# Multiplicative Preservers on Semigroups of Matrices

Wai-Shun Cheung [1], Shaun Fallat [2] [3] and Chi-Kwong Li [4]

## Abstract

We characterize multiplicative maps $\phi$ on semigroups of square matrices satisfying $\phi(\mathcal{P}) \subseteq \mathcal{P}$ for matrix sets $\mathcal{P}$, such as rank $k$ (idempotent) matrices, totally nonnegative matrices, $P_0$ matrices, $M_0$ matrices, positive semidefinite matrices, Hermitian matrices, normal matrices, and contractions. We also characterize multiplicative maps $\phi$ satisfying $\phi(g(X)) = \phi(X)$ for various functions $g$ on square matrices, such as the spectrum, spectral radius, numerical range, numerical radius, and matrix norms.

AMS Classification: 15A18, 15A33, 15A48, 15A57, 15A60.
Key words: Multiplicative map, semigroup, matrix algebra, matrix sets.

# 1 Introduction

An active research area in matrix theory is the study of linear preservers, which concerns the characterization of linear maps on matrix spaces with certain special properties; see [11, 14] and their references for some general background. Typically, linear preservers on square matrices have the form

$$A \mapsto PAQ \qquad \text{or} \qquad A \mapsto PA^tQ,$$

for some invertible matrices $P$ and $Q$. In particular, if the preserver is unital, i.e., it maps the identity matrix to identity matrix, then $P$ is the inverse of $Q$, and the preserver is an algebra isomorphism or anti-isomorphism. It is interesting to observe that the linearity assumption on the preservers leads to some interesting multiplicative properties. This naturally suggests the study of multiplicative preservers, i.e., multiplicative maps with some special properties on square matrices. In fact, such problems have attracted researchers in recent years; see, for example, [2, 7, 9, 13].

In [10], the authors obtained general results on multiplicative maps $\phi : M_n(\mathbf{D}) \to M_n(\mathbf{D})$, where $M_n(\mathbf{D})$ is the set of $n \times n$ matrices over a principal ideal domain $\mathbf{D}$. They characterized those multiplicative maps $\phi : M_n(\mathbf{D}) \to M_n(\mathbf{D})$ such that $\phi(A) \neq 0$ for some $A \in M_n(\mathbf{D})$ with $\det(A) = 0$. In [12], the author determined when a multiplicative map between associative rings $\mathcal{R}$ is additive. The results in these two papers are very useful in studying multiplicative preservers on algebras of square matrices or operators. However, if $\phi$ is just defined on a group or a semigroup of square matrices, additional techniques are required to characterize the associated multiplicative preservers; see [7].

[1]Department of Mathematics, University of Calgary, Calgary, Alberta T2N 1N4, Canada (wshun@math.ucalgary.ca)
[2]Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan S4S 0A2, Canada (sfallat@math.uregina.ca)
[3]Corresponding Author
[4]Department of Mathematics, College of William and Mary, P.O. Box 8795, Williamsburg, Virginia 23187-8795, USA (ckli@math.wm.edu)

The purpose of this paper is to establish some general techniques that are used to prove results for multiplicative maps on semigroups of square matrices. We refine the proofs in [10] to obtain some basic results in Section 2. Then we apply these results to study multiplicative preservers in Section 3, including the preservers of rank $k$ matrices, rank $k$ idempotent matrices, totally nonnegative matrices, $M_0$ matrices, $P_0$ matrices, normal matrices, Hermitian matrices, contractions, etc. We also include characterizations for the multiplicative preservers of spectrum, spectral radius, numerical range, numerical radius, norms, etc.

Our results can also be applied in the study anti-multiplicative maps (i.e., those $\psi$ such that $\psi(AB) = \psi(B)\psi(A)$), by considering the multiplicative map $X \mapsto \psi(A^t)$.

In the following discussion, we always assume that $\mathbf{D}$ is a principal ideal domain, which can be a field $\mathbb{F}$. Let $\mathbb{C}, \mathbb{R}, \mathbb{Q}$ denote the complex, real, and rational numbers, respectively. Denote by $\{e_1, \dots, e_n\}$ a basis for the module $\mathbf{D}^n$, and $E_{ij} = e_i e_j^t$ the standard matrix unit in $M_n(\mathbf{D})$. We always consider multiplicative maps $\phi : \mathcal{R} \to \mathcal{S}$, where $\mathcal{R}, \mathcal{S} \in M_n(\mathbf{D})$ are semigroups.

## 2 Basic Results

Recall that a square matrix $X$ is an *idempotent* if $X^2 = X$; two idempotents $X$ and $Y$ are orthogonal if $XY = YX = 0$. Clearly, if $\phi$ is a multiplicative map on square matrices, then $\phi(X)^2 = \phi(X^2) = \phi(X)$ for any idempotent $X$, and if $\phi(0) = 0$ then for any $X, Y$ satisfying $XY = 0$ we have $0 = \phi(0) = \phi(XY) = \phi(X)\phi(Y)$; so, in particular, $\phi$ will send orthogonal idempotents to orthogonal idempotents. It turns out that we can say much more.

**Lemma 2.1** *The matrices $X_1, \dots, X_n \in M_n(\mathbf{D})$ are mutually orthogonal nonzero idempotents if and only if there exists an invertible $S \in M_n(\mathbf{D})$ such that $X_j = S^{-1}E_{jj}S$ for $j = 1, \dots, n$.*

*Proof.* We follow the proof of [10, Theorem 1]. Since the ranks of orthogonal idempotents add ([1, p.89]), we see that $X_i = c_i d_i^t$ for some $c_i, d_i \in \mathbf{D}^n$ such that $d_i^t c_i = 1$. Let $S \in M_n(\mathbf{D})$ be such that $S = [d_1|\cdots|d_n]^t$. Then the given assumption on $X_i$ implies that $S[c_1|\cdots|c_n] = I_n$, i.e., $S^{-1} = [c_1|\cdots|c_n]$, and $X_i = S^{-1}E_{ii}S$ for $i = 1, \dots, n$, as asserted. □

Suppose $\mathcal{R} \subseteq M_n(\mathbf{D})$ is a semigroup such that $\{E_{ii} \mid 1 \le i \le n\} \subseteq \mathcal{R}$. For any $i, j \in \{1, \dots, n\}$, let

$$E_{ii}\mathcal{R}E_{jj} = \{E_{ii}XE_{jj} : X \in \mathcal{R}\} \subseteq \mathcal{R} \cap \{\alpha E_{ij} : \alpha \in \mathbf{D}\},$$

and consider the set of scalars

$$\mathbf{S}_{ij} = \{\alpha \in \mathbf{D} : \alpha E_{ij} \in \mathcal{R}\}. \qquad (2.1)$$

We have the following result.

**Proposition 2.2** *Suppose $\mathcal{R}, \mathcal{S}$ are semigroups of $M_n(\mathbf{D})$ such that*

$$\{E_{ii} \mid 1 \leq i \leq n\} \subseteq \mathcal{R}, \tag{2.2}$$

*and $\mathbf{S}_{ij}$ is defined as in (2.1). Let $\phi : \mathcal{R} \to \mathcal{S}$ be a multiplicative map. Then the implications $(a) \Rightarrow (b) \iff (c) \iff (d) \Rightarrow (e)$ hold for the following conditions.*

(a) *$\phi$ is injective.*

(b) *$\phi(0), \phi(E_{11}), \ldots, \phi(E_{nn})$ are distinct.*

(c) *$\phi(0) \neq \phi(E_{jj})$ for all $j = 1, \ldots, n$.*

(d) *$\{\phi(E_{ii}) : 1 \leq i \leq n\}$ is a set of nonzero orthogonal idempotents in $M_n(\mathbf{D})$.*

(e) *There is an invertible $S \in M_n(\mathbf{D})$ such that $\phi((a_{ij})) = S^{-1}(f_{ij}(a_{ij}))S$ for all $(a_{ij}) \in \mathcal{R}$, where $f_{ij} : \mathbf{S}_{ij} \to \mathbf{D}$ satisfies*

$$f_{ik}(ab) = f_{ij}(a)f_{jk}(b) \quad \text{for any } a \in \mathbf{S}_{ij}, b \in \mathbf{S}_{jk},$$

*and for $i \neq j$*

$$f_{ij}(a+b) = f_{ij}(a) + f_{ij}(b) \quad \text{whenever } E_{ii} + aE_{ij}, E_{jj} + bE_{ij} \in \mathcal{R}.$$

*Proof.* $(d) \Rightarrow (b)$ and $(a) \Rightarrow (b) \Rightarrow (c)$: Clear. $(c) \Rightarrow (d)$: Let $\phi(0) = X$. Then

$$X\phi(E_{ii}) = \phi(0)\phi(E_{ii}) = \phi(0) = X, \qquad \phi(E_{ii})X = \phi(E_{ii})\phi(0) = \phi(0) = X,$$

$$\phi(E_{ii})\phi(E_{jj}) = \phi(E_{ii}E_{jj}) = \phi(0) = X \quad \text{for } i \neq j, \quad \text{and} \quad X^2 = X.$$

Thus,

$$\mathcal{P} = \{\phi(E_{ii}) - X \ : \ 1 \leq i \leq n\}$$

is a set of $n$ nonzero mutually orthogonal idempotents in $M_n(\mathbf{D})$. By Lemma 2.1, we have $\sum_{i=1}^n(\phi(E_{ii}) - X) = I$. Since $X(\phi(E_{ii}) - X) = 0$ for all $i$, we see that $X = 0$. Hence, $\mathcal{P} = \{\phi(E_{ii}) \ : \ 1 \leq i \leq n\}$ is a set of nonzero orthogonal idempotents in $M_n(\mathbf{D})$.

$(d) \Rightarrow (e)$: By Lemma 2.1, there exists an invertible $S$ such that $\phi(E_{jj}) = S^{-1}E_{jj}S$ for $j = 1, \ldots, n$.

Replacing $\phi$ by the mapping $X \mapsto S\phi(X)S^{-1}$, we may assume that $\phi(E_{jj}) = E_{jj}$ for all $j = 1, \ldots, n$. For any $A = (a_{ij})$, $a_{ij}E_{ij} = E_{ii}AE_{jj} \in \mathcal{R}$ for all $i, j$ and the $(i, j)$ entry of $\phi(A)$ can be deduced from

$$E_{ii}\phi(A)E_{jj} = \phi(E_{ii})\phi(A)\phi(E_{jj}) = \phi(E_{ii}AE_{jj}) = \phi(a_{ij}E_{ij}).$$

Hence $\phi(A) = (f_{ij}(a_{ij}))$ for some $f_{ij} : \mathbf{S}_{ij} \to \mathbf{D}$.

If $a \in \mathbf{S}_{ij}$ and $b \in \mathbf{S}_{jk}$, then $(ab)E_{ik} = (aE_{ij})(bE_{jk})$ and $f_{ik}(ab) = f_{ij}(a)f_{jk}(b)$. Moreover, if $E_{ii} + aE_{ij}, E_{jj} + bE_{ij} \in \mathcal{R}$, then $(E_{ii} + aE_{ij})(E_{jj} + bE_{ij}) = (a + b)E_{ij}$. Applying $\phi$ on both sides, we conclude that $f_{ij}(a + b) = f_{ij}(a) + f_{ij}(b)$. □

If $\phi$ satisfies Proposition 2.2 (e) and if there are additional assumptions on $\mathbf{S}_{ij}$, then one can deduce more about the map $\phi$.

**Proposition 2.3** *Let $\mathcal{R}$ and $\mathcal{S}$ be semigroups of $M_n(\mathbf{D})$ such that (2.2) holds. Suppose $\phi : \mathcal{R} \to \mathcal{S}$ is a multiplicative map satisfying Proposition 2.2 (e).*

(1) *If $1 \in \mathbf{S}_{ij}$ and $f_{ij}(1) = 1$, then we have $f_{ki}(b) = f_{kj}(b)$ for $b \in \mathbf{S}_{ki} \cap \mathbf{S}_{kj}$ and $f_{ik}(b) = f_{jk}(b)$ for $b \in \mathbf{S}_{ik} \cap \mathbf{S}_{jk}$; in particular, $f_{ij}(b) = f_{ii}(b) = f_{jj}(b)$ for $b \in \mathbf{S}_{ij} \cap \mathbf{S}_{ii} \cap \mathbf{S}_{jj}$.*

(2) *If all the nonzero $\mathbf{S}_{ij}$ are the same, say, equal to $\mathbf{S}$, and if there is a collection of $(p, q)$ pairs covering the edges of a spanning tree of a graph with vertices in the index set $J \subseteq \{1, \ldots, n\}$, so that $f_{pq}(1)$ is invertible in $\mathbf{D}$ for these $(p, q)$ pairs, then there exist a $|J| \times |J|$ diagonal matrix $D$ and a multiplicative map $f : \mathbf{S} \to \mathbf{D}$ such that for any $|J| \times |J|$ matrix $(a_{ij})_{i,j \in J}$ over $\mathbf{S}$, we have*

$$(f_{ij}(a_{ij})) = D^{-1}(f(a_{ij}))D. \tag{2.3}$$

*Furthermore, if $|J| > 1$, then $f$ is additive.*

*Proof.* The first assertion follows from the facts that $f_{ki}(b)f_{ij}(1) = f_{kj}(b)$ and $f_{ik}(b) = f_{ij}(1)f_{jk}(b)$.

For the second assertion, we assume $J = \{1, \ldots, n\}$ for simplicity, and $\Gamma$ is the graph. Pick a collection $T$ of $n - 1$ $(i, j)$ pairs corresponding to the edges of a tree in $\Gamma$ so that $f_{ij}(1)$ is invertible for each $(i, j)$ pair in the collection. Note that if $(i, j) \in T$, then either $\mathbf{S}_{ji} = \{0\}$ or $f_{ji}(1)f_{ij}(1) = 1$. Now, construct the matrix $B \in M_n(\mathbf{D})$ so that $b_{ij} = f_{ij}(1)$ and $b_{ji} = f_{ij}(1)^{-1}$ if $(i, j) \in T$, and $b_{ij} = 0$ otherwise. Then there exists a diagonal $D \in M_n(\mathbf{D})$ such that all the nonzero entries of $DBD^{-1}$ equal $1 \in \mathbf{D}$. If $D(f_{ij}(a_{ij}))D^{-1} = (g_{ij}(a_{ij}))$, then $g_{ij}(1) = 1$ for those $(i, j) \in T$. Using the first assertion, we see that all $g_{ij}$ are equal, say, to $f$. If $|J| > 1$, then there is $p \neq q$ in $K$. Since $(E_{pp} + aE_{pq})(E_{qq} + bE_{pq}) = (a + b)E_{pq}$, applying $\phi$ on both sides, we see that $f_{pq}$ is additive, and so is $f$. $\square$

**Remark 2.4** Instead assuming (2.2) for $\mathcal{R}$, we can assume in Propositions 2.2 and 2.3 that $\mathcal{R}$ contains a set of mutually orthogonal nonzero (or rank one) idempotents $X_1, \ldots, X_n$. By Lemma 2.1, we have $X_j = S^{-1}E_{jj}S_j$ for $j = 1, \ldots, n$. One can then apply Propositions 2.2 and 2.3 to the mapping $X \mapsto \phi(S^{-1}XS)$ for $X$ such that $S^{-1}XS \in \mathcal{R}$. Also, in many applications, the index set $J$ referred to in Proposition 2.3 (2) is typically the entire index set $\{1, 2, \ldots, n\}$.

For our study of multiplicative preservers, we prove another general result.

**Proposition 2.5** *Let $1 \leq m < n$ and $\mathcal{R}$ be a semigroup of $M_n(\mathbf{D})$ containing all the diagonal matrices $E_J = \sum_{j \in J} E_{jj}$ whenever $J \subseteq \{1, \ldots, n\}$ satisfying $|J| \leq m$, here $E_\emptyset \equiv 0$. Suppose that for any two index sets $J$ and $K$ with $|J| = |K| \leq m$, there exists $A, B \in \mathcal{R}$ such that*

$$\sum_{j \in J} E_{jj} = A(\sum_{k \in K} E_{kk})B \in \mathcal{R}. \tag{2.4}$$

*If $\phi : \mathcal{R} \to M_n(\mathbf{D})$ is a multiplicative map, then one of the following conditions hold.*

(1) $\phi(E_J) = \phi(0) = 0$ *for all (some)* $J \subseteq \{1, \ldots, n\}$, *with* $|J| \le m$.

(2) $\phi(E_J) = \phi(0) \neq 0$ *for all* $J \subseteq \{1, \ldots, n\}$, *with* $|J| \le m$. *Moreover, if* $S \in M_n(\mathbf{D})$ *is invertible such that* $S\phi(0)S^{-1} = I_r \oplus 0_{n-r}$ *with* $r > 0$, *then there exists a multiplicative map* $\psi : \mathcal{R} \to M_{n-r}(\mathbf{D})$, *such that* $S\phi(X)S^{-1} = I_r \oplus \psi(X)$ *for any* $X \in \mathcal{R}$.

(3) $\phi(0) \neq \phi(E_{jj})$ *for all (any)* $j \in \{1, \ldots, n\}$. *In this case, Proposition 2.2 (b) – (e) hold.*

(4) $m = n - 1$, $\phi(E_J) = 0$ *for all* $J \subseteq \{1, \ldots, n\}$ *with* $|J| < n - 1$, *and*

$$\{\phi(I - E_{jj}) : 1 \le j \le n\}$$

*is a set of nonzero mutually orthogonal idempotents. Hence, there exists an invertible* $S \in M_n(\mathbf{D})$ *such that* $S\phi(I - E_{jj})S^{-1} = E_{jj}$ *for each* $j \in \{1, \ldots, n\}$, *and* $\phi(A(I - E_{ii}))$ *and* $\phi((I - E_{ii})A)$ *have rank at most one, for all* $A \in \mathcal{R}$.

*Proof.* Assume that there exists $J \subseteq \{1, \ldots, n\}$ with $|J| \le m$ such that $\phi(E_J) = 0$. Then for any $K \subseteq \{1, \ldots, n\}$ ($|K| = |J|$) not equal to $J$, there exist $A, B \in \mathcal{R}$ such that $E_K = AE_JB$, and thus $\phi(E_K) = \phi(A)\phi(E_J)\phi(B) = 0$. Now, for any $K \subseteq \{1, \ldots, n\}$ with $|K| < |J|$, we can write $E_K = E_{J_1} \cdots E_{J_r}$ for some $J_1, \ldots, J_r \subseteq \{1, \ldots, n\}$ with $|J_1| = \cdots = |J_r| = |J|$, and thus $\phi(E_K) = 0$. In particular, $\phi(0) = 0$.

Now we show by induction that $\phi(\sum_{j \in J} E_{jj}) = 0$ for each index set $J$ with $|J| < n - 1$. Suppose it is true for index set of order less than $k < n - 1$. Then $\{\phi(\sum_{j \in J} E_{jj}) \ : \ |J| = k\}$ is a set of $\binom{n}{k}$ idempotents. Since $\binom{n}{k} > n$, there must exist at least one index set $J$ with $\phi(\sum_{j \in J} E_{jj}) = 0$. By the assumption on $E_J$ and $E_K$ with $|J| = |K|$, we have $\phi(\sum_{j \in J} E_{jj}) = 0$ for all $|J| = k$.

If $m < n - 1$, we arrive at condition (1). If $m = n - 1$, then either condition (1) holds, or $\{\phi(I - E_{ii}) \ : \ 1 \le j \le n\}$ is a set of $n$ mutually orthogonal idempotents. By the assumption on $E_J$ and $E_K$ with $|J| = |K|$ and Lemma 2.1, either this set is zero or contains rank one idempotents. Since $\phi(I - E_{ii}) \neq 0$ for some $i \in \{1, \ldots, n\}$, we see that the latter condition holds. Thus condition (4) follows.

Next, suppose there exists $i \in \{1, \ldots, n\}$ such that $\phi(E_{ii}) \neq 0$. By the assumption on $E_J$ and $E_K$ with $|J| = |K|$, we have $\phi(E_{ii}) \neq 0$ for each $i \in \{1, \ldots, n\}$. If $\phi(0) = 0$, then Proposition 2.2 (c) holds, and hence condition (3) follows.

If $\phi(0) = P \neq 0$, then $P^2 = P$ and $\phi(X)P = P\phi(X)$ for all $X \in \mathcal{R}$. Thus the mapping $\Phi(X) = \phi(X) - P$ is multiplicative, such that $\Phi(0) = 0$. If $\Phi(E_{ii}) \neq 0$ for some $i$, then condition (3) holds for $\Phi$ and hence $\phi(E_{11}), \ldots, \phi(E_{nn})$ are idempotents such that $\{\phi(E_{ii}) - P \ : \ 1 \le i \le n\}$ is a set of nonzero mutually orthogonal idempotents, which is impossible. Therefore $\Phi(E_{ii}) = 0$ for all $i$ and the first assertion of (2) holds for $\phi$. Now, the last assertion of (2) follows from the fact that $\phi(X)P = P\phi(X) = P$ for all $X \in \mathcal{R}$. $\square$

We conclude this section with the following result.

**Proposition 2.6** *Suppose $\mathcal{R} \subseteq M_n(\mathbf{D})$ is a semigroup containing all the singular matrices (or nonnegative singular matrices if $\mathbf{D} = \mathbb{R}$ or $\mathbb{Q}$). If Proposition 2.5 (4) holds for a multiplicative map $\phi : \mathcal{R} \to M_n(\mathbf{D})$, then there exist an invertible $S \in M_n(\mathbf{D})$ and an additive multiplicative map $f : \mathbf{D} \to \mathbf{D}$ such that $\phi$ has the form*

$$A \mapsto S^{-1}(f(\det A_{ij}))S,$$

*where $A_{ij}$ is the submatrix of $A$ obtained by deleting the $i$th row and $j$th column.*

*Proof.* If $\mathcal{R}$ contains all singular matrices, then the result follows from [10, Theorem 1]. One can modify the proof to cover the case of nonnegative matrices. □

# 3 Multiplicative Preservers

In this section, we use the results in the previous section to study multiplicative preservers on semigroups of $n \times n$ matrices with $n \geq 2$. Let $\mathbf{S} = [0, \infty)$, $\mathbb{Q} \cap [0, \infty)$, or a field $\mathbb{F}$. We often consider the following semigroups:

$M_n(\mathbf{S})$: $n \times n$ matrices with entries in $\mathbf{S}$,

$M_n^m(\mathbf{S})$: matrices in $M_n(\mathbf{S})$ with rank at most $m \in \{1, \ldots, n\}$.

Usually, the multiplicative preservers $\phi$ have one of the following standard forms for some $p, q \in \{0, 1, \ldots, n\}$:

(I) There exist an invertible matrix $S \in M_n(\mathbf{S})$ and a nonzero additive multiplicative mapping $f : \mathbf{S} \to \mathbf{S}$ such that $\phi$ has the form

$$(a_{ij}) \mapsto S^{-1}(f(a_{ij}))S. \tag{3.1}$$

(II) There exist an invertible $S \in M_n(\mathbf{S})$ and a multiplicative map $\psi : M_n(\mathbf{S}) \to M_{n-q}(\mathbf{S})$ such that $S\phi(X)S^{-1} = I_q \oplus \psi(X)$ for any $X \in M_n(\mathbf{S})$, where $\psi(X) = 0$ if $\operatorname{rank} X \leq p$.

(III) There exist an invertible $S \in M_n(\mathbf{S})$ and a nonzero additive multiplicative map $f : \mathbf{S} \to \mathbf{S}$ such that $\phi$ has the form

$$A \mapsto S^{-1}(f(\det A_{ij}))S, \tag{3.2}$$

where $A_{ij}$ is the submatrix of $A$ obtained by deleting the $i$th row and $j$th column.

(IV) $\phi(X) = 0$ whenever $\operatorname{rank} X \leq p$.

In [10], multiplicative maps $\phi$ on $M_n(\mathbf{D})$ satisfying (IV) with $p = n - 1$ are called *degenerate mappings*.

**Remark 3.1** More can be inferred about the function $f$ in (3.1) and (3.2) if more is known about $\mathbf{S}$. We have the following statements:

(A) if $\mathbf{S}$ is a field, then $f$ field isomorphism;

(B) if $\mathbf{S}$ is the (nonnegative) reals or (nonnegative) rationals, then $f$ is the identity map;

(C) if $\mathbf{S}$ is a subfield of $\mathbb{C}$, then $f$ is a unital injective field homomorphism fixing all elements in $\mathbb{Q} \cap \mathbf{S}$;

(D) if $\mathbf{S}$ is a subfield of $\mathbb{C}$ satisfying $\mathbf{S} \subseteq \mathbb{Q} + i\mathbb{Q}$ or $\mathbb{R} \subseteq \mathbf{S}$ such that $f(\mathbb{R}) \subseteq \mathbb{R}$ in the latter case (for instance, this happens if $f$ is continuous), then $f$ must be of the form

$$z \mapsto z \qquad \text{or} \qquad z \mapsto \bar{z}.$$

(E) If $\phi$ maps nonnegative matrices to nonnegative matrices, then $S$ is a product of a permutation matrix and a diagonal matrix with positive diagonal entries.

*Proof.* Condition (A) follows since any nonzero field homomorphism will be an isomorphism. To see (B), note that a nonzero additive map $f$ on $\mathbb{R}$ satisfying $f(1) = 1$ will, in fact, fix all rational numbers. Moreover, if $x \geq 0$ then $\phi(x) = (\phi(\sqrt{x}))^2 \geq 0$. Thus, if $x_1 - x_2 \geq 0$ then $f(x_1) - f(x_2) = f(x_1 - x_2) \geq 0$, so $f$ is increasing. Furthermore, for any real number $x$ and rational numbers $x_1, x_2$ such that $x_1 \geq x \geq x_2$, we have $f(x_1) \geq f(x) \geq f(x_2)$; so, $f(x) = x$. For (C) and (D), see [15].

For (E), if (3.1) holds and $\phi(E_{jj})$ are nonnegative for all $j = 1, \ldots, n$, then $S$ and $S^{-1}$ can be chosen so both are nonnegative or both nonpositive. We can assume the former case holds and thus $S$ has the asserted form; if (3.2) holds and $\phi(I - E_{jj})$ are nonnegative for all $j = 1, \ldots, n$, then again we may conclude that $S$ and $S^{-1}$ can be chosen to be nonnegative, and thus $S$ has the asserted form. $\square$

## 3.1 Matrix Set Preservers

In this subsection, we study multiplicative maps $\phi : \mathcal{R} \to \mathcal{R}$ that preserve certain subsets $\mathcal{P}$ of a semigroup $\mathcal{R}$, i.e.,

$$\phi(\mathcal{P}) \subseteq \mathcal{P}.$$

**Theorem 3.2** *Let $\mathbf{S} = [0, \infty)$, $\mathbb{Q} \cap [0, \infty)$, or a field. Suppose $1 \leq m < n$ and $\mathcal{R} \subseteq M_n(\mathbf{S})$ is a semigroup containing $M_n^m(\mathbf{S})$, and suppose $\mathcal{P}$ is the set of rank $m$ matrices or the set of rank $m$ idempotent matrices. Then a multiplicative map $\phi : \mathcal{R} \to \mathcal{R}$ satisfies $\phi(\mathcal{P}) \subseteq \mathcal{P}$ if and only if $\phi$ has the standard form* (I) *so that* (A) – (E) *of Remark 3.1 hold, or $\phi$ has the standard form* (II) *with $m = p = q$.*

*Proof.* The sufficiency is clear. Conversely, suppose $\phi(\mathcal{P}) \subseteq \mathcal{P}$. Then either Proposition 2.5 (2) holds with $r = m$ or Proposition 2.5 (3) holds. In the former case, we see that $\phi$ has standard form (II) with $m = p = q$; In the latter case, we can apply Propositions 2.2 and 2.3 to demonstrate that $\phi$ has the standard form (I). $\square$

Using arguments similar to those in the proof of the previous theorem, and Propositions 2.2 – 2.5, we have the next result concerning multiplicative maps having special properties on rank $p$ matrices.

**Theorem 3.3** *Let* $\mathbf{S} = [0,\infty)$, $\mathbb{Q} \cap [0,\infty)$, *or a field. Suppose* $1 \le p, q < n$ *and* $\mathcal{R}, \mathcal{S} \subseteq M_n(\mathbf{S})$ *such that* $M_n^p(\mathbf{S}) \subseteq \mathcal{R}$. *The following conditions are equivalent for a multiplicative map* $\phi : \mathcal{R} \to \mathcal{S}$.

(a) *$\phi$ maps rank $p$ matrices to rank at most $q$ matrices.*

(b) *$\phi$ maps rank $p$ idempotents to rank at most $q$ idempotents.*

(c) *One of the following holds:*

(i) $p = q$ *and $\phi$ has the standard form* (I) *so that* (A) – (E) *hold.*

(ii) *$\phi$ has the standard form* (II).

(iii) $n - 1 = p > q = 1$ *and $\phi$ has the standard form* (III) *so that* (A) – (E) *hold.*

(iv) *$\phi$ has the standard form* (IV).

Next, we consider multiplicative preservers of other matrix sets $\mathcal{P}$. Very often, it is easy to construct degenerate multiplicative maps $\phi$ having a desired property. For example, if there is an idempotent $P \in \mathcal{P}$, we can have degenerate multiplicative maps of the form

$$A \mapsto P \qquad \text{or} \qquad A \mapsto \det(A)A.$$

So, we impose some additional assumptions on $\phi$ or $\mathcal{R}$ to obtain more reasonable results. If we assume that $\phi(0) \neq \phi(X)$ for some (or for all) rank one idempotent matrices, then we can invoke Proposition 2.2 right away. Of course, one may replace this assumption by any of the conditions (a) – (d) in Proposition 2.2, for example, we may assume that $\phi$ is injective and $\phi(\mathcal{P}) \subseteq \mathcal{P}$. Another natural assumption is that $\phi(\mathcal{P}) = \mathcal{P}$. It turns out that any one of these assumptions lead to nice characterization theorems for multiplicative preservers of many important classes of matrices.

An $n \times n$ matrix is called *totally nonnegative* (TN) if all of its minors of all sizes are nonnegative. From the classical identity of Cauchy and Binet, it follows that the product of any two TN matrices is again a TN matrix. Here we study multiplicative preservers of TN matrices. Denote by **TN** the set of TN matrices.

**Theorem 3.4** *Let* $m \in \{1, \dots, n\}$, *and let $\mathcal{P}$ be the set of* TN *matrices with rank at most $m$. Suppose* $\mathcal{R} \subseteq M_n(\mathbb{R})$ *is a semigroup containing $\mathcal{P}$. The following are equivalent for a multiplicative map* $\phi : \mathcal{R} \to \mathcal{R}$.

(a) $\phi(0) \neq \phi(X)$ *for some rank one (idempotent) matrix* $X \in \mathcal{P}$ *and* $\phi(\mathcal{P}) \subseteq \mathcal{P}$.

(b) $\phi(\mathcal{P}) = \mathcal{P}$.

(c) *There exist* $S = \sum_{j=1}^n s_j E_{jj}$ *or* $\sum_{j=1}^n s_j E_{j,n-j+1}$ *with* $s_j > 0$ *for all $j$ such that $\phi$ has the form* $A \mapsto S^{-1}AS$.

*Proof.* It is clear that (c) implies (a) and (b). Note that for any semigroup that contains $\mathcal{P}$, condition (2.4) holds for any $J, K \in \{1, \ldots, n\}$ with $|J| = |K| \leq m$. To this end, consider $1 \leq i_1 < i_2 < \ldots < i_k \leq m$ and $1 \leq j_1 < j_2 < \ldots < j_k \leq m$. Then $E_{i_1 j_1} + \cdots + E_{i_k j_k}$ and $E_{j_1 j_1} + \cdots + E_{j_k j_k}$ are both easily seen to be TN. Moreover,

$$E_{i_1 i_1} + \cdots + E_{i_k i_k} = (E_{i_1 j_1} + \cdots + E_{i_k j_k})(E_{j_1 j_1} + \cdots + E_{j_k j_k})(E_{j_1 i_1} + \cdots + E_{j_k i_k}).$$

Thus, Proposition 2.5 can be applied.

If (a) holds or if $m < n$ and (b) holds, then Proposition 2.5 (3) follows. By Propositions 2.2 and 2.3, $\phi((a_{ij})) = S^{-1}(f(a_{ij}))S$ for some invertible $S$, and an additive multiplicative map $f$ on the nonnegative real numbers with $f(1) = 1$. We then conclude that $f$ is the identity map and the invertible matrix $S$ can only be of the required form.

Next, assume $m = n$ and (b) holds. By Proposition 2.5 one of the conditions (1) with $m = n - 1$, or (3), or (4) holds. If (3) holds, then $\phi$ has the desired form. So suppose (1) or (4) holds. Observe that if $A$ is TN and singular, then by [4] $A$ has a factorization into TN bidiagonal matrices. Since $A$ is singular, at least one of the bidiagonal factors, call it $L$, is singular. Since bidiagonal matrices are triangular, it follows that at least one main diagonal entry of $L$ is zero. Then either $L$ has a zero row or column, which includes a zero main diagonal entry.

Thus if either (1) or (4) in Proposition 2.5 holds, then the multiplicative map $\phi^2$ will map the set of TN matrices onto itself and map singular matrices to zero. Consider the family

$$\mathcal{S} = \{I + E_{ij} : |i - j| = 1, \alpha > 0\} \cup \mathcal{D},$$

where $\mathcal{D}$ is the set of positive diagonal matrices. Then $\mathcal{S}$ generates all the invertible TN matrices [6], and thus $\phi(\mathcal{S})$ will generate all the nonzero matrices in the range. But then $\phi^2(\mathcal{S})$ contains matrices diagonally equivalent to those in $\mathcal{S}$. Now, if $A \in \mathbf{TN}$ is invertible, then $A$ is a product of matrices in $\mathcal{S} \cup \mathcal{D}$, and so must its image. So, $\phi^2(A)$ is invertible. Thus, non-zero singular TN matrices have no preimages, which contradicts the fact that $\phi^2(\mathbf{TN}) = \mathbf{TN}$. $\square$

A superset of the TN matrices is the well-studied $P_0$-matrices. An $n \times n$ matrix is called a $P_0$ matrix if all of its principal minors (i.e., the minors whose row and column index sets are the same) are nonnegative. Another class of matrices of interest is the $M_0$ matrices, i.e., matrices of the form $A = rI - N$ for a nonnegative matrix $N$ and a positive number $r$ which is larger than or equal to the Perron (largest positive) eigenvalue of $N$; see [8, Chapter 2] for background. We have the following preserver result.

**Theorem 3.5** *Let $m \in \{1, \ldots, n\}$, and let $\mathcal{P}$ be the set of $P_0$ matrices or the set of $M_0$ matrices with rank at most $m$. Suppose $\mathcal{R} \subseteq M_n(\mathbb{R})$ is a semigroup containing $\mathcal{P}$. The following are equivalent for a multiplicative map $\phi : \mathcal{R} \to \mathcal{R}$.*

(a) *$\phi(0) \neq \phi(X)$ for some rank one (idempotent) $X$ in $\mathcal{P}$, and $\phi(\mathcal{P}) \subseteq \mathcal{P}$.*

(b) *$\phi(\mathcal{P}) = \mathcal{P}$.*

(c) *There exists $S$, which is a product of a positive diagonal matrix and a permutation matrix such that $\phi$ has the form $A \mapsto S^{-1}AS$.*

*Proof.* We first consider $\mathcal{P}$ to be the set $\mathbf{P_0}$ of $P_0$ matrices. Again (c) implies (a) and (b) is trivial. We can prove (2.4) for any $J, K \subseteq \{1, \ldots, n\}$ with the same argument in the proof of the previous theorem. Thus, Proposition 2.5 can be applied.

If (a) holds, or if $m < n$ and (b) holds, then Proposition 2.5 (3) follows. By Propositions 2.2 and 2.3, we see that $\phi((a_{ij}) = S^{-1}(f_{ij}(a_{ij}))S$ for some invertible $S$, $f_{ij}$'s are additive and $f_{ij}(a) = f_{11}(a)$ for all nonnegative real number $a$. Again $f_{11}$ is the identity map. Since $f_{ij}$ is additive, $f_{ij}(-a) = -f_{ij}(a) = -f_{11}(a) = -a$ for all nonnegative number $a$ and so $f_{ij}$ is the identity map also.

Assume $m = n$ and $\phi(\mathbf{P_0}) = \mathbf{P_0}$. We will follow along the same lines as in the proof of the previous theorem. By Proposition 2.5 and the fact that $\phi(\mathbf{P_0}) = \mathbf{P_0}$, one of the conditions (1) with $m = n - 1$, or (3), or (4) in Proposition 2.5 holds. If (3) holds, then $\phi$ has the desired form. So, as before, suppose (1) or (3) holds.

Suppose $A$ is a singular $P_0$-matrix. Then after simultaneous permutation of rows and columns, we may assume that the last row is a nontrivial linear combination of the first $n-1$ rows. Hence there exists an $(n-1)$-vector $x$ such that

$$B = \left[\begin{array}{c|c} I & 0 \\ \hline x^t & 1 \end{array}\right] \in \mathbf{P_0} \quad \text{and} \quad BA = C = \begin{bmatrix} C_1 & d \\ 0 & 0 \end{bmatrix}.$$

Hence $A = B^{-1}C$, where $B$ (and hence $B^{-1}$), and $C$ are both $P_0$-matrices. Moreover, $C$ has a zero row.

Thus, as before, if either (1) or (4) holds, then the multiplicative map $\phi^2$ will map the set of $P_0$ matrices onto itself, and map singular matrices to zero. Suppose $\phi^2$ takes singular matrices to zero. Since the inverse of nonsingular $P_0$ matrices are $P_0$ matrices and the fact that $\phi^2(\mathbf{P_0}) = \mathbf{P_0}$ implies $\phi^2(I) = I$, we have $\phi^2$ maps invertible matrices to invertible matrices. Thus $\phi^2(\mathbf{P_0}) \neq \mathbf{P_0}$, which is a contradiction.

The proof of the $M_0$ matrices preserving maps can be done similarly. The only additional observation needed is:

In the proof of $\mathbf{P_0}$ preservers, if $N$ is a nonnegative matrix and $r > 0$ is such that $A = rI - N$ is a singular $M_0$ matrix, then the last row $x$ of $B$ (above) satisfies $xA = 0$, i.e., $rx = xN$. So, $x$ is a left Perron vector of the nonnegative matrix $N$. Thus, $x$ is nonnegative. It follows that $A = B^{-1}C$ such that $B^{-1}$ and $C$ are $M_0$ matrices. □

Next, we consider the multiplicative maps that preserve the *positive semidefinite matrices* (PSD)*, Hermitian matrices, normal matrices, or contractions,* i.e., matrices with spectral norm not larger than one. In what follows for a given complex matrix $A$, $\overline{A}$ denotes the matrix obtained from $A$ by conjugating each entry of $A$.

**Theorem 3.6** *Let $m \in \{1, \ldots, n\}$, and let $\mathcal{P}$ be the set of positive semidefinite matrices, the set of Hermitian matrices, the set of normal matrices, or the set of contractions with rank at most $m$. Suppose $\mathcal{R} \subseteq M_n(\mathbb{C})$ is a semigroup containing $\mathcal{P}$. The following are equivalent for a multiplicative map $\phi : \mathcal{R} \to \mathcal{R}$.*

(a) $\phi(0) \neq \phi(X)$ *for some rank one (idempotent)* $X \in \mathcal{P}$, *and* $\phi(\mathcal{P}) \subseteq \mathcal{P}$.

(b) $\phi(\mathcal{P}) = \mathcal{P}$.

(c) *There exists a unitary matrix* $U \in M_n(\mathbb{C})$ *such that* $\phi$ *has the form*

$$A \mapsto U^*AU \qquad or \qquad A \mapsto U^*\overline{A}U.$$

*Proof.* We first consider the case of PSD matrices. Denote by **PSD** the set of PSD matrices. Again, (c) implies (a) and (b) is trivial.

First, we show that if $k \leq \min\{m, n-1\}$, then (2.4) holds for any $J, K \subseteq \{1, \ldots, n\}$ with $|J| = |K| = k$. We first consider the case when $|J \cap K| = k - 1$. Assume that $p \in J \setminus K$ and $q \in K \setminus J$. Then $P = \sum_{j \in J \cup K} E_{jj} + E_{pq} + E_{qp}$ is a rank $k$ positive semi-definite Hermitian matrix. Thus $A = E_K P, B = PE_K \in \mathcal{R}$ satisfy $E_K = AE_J B$. For general $E_J$ and $E_K$, we can construct a finite sequence of index sets $J_1, \ldots, J_r \subseteq \{1, \ldots, n\}$ so that $J = J_1$, $J_r = K$, $|J_1| = \cdots = |J_r|$, and $|J_s \cap J_{s+1}| = k - 1$ for all $s = 1, \ldots, r-1$. We can then apply the above results to show that

$$E_K = A_{r-1} \cdots A_1 E_j B_1 \cdots B_{r-1}$$

for some $A_1, \ldots, A_{r-1}, B_1, \ldots, B_{r-1} \in \mathcal{R}$. Now, Proposition 2.5 can be applied.

Assume (a) holds, or (b) holds with $m < n$. Then Proposition 2.5 (c) holds. By Propositions 2.2 and 2.3, we see that $\phi((a_{ij}) = S^{-1}(f_{ij}(a_{ij}))S$ for some invertible $S$, $f_{ij}$'s are additive and $f_{ij}(a) = f_{11}(a)$ for all nonnegative real number $a$. Again $f_{11}$ is the identity map on $\mathbb{R}$, and thus $\phi$ has the form $X \mapsto S^{-1}XS$ or $X \mapsto S^{-1}\overline{X}S$. Considering the images of $X = vv^*$, where $v \in \mathbb{C}^n$ is a unit vector, we see that $S$ can be chosen to be unitary. Hence condition (c) holds.

Assume $\phi(\mathbf{PSD}) = \mathbf{PSD}$. Then again, by Proposition 2.5 and the fact that $\phi(\mathbf{PSD}) = \mathbf{PSD}$, one of Proposition 2.5 (1) with $m = n - 1$, or (3), or (4) holds. If (3) holds, then $\phi$ has the desired form. So, as before, suppose (1) or (4) of Proposition 2.5 holds. Now assume $A$ is a singular PSD matrix. Then, following the argument in the $P_0$ matrix case and taking into account of symmetry, it follows that there exists an $n \times n$ invertible $P_0$ matrix $B$ such that

$$A = B^{-1}C(B^{-1})^t = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where $C_1$, and $C$ are both PSD matrices. Applying a result of Ballantine [3], $B^{-1}$ can be written as a product of at most five positive definite matrices.

Thus if either (1) or (4) in Proposition 2.5 holds, then the multiplicative map $\phi^2$ will satisfy $\phi^2(\mathbf{PSD}) = \mathbf{PSD}$ and map singular matrices to zero. Since the inverse of nonsingular PSD matrices are PSD matrices and the fact that $\phi^2(\mathbf{PSD}) = \mathbf{PSD}$ implies $\phi^2(I) = I$, we have $\phi^2$ maps invertible matrices to invertible matrices. Thus $\phi^2(\mathbf{PSD}) \neq \mathbf{PSD}$, which is a contradiction.

The proofs for the cases of normal matrices, Hermitian matrices, and contractions can be carried out similarly. Note that in each of these cases, condition (2.4) can be easily verified. $\square$

## 3.2 Preservers of Functions

Suppose $g$ is a (scalar, vector, or set valued) function on matrices in $M_n(\mathbf{S})$. We consider multiplicative maps $\phi : \mathcal{R} \to M_n(\mathbf{S})$ that preserve $g$, i.e.,

$$g(\phi(X)) = g(X) \qquad \text{for all } X.$$

If

$$g(0) \neq g(X) \quad \text{for all (or for some) rank one idempotent } X, \tag{3.3}$$

then one can always conclude that $\phi$ has standard form (I) so that (A) – (E) of Remark 3.1 hold. Very often, one can deduce additional conditions on $S$. For example, one can use Theorem 3.2 to verify that the multiplicative preservers of the rank function have the standard form (I). We illustrate this scheme in the following. Note that these results cover some of those in [2, 7], and actually, many of the results can be deduced from those in [10].

The following result concerns preservers of spectra (counting or not counting multiplicities) of matrices over a field.

**Proposition 3.7** *Suppose $\mathcal{R} \subseteq M_n(\mathbb{F})$ is a semigroup containing $M_n^1(\mathbf{S})$. Then a multiplicative map $\phi : \mathcal{R} \to \mathcal{R}$ preserves the spectrum of (rank one idempotent) matrices in $\mathcal{R}$ if and only if there exists an invertible $S \in M_n(\mathbb{F})$ such that $\phi$ has form*

$$X \mapsto S^{-1}XS.$$

*Proof.* The "if" part is clear. For the converse, if $\phi$ preserves the spectrum, then $\phi(0) = 0$, and $\phi$ maps $n$ mutually orthogonal rank one idempotents to $n$ mutually orthogonal rank one idempotents. By Theorem 3.2, $\phi$ has the standard form (I). Considering the image of $X = aE_{11}$, we see that $f$ must be the identity map. The result then follows. □

Using similar arguments, we have the following two results concerning multiplicative preservers of the spectrum or Perron root (which is the same as the spectral radius or the largest positive eigenvalue) of (rational) nonnegative matrices, and spectral radii of complex matrices.

**Proposition 3.8** *Let $\mathbf{S} = [0, \infty)$ or $\mathbb{Q} \cap [0, \infty)$. Suppose $\mathcal{R} \subseteq M_n(\mathbf{S})$ is a semigroup containing $M_n^1(\mathbf{S})$. Then a multiplicative map $\phi : \mathcal{R} \to \mathcal{R}$ preserve the spectrum or spectral radius of (rank one idempotent) matrices in $\mathcal{R}$ if and only if there exists an invertible $S \in M_n(\mathbb{R})$ such that $\phi$ has the form*

$$X \mapsto S^{-1}XS,$$

*where $S \in M_n(\mathbf{S})$ is a product of a permutation matrix and a diagonal matrix with positive diagonal entries.*

**Proposition 3.9** *Suppose $\mathcal{R} \subseteq M_n(\mathbb{C})$ is a semigroup containing $M_n^1(\mathbb{C})$. Then a multiplicative map $\phi : \mathcal{R} \to \mathcal{R}$ preserve the spectral radius of (rank one or just rank one idempotent) matrices in $\mathcal{R}$ if and only if there exists an invertible $S \in M_n(\mathbb{C})$ such that $\phi$ has the form*

$$X \mapsto S^{-1}XS \qquad \text{or} \qquad X \mapsto S^{-1}\overline{X}S.$$

The *numerical range* and *numerical radius* of $A \in M_n(\mathbb{C})$ are defined and denoted by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n,\ x^*x = 1\} \qquad \textit{and} \qquad r(A) = \max\{|z| : z \in W(A)\}.$$

The *spectral norm* on $M_n(\mathbb{C})$ is defined by

$$\|A\| = \max\{(x^*A^*Ax)^{1/2} : x \in \mathbb{C}^n,\ x^*x = 1\}.$$

For these notions we have the following result.

**Proposition 3.10** *Suppose $\mathcal{R}, \mathcal{S}$ are semigroups of $M_n(\mathbb{C})$ such that $M_n^1(\mathbb{C}) \subseteq \mathcal{R}$. Let $g(A)$ denote $W(A), r(A)$, or $\|A\|$. Then a multiplicative map $\phi : \mathcal{R} \to \mathcal{S}$ satisfies $g(\phi(A)) = \phi(A)$ for all $A \in \mathcal{R}$ if and only if there is a unitary $U$ such that*

(i) *$\phi$ has the form $X \mapsto U^*XU$, or*

(ii) *$g(A) = r(A)$ or $\|A\|$, and $\phi$ has the form $X \mapsto U^*\overline{X}U$.*

*Proof.* The "if" part can be easily checked. For the converse, note that the given condition ensures that $\phi(0) = 0$ and rank one idempotents are mapped to nonzero idempotents. By Propositions 2.2 and 2.3, $\phi$ has the standard form (I). Since $g(\phi(A)) = g(A)$ for all rank one Hermitian matrices, we see that $S$ is unitary, and $f(\mathbb{R}) \subseteq \mathbb{R}$. So, $\phi$ has the form $A \mapsto U^*AU$ or $A \mapsto U^*\overline{A}U$. If $g(A) = W(A)$, one can consider $W(\phi(A))$ for $A = iE_{11}$ and conclude that the latter case cannot occur. □

**Remark 3.11** Similar results hold for many other functions $g$ on square matrices satisfying condition (3.3), including various functions on eigenvalues, singular values, and other many different types of numerical ranges, numerical radii, and norms; see [14].

## Acknowledgment

Research of the first author was partially supported by a PIMS Postdoctoral Fellowship, the research of the second author was partially supported by an NSERC research grant, and the third author was partially supported by an NSF research grant, and an ILAS lecture fund. This research was conducted while the first and third authors were visiting the Department of Mathematics and Statistics of University of Regina after the CMS summer meeting, 2001, and they would like to thank the colleagues at the University of Regina for their hospitality.

# References

[1] A.A. Albert, Modern Higher Algebra, University of Chicago Press, Chicago, 1997.

[2] G. An and J. Hou, Multiplicative maps on matrix algebras, preprint.

[3] C.S. Ballantine, Product of positive definite matrices. III, *J. Algebra* **10** (1968) 174-182.

[4] C.W. Cryer, Some properties of totally positive matrices, *Linear Algebra Appl.* **15** (1976) 1-25.

[5] M. Gasca and C.A. Micchelli, eds., *Total Positivity and its Applications*, Mathematics and its Applications **359**, Kluwer Academic, Dordrecht, 1996.

[6] M. Gasca and J.M. Peña, On factorizations of totally positive matrices, [5, pp. 109-130].

[7] S.H. Hochwald, Multiplicative maps on matrices that preserve spectrum, *Linear Algebra Appl.* **212/213** (1994) 339-351.

[8] R. Horn and C. Johnson, *Topics in Matrix Analysis,* Cambridge University Press, Cambridge, 1991.

[9] J. Hou, Multiplicative maps on $B(X)$, *Sci. China Ser. A* **41** (1998), no. 4, 337–345.

[10] M. Jodeit Jr. and T.Y. Lam, Multiplicative maps of matrix semigroups, *Arch. Math.* **20** (1969) 10-16.

[11] C.K. Li and S. Pierce, A survey on linear preserver problems, *Amer. Math. Monthly* **108** (2001), 591-605.

[12] W.S. Martindale III, When are multiplicative mappings additive? *Proc. Amer. Math. Soc.* **21** (1969) 695-698.

[13] L. Molnár, Some multiplicative preservers on $B(H)$, *Linear Algebra Appl.* **301** (1999), 1–13.

[14] S. Pierce et al., A survey of linear preserver problems, *Linear and Multilinear Algebra* **33** (1992) 1-129.

[15] P.B. Yale, Automorphisms of the complex numbers, *Math. Magazine* **39** (1966), 135–141.