C}^{n\times n}$ defined by $\phi(A) = U^*AU$ or $\phi(A) = U^*A^tU$ for some unitary matrix U. Then clearly ϕ preserves eigenvalues, the determinant, the spectral norm, the unitary group, hermitian matrices, normal matrices, the numerical range, inertia, etc. The interesting question is whether any one of these properties is strong enough to force ϕ to be of the form $\phi(A) = U^*AU$ or $\phi(A) = U^*A^tU$ for some unitary matrix U. To answer this question one clearly has to study various LPPs.

In some cases, the solution of the LPP would suggest a practical tool for simplifying some other mathematical problems. Historically, people were once interested in knowing whether there exists a linear operator ϕ on $\mathbb{R}^{n\times n}$ that satisfies per $\phi(A) = \det A$ [45]. This interest may have stemmed from the observation that the computation of the permanent is in general more difficult than that of the determinant when the order of the matrix is high. If there were to exist such a linear ϕ that satisfies per $\phi A = \det A$, then the computation of the permanent would be made much easier via the linear transformation ϕ. However, it was shown later that such a linear operator cannot exist [41]. Another example concerns the problem of solving a system of differential equations. To simplify the problem, people would like to apply certain transformations to the system before solving it. The transformation should be simple and have some nice properties. For example, one might want to use a linear transformation on a linear differential system and hope that it preserves the eigenmodes or the stability of the system. This naturally gives rise to a linear preserver problem.

Sometimes the aim of studying a LPP on a certain topic is to better understand the subject under consideration. In many situations (such as the study of systems theory or the theory of canonical forms of matrices) people have to consider certain group actions on a matrix space. In such cases, the focus will be on the orbits of matrices under the specific group action rather than on a single matrix. One basic question would then be how one could differentiate different orbits. A natural way to study this problem is to consider functions between two given orbits. For example, one may ask whether there is a function from the matrix space to itself that maps one orbit onto or into another orbit. In particular, if one further restricts the functions to be linear, then the question becomes a LPP. Such examples will be further discussed in the final section when we describe some special techniques.

Some LPPs may also appear as particular cases of some more general questions. For example, in studying Banach spaces one would like to know the structure of the linear isometries on them. If the Banach space under consideration is a matrix space, then the question can be regarded as a LPP. This shows that a mathematical problem in a more general setting can sometimes motivate the study of a particular LPP.

4. SOME ACTIVE TOPICS

As already indicated in the beginning of Section 3, any function on matrices, subset of matrices, relation on matrices, or transformation of matrix spaces will induce a LPP, once the term "preserve" is defined. Also, by varying the underlying matrix space (say, by considering the different cases of

$\mathcal{M} = \mathbb{C}^{m\times n}$, $\mathbb{R}^{m\times n}$, H_n, etc.), one can generate several different LPPs. Moreover, since LPPs can be considered in various contexts, the source of problems is even richer. This may explain the large volume of research literature and the intense research efforts on this subject. The following is a brief list of some active research topics.

A. *Rank Preservers*

A linear operator ϕ on $\mathcal{M}$ is a *rank k preserver* for a positive integer k if ϕ maps the set of rank k matrices into itself. It is worth mentioning that the proofs of many LPPs depend on the structure of rank one preservers. The problem for rank k preservers is completely solved when $\mathcal{M} = \mathbb{C}^{m\times n}$ [3], or when $k = 1$, $\mathcal{M} = \mathbb{F}^{m\times n}$, and $\mathbb{F}$ is an algebraically closed field of characteristic 0 [42]. In both cases, a rank k preserver must have the form (1) or (2) for some nonsingular matrices M and N. A related problem is the study of *rank k nonincreasing* maps, i.e., those linear operators on $\mathcal{M}$ that map rank k matrices to matrices of rank less than or equal to k, for a positive integer k (see, e.g., [2]). These problems can also be extended from matrix spaces to tensor spaces (see, e.g., [11, 16, 42]).

B. *Inertia Preservers*

Let $\mathcal{M} = H_n$ or $S_n(\mathbb{R})$. We say that a matrix $A \in \mathcal{M}$ has inertia (r, s, t) if A has r positive eigenvalues, s negative eigenvalues, and t zero eigenvalues. Denote by $G(r, s, t)$ the set of matrices in $\mathcal{M}$ with inertia (r, s, t). A linear operator ϕ on $\mathcal{M}$ is a $G(r, s, t)$ preserver if ϕ maps $G(r, s, t)$ into itself. The problem of characterizing the $G(n, 0, 0)$ preservers is open and generally considered to be difficult. The following conjecture is due to Johnson and Pierce [23]: For $n \geqslant 3$, $r > 0$, $s > 0$, ϕ is a $G(r, s, t)$ preserver if and only if (1) or (2) holds with $M = \mu N^*$, where $\mu = 1$ if $r \neq s$ and $\mu = \pm 1$ if $r = 2$. This conjecture is true if one assumes that ϕ is nonsingular. Without this additional nonsingularity assumption, it was confirmed by Loewy in [36] only for the cases when $r \neq s$. For other related results on this topic, we refer to the report of Loewy [37].

C. *Algebraic Set Preservers*

A subset $\mathcal{S}$ of $\mathcal{M}$ is an *algebraic set* if it is the set of common zeros of a finite collection of polynomials in the entries of matrices. For example, the set of all matrices with zero determinant is an algebraic set. A linear operator ϕ on $\mathcal{M}$ preserving $\mathcal{S}$ will then be an algebraic set preserver. For square matrices, the set $\mathcal{S}$ may be a multiplicative group. In this case ϕ is an algebraic group preserver. It is also interesting to determine the structure of

those linear operators ϕ on $\mathcal{M}$ that preserve a fixed polynomial function on the entries of matrices. For all these problems, it is natural to use the algebraic geometry and algebra techniques. It turns out that the results of many problems of this type are that ϕ must be of the form (1) or (2) with certain additional conditions on M and N. We refer to the report [43] of Pierce for further details on this topic.

D. *Functions of Singular Values*

A function F on $\mathcal{M} = \mathbb{C}^{m\times n}$ or $\mathbb{R}^{m\times n}$ is said to be *unitarily invariant* if its function value depends only on the singular values of its argument, i.e., if $F(A) = F(B)$ whenever A and B have the same singular values. Of particular interest is the study of unitarily invariant norms on $\mathcal{M}$. There has been a great deal of interest in determining the structure of those linear operators preserving certain functions on singular values of matrices. Clearly, if F is a unitarily invariant function and if ϕ is of the form (1) or (2) for some unitary matrices M and N, then ϕ preserves F. It is interesting to note that the converse also holds for many unitarily invariant functions F. For a brief survey and a unifying result, we refer the reader to [32]. An interesting problem in this area is to determine the conditions on a unitarily invariant function F such that ϕ preserves F if and only if ϕ preserves the singular values of matrices. In this case, ϕ must preserve $\mathcal{S}(c)$ for all $c \in \mathbb{R}^m_\downarrow$ and hence will be of the form (1) or (2) for some unitary matrices M and N (see the discussion after Section 2, Problem II, and the results in [30] and [35]).

E. NUMERICAL RANGE AND NUMERICAL RADIUS PRESERVERS

Suppose $\mathcal{M} = \mathbb{C}^{n\times n}$ or H_n. Let

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n,\ x^*x = 1\}$$

be the *numerical range* (or *field of values*) of A, which can be regarded as a set valued function on matrices. Associated with the numerical range is the numerical radius of A defined by

$$r(A) = \max\{|z| : z \in W(A)\},$$

which is a scalar function on matrices. It was shown that a linear operator ϕ on $\mathcal{M}$ preserves the numerical range of matrices if and only if ϕ is of the form (1) or (2) with $M = N^*$ and $MN = I$, and a linear operator ϕ preserves the

numerical radius of matrices if and only if ϕ is a unit scalar multiple of a linear operator that preserve the numerical range [26]. There are many generalizations of the numerical range and the numerical radius. The corresponding LPPs are of wide interest. One may see [28] and its references for more details. While many results on generalized numerical range preservers are known, it would be interesting to prove or disprove that the corresponding numerical radius preservers are precisely the unit multiples of them.

F. *Linear Preservers of Relations*

In Section 2 we have mentioned the problem concerning preservers of commutativity. Some authors have considered LPPs on other relations which are equivalence relations and are related to canonical forms of matrices. For example, Hiai [21] gives a complete characterization for the linear preservers of similarity on $\mathbb{C}^{n\times n}$. A unified treatment for many similar problems was given in [22]. It is observed that many LPPs on equivalence relations are connected to certain group actions on matrices, and differential geometry techniques are useful in solving them. A more detailed discussion of this is in Section 5.

G. *Matrices over Rings and Boolean Algebras*

Although the LPPs discussed so far are mostly on matrices over fields, there is no reason why the same problem cannot be transported to matrices over rings or boolean algebras. In fact, many interesting questions and techniques have been brought into the subject when LPPs over such algebraic structures have been considered. There is a fair amount of literature and many current developments on these problems. One may see [4, 5, 49] and their references.

5. DIVERSIFICATION AND UNIFICATION

The list of topics in the previous section, though brief, shows clearly that the LPP has been divided into many different research areas, each with a number of variations of problems. This is to be expected, because one can easily generate new LPPs by considering different functions, subsets, relations, transformations, etc., or by changing the underlying matrix spaces, or by modifying certain conditions in the existing results. Moreover, one can generalize a LPP to operator algebras or tensor spaces instead of staying within matrix spaces.

Certainly, different LPPs require different tools or techniques in solving them. For example, the solution of the LPP on rank k preservers on $\mathbb{C}^{m\times n}$ depends on the structures of matrix subspaces containing only rank k or zero matrices [3], while the proof of Hiai's result on similarity preservers uses a tangent space argument [21]. Hence the diversification of an LPP leads to a variety of different tools and techniques. Yet this same diversification also points out some possibilities of the unification of the various LPPs. Take the LPP on commutativity for example. The problem has been considered on different matrix spaces [1, 10, 14, 44, 46, 48]. Only by knowing the proofs of the various results (on different matrix spaces) can one identify the ingredients of a proof of a general commutativity preserver problem.

There are several approaches to unifying different LPPs. The first is to find a general technique or method to treat different problems. A good illustration of this idea is the paper [44] by Pierce and Watkins, in which they use projective geometry to solve two unrelated LPPs, namely, the characterizations of commutativity preservers and k-numerical range preservers on $\mathbb{C}^{n\times n}$. Another example is the several papers [7, 8] by Botta, who uses results in algebraic geometry to prove (or re-prove) many linear preserver results. Recently, we have used a duality technique together with some results from differential geometry (see the next section) to solve several linear isometry problems [29–31, 33]. In fact, various tools such as operator theory, combinatorics, graph theory, abstract algebra, and multilinear algebra, have been used to handle different LPPs. As pointed out by C. R. Johnson, there even exist some proofs of LPPs that do not depend on any other known linear preserver results; the techniques used are different from those used in all other problems (see, e.g., [6, 20]). However, despite the fact that there are many different approaches, the results of most LPPs look very similar. One may wonder whether there is a general principle behind all the proofs, and whether there is a general method to treat all or most of the LPPs.

Another way to unify different LPPs is to find a general formulation for them. Having a general formulation for several problems might lead to some uniform technique or strategy to solve them. For example, when studying algebraic set preservers it is natural to use algebraic geometry (see, e.g., [9]); to study linear isometries it is natural to study the unit balls with respect to the norms (see, e.g., [19, 30]); to study linear operators preserving certain equivalence relations it is natural to study the geometrical properties of those equivalence classes (see, e.g., [22, 27]), etc.

Besides using general techniques or general formulations, there are other ways to relate different LPPs. For example, as we proposed in [29], one nice way to relate different LPPs is to study the dual transformation of a linear preserver. Sometimes, the dual transformation gives rise to another type of linear preserver. Thus one may increase the variety of tools that can be used to

tackle the problem, and one may double the number of results obtained. More details of this idea will be discussed in the next section.

6. SOME SPECIAL TECHNIQUES

A. *Duality Techniques*

One principle that we used in our recent papers on LPPs is the *duality technique*. The idea is simply to study the dual transformation ϕ^* of ϕ as well as ϕ itself. The idea of using the dual transformation to study the linear preserver has also been used by other authors (see, e.g., [25]). We found that this principle is especially useful when dealing with linear isometry problems. Notice that a linear operator ϕ preserves a certain norm on $\mathcal{M}$ if and only its dual transformation ϕ^* preserves the dual norm. While the norm or the unit norm ball under consideration may be complicated, the dual norm or the unit dual norm ball may have simpler structures. So it might be easier to characterize the dual transformation and then determine the structure of ϕ. Furthermore, after solving a LPP, one might get several additional results because of the duality relations. To illustrate this we give the following result.

PROPOSITION 1. *Let ϕ be a linear operator on $\mathbb{C}^{n\times n}$. The following conditions are equivalent*:

(a) *ϕ preserves the spectral norm.*
(b) *ϕ maps the set of unitary matrices onto itself.*
(c) *ϕ^* preserves the trace norm (i.e., the Ky Fan n-norm).*
(d) *ϕ^* maps the set of matrices with singular values $1, 0, \ldots, 0$, onto itself.*
(e) *ϕ is of the form*

$$A \to UAV \quad or \quad A \to UA^tV$$

for some unitary matrices U and V.

(f) *ϕ^* is of the form*

$$A \to UAV \quad or \quad A \to UA^tV$$

for some unitary matrices U and V.

The equivalence of the first four conditions depends on three facts: unitary matrices are the extreme points of the unit ball with respect to the spectral norm; the trace norm is the dual norm of the spectral norm; and the matrices

with singular values $1, 0, \ldots, 0$, are the extreme points of the unit ball with respect to the trace norm. The equivalence of the last two conditions can be verified readily. The equivalence of conditions (e) and (b) can be easily deduced from the result by Marcus in [38] (see our discussion on Problem II in Section 2).

In the above example, with the given structure (e) of those linear operators that satisfy condition (b), the other three types of linear preservers satisfying (a), (c), or (d) are also characterized because of the duality relations. Moreover, since conditions (a) to (d) are equivalent, we have flexibility in choosing any one of the conditions to work with in order to get the characterization of ϕ.

Some other remarks are in order. First, sometimes one may concentrate on either ϕ or ϕ^*, but sometimes one may need to consider both ϕ and ϕ^* simultaneously throughout the proof of certain results (see, e.g., [30]). Secondly, if we are considering a linear isometry ϕ, then naturally it preserves some bounded sets such as the unit ball, or the set of extreme points of the unit ball. On the other hand, if we are considering a set preserver problem, especially when the set is compact, we may create a norm by generating a norm ball form the set in a certain way. Then we may use the technique of treating linear isometries to solve the problem. For example (see [30]), consider $c \in \mathbb{R}^m_\downarrow$ and $\mathscr{S}(c)$ as defined in Section 2. Then the convex hull of $\mathscr{S}(c)$ can be regarded as the unit ball of a certain norm on $\mathbb{C}^{m\times n}$, and the corresponding dual norm is just the c-spectral norm $F_c(A)$ of A defined by $F_c(A) = c^t\sigma(A)$, where $\sigma(A) \in \mathbb{R}^m_\downarrow$ is the vector of singular values of A. Thus a linear operator ϕ satisfies $\phi(\mathscr{S}(c)) = \mathscr{S}(c)$ if and only if ϕ^* preserves the c-spectral norm, and we could use some standard techniques of studying isometries in order to study ϕ. For another example (see [29]), let $c \in \mathbb{R}^n$, and $\mathscr{U}(c)$ denote the set of all $n \times n$ hermitian matrices whose vector of eigenvalues (arranged in a certain order) equals c. Then a linear operator ϕ on $\mathbb{C}^{n\times n}$ preserves $\mathscr{U}(c)$ if and only if ϕ^* preserves the *c-numerical range* of A, which is defined as

$$W_c(A) = \{\operatorname{tr}(\operatorname{diag}(c_1, \ldots, c_n)\, UAU^*) : U^*U = I_n\}$$

for all $A \in \mathbb{C}^{n\times n}$. Moreover, ϕ preserves the set $\mathscr{V}(c) = \bigcup_{|\mu|=1} \mu\mathscr{U}(c)$ if and only if ϕ^* preserves the *c-numerical radius* of A, which is defined as

$$r_c(A) = \max\{|z| : z \in W_c(A)\}$$

for all $A \in \mathbb{C}^{n\times n}$. Thus all these linear preservers are related, and any information on one problem would be useful for the other problems.

B. Differential Geometry and Other Techniques

Suppose $\sim$ is an equivalence relation on $\mathcal{M}$. For any $A \in \mathcal{M}$, let $[A]$ denote the equivalence class of A under $\sim$. Two major types of LPPs concerning the equivalence relation $\sim$ are the following:

(i) Characterize those linear operators ϕ on $\mathcal{M}$ that satisfy $\phi(A) \sim \phi(B)$ whenever $A \sim B$, or, equivalently,

$$\phi([A]) \subset [\phi(A)] \qquad \text{for all} \quad A \in \mathcal{M}. \tag{6}$$

(ii) Given fixed $A_0, B_0 \in \mathcal{M}$, characterize those linear operators ϕ on $\mathcal{M}$ that satisfy $\phi(X) \sim B_0$ whenever $X \sim A_0$, or, equivalently,

$$\phi([A_0]) \subset [B_0]. \tag{7}$$

Notice that problem (i) is Problem III described in Section 2, and problem (ii) is a particular case of Problem II. Many variations of the problems are possible. For examples, one may restrict ϕ to be nonsingular in the above, or replace the inclusions in (6) or (7) by equalities. Or, one may replace $[A_0]$ and $[B_0]$ in (7) by subsets which are unions of equivalence classes.

In view of (6) and (7), one sees that the LPPs at hand are concerned with linear imbeddings of one equivalence class into another. Thus the linear algebraic and the geometric structure of the equivalence class are of great importance. In many cases the equivalence classes are differentiable manifolds, and thus their dimensions or their tangent spaces may be considered to help solve the LPPs. In fact, the equivalence classes are usually orbits of matrices under a certain group action of an algebraic Lie group. For example, let $\sim_{\text{rank}}$ be the equivalence relation on $\mathbb{C}^{m\times n}$ defined by $A \sim_{\text{rank}} B$ if and only if rank A = rank B, or equivalently, $B = MAN$ for some nonsingular $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$. Then (see Section 4 of [22]) for any $A \in \mathbb{C}^{m\times n}$, the equivalence class $[A]$ is the orbit of A under the group action of

$$A \to MAN$$

with nonsingular $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$, and the orbit is a differentiable manifold of real dimension $2k(m + n - k)$, where k = rank A, and the tangent space to $[A]$ at A is the complex subspace

$$\{XA + AY : X \in \mathbb{C}^{m\times m}, Y \in \mathbb{C}^{n\times n}\}.$$

For the sake of brevity, let us consider problem (i) with the restriction that ϕ is nonsingular. The following result, the proof of which is straightforward, holds for all equivalence relations $\sim$ for which the equivalence classes are differentiable manifolds.

PROPOSITION 2. *Suppose ϕ is a linear operator on $\mathscr{M}$ that satisfies $\phi([A]) \subset [\phi(A)]$ for all $A \in \mathscr{M}$. Let $\mathscr{T}_A$ denote the tangent space to $[A]$ at A. Then*

(a) $\phi(\mathscr{T}_A) \subset \mathscr{T}_{\phi(A)}$, *and*
(b) *if ϕ is nonsingular then* $\dim[A] = \dim \mathscr{T}_A \leqslant \dim \mathscr{T}_{\phi(A)} = \dim[\phi(A)]$.

It turns out that the result in Proposition 2, though simple, is very useful in solving some LPPs (see, e.g., [22, 27]). As a demonstration, we apply it to prove the following.

PROPOSITION 3. *Let ϕ be a nonsingular linear operator on $\mathbb{C}^{m\times n}$ that satisfies* $\operatorname{rank} \phi(A) = \operatorname{rank} \phi(B)$ *whenever* $\operatorname{rank} A = \operatorname{rank} B$. *Then ϕ is of the form*

(a) $\phi(A) = MAN$ *for all* $A \in \mathbb{C}^{m\times n}$, *or*
(b) $m = n$ *and* $\phi(A) = MA^tN$ *for all* $A \in \mathbb{C}^{m\times n}$

for some nonsingular matrices $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$.

Proof. Suppose ϕ satisfies the hypotheses of the proposition. For $k = 0, \ldots, l$, where $l = \min\{m, n\}$, let A_k be a matrix in $\mathbb{C}^{m\times n}$ of rank k. Then $\mathbb{C}^{m\times n}$ is partitioned by the equivalence relation $\sim_{\text{rank}}$ into $l + 1$ equivalence classes $[A_0], \ldots, [A_l]$. Notice that

$$\dim[A_1] = 2(m + n - 1) < 2k(m + n - k) = \dim[A_k]$$

for all $k > 1$, where $\dim[A]$ stands for the real dimension of the manifold $[A]$. Since ϕ satisfies (6), by Proposition 2(b) we have

$$\dim\left[\phi^{-1}(A_1)\right] \leqslant \dim[A_1] < \dim[A_k]$$

for all $k > 1$. It is clear that $\phi^{-1}(A_1) \notin [A_0] = \{0\}$. Therefore we conclude that $[\phi^{-1}(A_1)] = [A_1]$. Then, by (6),

$$\phi\left([A_1]\right) = \phi\left(\left[\phi^{-1}(A_1)\right]\right) \subset \left[\phi\circ\phi^{-1}(A_1)\right] = [A_1],$$

or, equivalently, ϕ maps the set of all rank 1 matrices in $\mathbb{C}^{m\times n}$ into itself. By a result of Marcus and Moyls on rank 1 preservers (Theorem 1 in [42]), ϕ must be of the form (a) or (b) described in the proposition. ■

In the above proof, one sees that the original LPP, which is of problem (i) type, is transformed with the help of Proposition 2 to become a type (ii) problem, namely, of characterizing those ϕ that satisfy

$$\phi\big([A_1]\big) \subset [A_1].$$

In fact, this is the basic strategy in the proofs of the main results in [22, 27]: apply the dimension and tangent space arguments to transform a problem of type (i) to a problem of type (ii) for which the answer is already known or is easier to obtain. We got this idea of using the tangent space and the dimension argument from [21].

Other properties of the equivalence class may also be useful in solving the LPP. We list some of them here. Techniques in (a) and (b) below have been used in [27]:

(a) Suppose ϕ satisfies (6). Then $\phi(\mathcal{T}_A) \subset \mathcal{T}_{\phi(A)}$ by Proposition 2. This can be combined with (6) to deduce that

$$\phi\big([A] \cap \mathcal{T}_A\big) \subset \big[\phi(A)\big] \cap \mathcal{T}_{\phi(A)}$$

or other similar formulas. Then one may consider the linear algebraic or geometric structure of $[A] \cap \mathcal{T}_A$ or other similar subsets to solve the LPP.

(b) As indicated above, the dimension of $[A]$ is a useful tool in distinguishing between different equivalence classes. Other topological properties, such as connectedness and boundedness, of $[A]$ may also be considered. For example, let $\sim_{\text{sim}}$ be the equivalence relation on $\mathbb{C}^{n\times n}$ defined by $A \sim_{\text{sim}} B$ if A is similar to B. Let E_{ij} denote the $n \times n$ matrix with 1 at the (i, j) entry and elsewhere zero, and $[A]$ the equivalence class of A under $\sim_{\text{sim}}$. Then $\dim[E_{11}] = \dim[E_{12}]$, and hence the dimension alone cannot distinguish $[E_{11}]$ from $[E_{12}]$. However, the zero matrix is in the closure of $[E_{12}]$ but not $[E_{11}]$ (because every element in $[E_{11}]$ must have eigenvalues $1, 0, \ldots, 0$, and thus $[E_{11}]$ is bounded away from 0). As a result, there cannot exist a nonsingular linear operator ϕ that satisfies

$$\phi\big([E_{12}]\big) \subset [E_{11}].$$

(c) Let $\mathcal{S}(A)$ be the class of all subspaces contained in $[A]$ (or the closure of $[A]$, or the pencil $\bigcup_{r\in\mathbb{R}} r[A]$, etc.) with maximal dimension $d(A)$. If ϕ is nonsingular and satisfies

$$\phi\big([A]\big) \subset [B],$$

then

$$\phi(\mathscr{S}(A)) \subset \mathscr{S}(B) \quad \text{and} \quad d(A) \leqslant d(B).$$

These can be utilized to solve the LPP. In [3], maximal subspaces contained in $\{0\} \cup R_k$, where R_k is the set of all rank k matrices in $\mathbb{C}^{m\times n}$, are considered in characterizing the linear preservers of R_k.

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