On the Hu-Hurley-Tam Conjecture Concerning The Generalized Numerical Range

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Abstract

Suppose m and n are integers such that $1 \leq m \leq n$, and H is a subgroup of the symmetric group S_m of degree m. Define the *generalized matrix function* associated with the principal character of the group H on an $m \times m$ matrix $B = (b_{ij})$ by

$$d^H(B) = \sum_{\sigma \in H} \prod_{j=1}^{m} b_{j\sigma(j)},$$

and define the *generalized numerical range* of an $n \times n$ matrix A associated with d^H by

$$W^H(A) = \{d^H(V^*AV) : V \text{ is } n \times m \text{ such that } V^*V = I_m\}.$$

It is known that $W^H(A)$ is convex if $m = 1$ or if $m = n = 2$. Hu, Hurley and Tam made the following conjecture:

Suppose $H = S_m$, $2 \leq m \leq n$ with $(m, n) \neq (2, 2)$. Let $A \in M_n$ be a normal matrix.

Then $W^H(A)$ is convex if and only if A is a multiple of a Hermitian matrix.

In this note, counter-examples are given to show that the conjecture is not true when $m < n$. Some techniques are developed to show that the conjecture is valid under more restrictive assumptions.

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1 Introduction

Let M_n be the algebra of $n \times n$ complex matrices. Suppose m is a positive integer such that $1 \leq m \leq n$, and H is a subgroup of the symmetric group S_m of degree m. Define the *generalized matrix function* associated with the principal character of the group H on an $m \times m$ matrix $B = (b_{ij})$ by

$$d^H(B) = \sum_{\sigma \in H} \prod_{j=1}^{m} b_{j\sigma(j)},$$

and define the *generalized numerical range* of an $A \in M_n$ associated with d^H by

$$W^H(A) = \{d^H(V^*AV) : V \text{ is } n \times m \text{ such that } V^*V = I_m\}.$$

Denote by $X[1,\ldots,m]$ the leading $m \times m$ principal submatrix of $X \in M_n$. It is easy to verify that

$$W^H(A) = \{d^H(U^*AU[1,\ldots,m]) : U \in M_n,\ U^*U = I\}.$$

When $H = S_m$, then $d^H(B)$ is the permanent of B, and $W^H(A)$ is known as the mth *permanental range* of $A \in M_n$. When $m = 1$, it reduces to the *classical numerical range* of A defined by

$$W(A) = \{x^*Ax : x \in \mathbf{C}^n,\ x^*x = 1\},$$

which has been studied extensively (see e.g. [6, Chapter 1]). The object $W^H(A)$ is one of the many generalizations of $W(A)$ involving multilinear algebraic structures introduced in [12], and has stimulated some research in the last decade [2, 3, 7, 8, 9].

The celebrated Toeplitz-Hausdorff theorem (see e.g. [6, Chapter 1]) asserts that the classical numerical range of a matrix is always convex. This result leads to many interesting consequences in theory and applications. It was proved in [9] that if $(m,n) = (2,2)$ then $W^H(A)$ is convex. However, it was shown in [7, 10] that there exists a normal matrix $A \in M_n$ such that the permanental range $W^H(A)$ is not convex if $2 \le m \le 3 \le n$. Moreover, the following conjecture was made in [7].

Conjecture 1.1 *Suppose $H = S_m$, $2 \le m \le n$ with $(m,n) \neq (2,2)$. Let $A \in M_n$ be a normal matrix. Then $W^H(A)$ is convex if and only if A is a multiple of a Hermitian matrix.*

In this note, we give counter-examples to show that the conjecture is not true when $m < n$ (see Section 2). Even though Conjecture 1.1 is not valid in general, we show that it holds under certain restriction (see Section 3). Some techniques are developed to prove the result that may lead to a better understanding of $W^H(A)$ and the correct condition for the convexity of $W^H(A)$.

2 Counter-Examples

The following facts are needed to verify our counter-examples of Conjecture 1.1.

Lemma 2.1 *If $B \in M_m$ is an $m \times m$ principal submatrix of A, then* $\det(B) \in W^H(A)$.

Proof. Let $U \in M_m$ be unitary such that U^*BU is in triangular form, and let P be the $n \times m$ submatrix of I so that $P^*AP = B$. Then $\det(B) = d^H(U^*P^*APU) \in W^H(A)$ as asserted. □

Lemma 2.2 *Suppose $B = \text{diag}\,(1, w, w^2)$, where w is the cubic root of unity. Then*

$$W(B) = W(B^{-1}) = \{\det(U^*BU[1,2]) : U \in M_3,\ U^*U = I_3\}$$

is the convex hull of $\{1, w, w^2\}$.

Proof. It is well-known that if A is normal, then $W(A)$ is the convex hull of the set of eigenvalues of A. Since $B = \text{diag}\,(1, w, w^2)$ and $B^{-1} = \text{diag}\,(1, w^2, w)$, we see that $W(B) = W(B^{-1})$ is the convex hull of $\{1, w, w^2\}$.

Now, suppose $U \in M_3$ is unitary and B_1 is the 2×2 submatrix lying in the left top corner of U^*BU. Then by the adjoint formula of the inverse of an invertible matrix, $\det(B_1)$ is just the $(3,3)$ entry of $U^*B^{-1}U$, which is an element in $W(B^{-1}) = W(B)$.

Conversely, for any unit vector $x \in \mathbf{C}^3$ and $\mu = x^*B^{-1}x \in W(B^{-1})$, we can find a unitary matrix $U \in M_3$ so that the third column of U is x. Then the $(3,3)$ entry of $U^*B^{-1}U$ equals $\mu = \det(B_1)$, where B_1 is the 2×2 submatrix lying in the left top corner of U^*BU. The result follows. $\square$

Example 2.3 *Let $2 \le m < n$. Suppose $A = I_{n-2} \oplus \text{diag}\,(w, w^2)$, where w is the cubic root of unity. Then $W^H(A) = W(A)$ is the triangular region with vertices $1, w, w^2$.*

Proof. To show that $W(A) \subseteq W^H(A)$, let $B = \text{diag}\,(1, w, w^2)$. By Lemma 2.2, for any $\mu \in W(B) = W(A)$, there exists $\tilde{B}$ unitarily similar to B so that μ equals to the leading 2×2 minor of $\tilde{B}$. Since A is unitarily similar to $\tilde{A} = I_{m-2} \oplus \tilde{B} \oplus I_{n-m-1}$, and $\det(\tilde{A}[1, \ldots, m]) = \mu$, it follows from Lemma 2.1 that $\mu \in W^H(A)$.

Next, we prove the reverse inclusion. Note that $W^H(A) \subseteq W(K(A))$, where $K(A)$ is the induced matrix associated with principal character. Since $A = I_{n-2} \oplus \text{diag}\,(w, w^2)$, $K(A)$ is also unitary with eigenvalues $1, w, w^2$ with certain multiplicities, see [11]. Hence $W(K(A)) = W(A)$. $\square$

Let χ be a degree one character on a subgroup H of the symmetric group of degree m. One may consider the generalized matrix function $d_\chi^H(B) = \sum_{\sigma \in H} \chi(\sigma) \prod_{j=1}^m b_{j\sigma(j)}$ and the decomposable numerical range

$$W_\chi^H(A) = \{d_\chi^H(V^*AV) : V \text{ is } n \times m \text{ such that } V^*V = I_m\}$$

associated with χ on H, see [1, 11, 12]. In fact, Lemmas 2.1 and 2.2 can be deduced from some general results on decomposable numerical ranges $W_\chi^H(A)$ associated with degree one characters χ, and Example 2.3 is valid if one replaces $W^H(A)$ by $W_\chi^H(A)$ for any χ.

It would be nice to prove or disprove Conjecture 1.1 when $m = n > 2$.

3 Related Results

In this section, we focus on the case when $H = \{e\}$ is the trivial subgroup. Note that in such case, $d^H(B)$ is just the product of the diagonal entries of B, but the shape of $W^H(A)$ is non-trivial even in this case, see e.g. [2, 9]. We will write $W^H(A)$ as $W_m^*(A)$. Our goal is to prove the following result showing that Conjecture 1.1 is valid under certain restriction.

Theorem 3.1 *Let $2 \le m \le n$ and $(m,n) \ne (2,2)$. Suppose $A \in M_n$ is a normal matrix with collinear eigenvalues. Then $W_m^*(A)$ is convex if and only if A is a multiple of a Hermitian matrix.*

Proof. Suppose $A \in M_n$ is a multiple of a Hermitian matrix. Then there exists a nonzero $\mu \in \mathbf{C}$ such that μA is Hermitian. Thus, $\mu^m W_m^*(A) = W_m^*(\mu A) \subseteq \mathbf{R}$. By the connectedness and compactness of $W_m^*(\mu A)$, it is a real line segment and hence is convex.

To prove the converse, suppose $A \in M_n$ is normal with collinear eigenvalues, but it is not a multiple of a Hermitian matrix. Then we can choose a suitable nonzero $\nu \in \mathbf{C}$ so that νA is unitarily similar to $\operatorname{diag}(\mu_1, \ldots, \mu_n) + iI_n$ for some $\mu_1 \ge \cdots \ge \mu_n$ with $\mu_1 > \mu_n$. Since $\nu^m W_m^*(A) = W_m^*(\nu A)$, it suffices to show that $W_m^*(\nu A)$ is not convex to get the desired conclusion.

The rest of this section is devoted to proving the following.

Assertion: *Suppose $1 < m \le n$ and $(m,n) \ne (2,2)$. If $A = \operatorname{diag}(\mu_1, \ldots, \mu_n) + iI_n$ is not a scalar matrix, then $W_m^*(A)$ is not convex.*

Given two vectors $x, y \in \mathbf{R}^n$, we say that x is *majorized* by y, denoted by $x \prec y$, if the sum of the k largest entries of x is not larger than that of y for $k = 1, \ldots, n$, and the sum of the entries of x is the same as that of y. By [2, Theorem 2.4], we have the following parametrization of $W_m^*(A)$. One may see [5] and [10] for parameterizations of other generalized numerical ranges.

Lemma 3.2 *Let $1 \le m \le n$, $A = \operatorname{diag}(\mu_1, \ldots, \mu_n) + iI$ with $\mu_1 \ge \cdots \ge \mu_n$. Then $z \in W_m^*(A)$ if and only if $z = \prod_{j=1}^m \operatorname{cosec}\theta_j e^{i(\theta_1+\cdots+\theta_m)}$, where $\theta_j \in (0,\pi)$ satisfy*

$$\sum_{j=1}^k \mu_{n-j+1} \le \sum_{j=1}^k \cot\theta_{s_j} \le \sum_{j=1}^k \mu_j \quad \textit{whenever} \quad 1 \le s_1 < \cdots < s_k \le n, \tag{1}$$

for any $1 \le k \le m$. In particular, if $m = n$, condition (1) becomes

$$(\cot\theta_1, \ldots, \cot\theta_n) \prec (\mu_1, \ldots, \mu_n).$$

Next, we have the following.

Lemma 3.3 *Let $1 \le m \le n$. Suppose $\theta_1, \ldots, \theta_m \in (0,\pi)$ satisfy (1) and $\theta_1 + \cdots + \theta_m = m\theta$. Then $\prod_{j=1}^m \operatorname{cosec}\theta_j \ge (\operatorname{cosec}\theta)^m$. The equality holds if and only if $\theta_j = \theta$, $j = 1, \ldots, m$.*

Proof. Consider the function $f(t) = \ln(1/\sin t)$ on $(0, \pi)$. Since $f''(t) = \operatorname{cosec}^2 t > 0$, we see that f is strictly convex. By [13, 3.C.1], we have $\sum_{j=1}^m f(\theta_j) \geq \sum_{j=1}^m f(\theta)$ and the equality holds if and only if $\theta_j = \theta$, $j = 1, \dots, m$. $\square$

The Proof of the Assertion when $m < n$.

Let $\alpha_1 = \cot^{-1}((\mu_1 + \cdots + \mu_m)/m)$, $\alpha_2 = \cot^{-1}((\mu_{n-m+1} + \cdots + \mu_n)/m)$. Since $\mu_1 > \mu_n$, we have $\alpha_1 < \alpha_2$. Then, $\alpha_j^m \in W_m^*(A)$ for $j = 1, 2$. By Lemma 3.2, we see that the curve $z(\theta) = (\operatorname{cosec}\theta)^m e^{im\theta}$, $\alpha_1 < \theta < \alpha_2$, lies in $W_m^*(A)$. Since all diagonal entries of U^*AU is nonzero, we see that $0 \notin W_m^*(A)$. Thus the difference of the arguments between α_1^m and α_2^m should be less than π. Furthermore, by Lemma 3.3 $z(\theta)$ in the nearest point to the origin among those points lying on the half ray from the origin passing through $z(\theta)$ for any $\theta \in [\alpha_1, \alpha_2]$. Hence it is a nonconvex boundary of $W_m^*(A)$. We conclude that $W_m^*(A)$ in not convex. $\square$

We need a few more lemmas to handle the case when $m = n$. The first one is a special case of the result in [9, Theorem 1 (b)]. We give a short proof of it for completeness.

Lemma 3.4 *Let $A = \operatorname{diag}(\mu_1, \mu_2) + iI$. Then $W_2^*(A)$ is a horizontal line segment with all the points having imaginary parts $(\mu_1 + \mu_2)$. Furthermore, suppose $B = (b_{ij})$ is unitarily similar to A and $z = b_{11}b_{22}$. Then z is the end point of $W_2^*(A)$ if and only if B is a diagonal matrix or $b_{11} = b_{22}$.*

Proof. From Lemma 3.2, we see that

$$W_2^*(A) = \{(t_1t_2 - 1) + (\mu_1 + \mu_2)i : (t_1, t_2) \prec (\mu_1, \mu_2)\}.$$

The result of the first part follows. The end points of $W_2^*(A)$ correspond to the extrema of the product t_1t_2 subject to $(t_1, t_2) \prec (\mu_1, \mu_2)$. They are attained only when $\{t_1, t_2\} = \{\mu_1, \mu_2\}$ or $t_1 = t_2$. Since $t_1 + i$ and $t_2 + i$ can be regarded as the diagonal entries of B, the result follows. $\square$

When $m = n = 3$, we also have a complete description of $W_3^*(A)$.

Lemma 3.5 *Let $A = \operatorname{diag}(\mu_1, \mu_2, \mu_3) + iI$.*

(a) *The set $W_3^*(A)$ is the closed region bounded by the following curves:*

(i) $\{\prod_{j=1}^3 (t_j + i) : (t_1, t_2, t_3) \prec (\mu_1, \mu_2, \mu_3), \mu_1 = t_1 \geq t_2 \geq t_3\}$,

(ii) $\{\prod_{j=1}^3 (t_j + i) : (t_1, t_2, t_3) \prec (\mu_1, \mu_2, \mu_3), t_1 \geq t_2 \geq t_3 = \mu_3\}$,

(iii) $\{\prod_{j=1}^3 (t_j + i) : (t_1, t_2, t_3) \prec (\mu_1, \mu_2, \mu_3), t_1 = t_2 \geq t_3\}$,

(iv) $\{\prod_{j=1}^3 (t_j + i) : (t_1, t_2, t_3) \prec (\mu_1, \mu_2, \mu_3), t_1 \geq t_2 = t_3\}$.

(b) *Suppose $B = (b_{ij})$ is unitarily similar to A and $z = b_{11}b_{22}b_{33} \in W_3^*(A)$. Then, the following are equivalent.*

(i) *The point z is an interior point of $W_3^*(A)$.*

(ii) *The diagonal entries b_{11}, b_{22} and b_{33} are distinct, and $\mu_3 < b_{jj} - i < \mu_1$, $j = 1, 2, 3$.*

(iii) *The diagonal entries b_{11}, b_{22} and b_{33} are distinct. Moreover, at least two of the numbers among b_{12}, b_{13} and b_{23} are nonzero, or $\mu_1 > \mu_2 > \mu_3$ and B is permutationally similar to $(\mu_2 + i) \oplus B'$ where B' is not a diagonal matrix.*

Proof. From Lemma 3.2, we know that

$$W_3^*(A) = \{(t_1 + i)(t_2 + i)(t_3 + i) : (t_1, t_2, t_3) \prec (\mu_1, \mu_2, \mu_3)\}.$$

Consider the continuous function $f : \mathbf{R}^3 \to \mathbf{C}$ given by

$$f(t_1, t_2, t_3) = (t_1 + i)(t_2 + i)(t_3 + i).$$

Suppose $t_1 + t_2 + t_3 = s_1 + s_2 + s_3$. If $f(t_1, t_2, t_3) = f(s_1, s_2, s_3)$, we easily check that

$$\left\{ \begin{array}{rcl} t_1 t_2 t_3 & = & s_1 s_2 s_3 \\ t_1 t_2 + t_1 t_3 + t_2 t_3 & = & s_1 s_2 + s_1 s_3 + s_2 s_3 \\ t_1 + t_2 + t_3 & = & s_1 + s_2 + s_3 \end{array} \right.$$

which implies that the t_j's and s_j's are the roots of the same cubic polynomial, and hence (t_1, t_2, t_3) is a permutation of (s_1, s_2, s_3).

We now reduce f to a 1-to-1 function as follows. Suppose $t_1 \geq t_2 \geq t_3$. For $(t_1, t_2, t_3) \prec (\mu_1, \mu_2, \mu_3)$, we see that (t_1, t_2) belongs to the region

$$D = \{(t_1, t_2) : \mu_1 \geq t_1 \geq t_2 \geq \mu_1 + \mu_2 + \mu_3 - t_1 - t_2, t_1 + t_2 \leq \mu_1 + \mu_2\}.$$

Consider the continuous function $g : \mathbf{R}^2 \to \mathbf{C}$ given by

$$g(t_1, t_2) = (t_1 + i)(t_2 + i)(\mu_1 + \mu_2 + \mu_3 - t_1 - t_2 + i).$$

From the above argument, it is clear that g is 1-to-1 on D and $g(D) = W_3^*(A)$.

Let ∂D and D^o denote the boundary and interior of D. Now $g(\partial D)$ is a closed curve in $\mathbf{C}$ without crossing. As D^o is connected and g is 1-to-1 on D, $g(D^o)$ must be either entirely inside or entirely outside the region bounded by $g(\partial D)$. However, in D, we may continuously contract ∂D to a point. Thus we can conclude that (I) $g(D^o)$ is the open region bounded by $g(\partial D)$ and also consequently (II) $g(\partial D)$ is the boundary of $g(D)$.

The result of (a) now follows from (II), and the fact that $\mu_1 + \mu_2 + \mu_3 = t_1 + 2t_2$ is equivalent to $t_2 = t_3$. For (b), as $t_1 + i$, $t_2 + i$ and $t_3 + i$ can be regarded as the diagonal entries of B, the equivalence of (i) and (ii) follows from (I) and (II). The equivalence of (ii) and (iii) is straight forward. $\square$

We remark that in Lemma 3.5(a), the curves (i) and (ii) are actually line segments. Computer plots show that the curves (iii) and (iv) are non-convex boundary intersecting at the point $(\frac{1}{3}\sum_{j=1}^{3} \mu_j + i)^3$. Our goal is to prove that, for $n \geq 3$ and under the condition $\mu_1 > \mu_n$, the point $(\frac{1}{n}\sum_{j=1}^{n} \mu_j + i)^n$ is special in the sense that it is a boundary point and the boundary sufficiently close to it must be non-convex.

Lemma 3.6 *Let $A = \operatorname{diag}(\mu_1, \ldots, \mu_n) + iI$. Then, there is an $\epsilon > 0$ such that for any $z \in W_n^*(A)$ satisfying $|z - (\frac{1}{n}\sum_{j=1}^n \mu_j + i)^n| < \epsilon$, the following condition holds:*

If $B = (b_{ij})$ is unitarily similar to A, $z = \prod_{j=1}^n b_{jj}$, and B is permutationally similar to $S \oplus T$ where S is 1×1 or all the diagonal entries of S are the same, then all the diagonal entries of S equal $(\frac{1}{n}\sum_{j=1}^n \mu_j + i)$.

Proof. Suppose $N \subseteq \{1, \ldots, n\}$ such that $c = \frac{1}{|N|}\sum_{j \in N} \mu_j \neq \frac{1}{n}\sum_{j=1}^n \mu_j$. Let $A(N)$ be the matrix formed by deleting row j and column j of A for all $j \in N$. Note that $A(N)$ is then a diagonal matrix with diagonal entries $\mu_j + i$ where $j \in \{1, \ldots, n\} \backslash N$. It is easy to see that the compact set $(c+i)^{|N|} W_{n-|N|}^*(A(N)) \subseteq W_n^*(A)$ and, by Lemma 3.3,

$$z_o = (\frac{1}{n}\sum_{j=1}^n \mu_j + i)^n \notin (c+i)^{|N|} W_{n-|N|}^*(A(N)).$$

Consequently the distance between z_o and $(c+i)^{|N|} W_{n-|N|}^*(A(N))$ is positive. Let

$$Q = \bigcup (c+i)^{|N|} W_{n-|N|}^*(A(N)),$$

where the union is over all $N \subseteq \{1, \ldots, n\}$ such that $\frac{1}{|N|}\sum_{j \in N} \mu_j \neq \frac{1}{n}\sum_{j=1}^n \mu_j$. As there are only finitely many choices of N, we see that z_o is of positive distance, say d, to the compact set Q. Let $\epsilon \in (0, d)$. We are going to show that the lemma holds.

Suppose $z \in W_n^*(A)$ satisfies $|z - z_o| < \epsilon$, $z = \prod_{j=1}^n b_{jj}$ where $B = (b_{ij})$ is unitarily similar to A, and B is permutationally similar to $S \oplus T$ where S is 1×1 or all the diagonal entries of S are the same, say equal s. Since B is similar to a direct sum and S is a summand, $s = \frac{1}{|N|}\sum_{j \in N} \mu_j + i$ for some $N \subseteq \{1, \ldots, n\}$. As $z \notin Q$, we see that the only possible value for s is $\frac{1}{n}\sum_{j=1}^n \mu_j + i$. The result follows. $\square$

We readily have the following lemma because the diagonal entries involved are non-zero.

Lemma 3.7 *Let $B = (b_{ij})$ be unitary similar to A and $z = \prod_{j=1}^n b_{jj} \in W_n^*(A)$. Suppose $B' = (b'_{ij})$ is a $p \times p$ principal submatrix of B and $z' = \prod_{j=1}^p b'_{jj} \in W_p^*(B')$. If z' is an interior point of $W_p^*(B')$, then z is an interior point of $W_n^*(A)$.*

Proof of the Assertion for $m = n \geq 3$.

As in the proof of our Assertion when $m < n$, using Lemma 3.3, we see that $z_0 = (\frac{1}{n}\sum_{j=1}^n \mu_j + i)^n$ is the point nearest to the origin among all those points in $W_n^*(A)$ lying on the half ray from the origin passing through z_0. Thus we know that z_0 is a boundary point of $W_n^*(A)$.

Assume the contrary that $W_n^*(A)$ is convex. Since $\mu_1 > \mu_n$, we see that $W_n^*(A)$ is a compact connected set but not a singleton. Let ϵ be given as in Lemma 3.6 and z a boundary point of $W_n^*(A)$ such that $0 < |z - (\frac{1}{n}\sum_{j=1}^n \mu_j + i)^n| < \epsilon$. Suppose $B = (b_{ij})$ is unitarily similar to A such that $z = \prod_{j=1}^{n} b_{jj}$. By a suitable permutation similarity, we may assume that B has r distinct diagonal entries and $B = (B_{ij})_{i,j=1,\ldots,r}$, where each B_{jj} block has equal diagonal entries. We prove the result by showing that for different values of r we have a contradiction.

If $r = 1$ all the diagonal entries of B are $\frac{1}{n}\sum_{j=1}^{n}\mu_j + i$ and thus $z = (\frac{1}{n}\sum_{j=1}^{n}\mu_j + i)^n$. This contradicts our choice of z and the result follows.

For $r \geq 2$, we first establish the following claim.

Claim: *If there exists $b_{jk} \neq 0$ with $b_{jj} \neq b_{kk}$, then b_{jj} and b_{kk} must be of multiplicity 1 among the diagonal entries of B.*

If the claim is not true, consider a 3×3 principal submatrix, say $B[j,k,l]$, with $b_{kk} = b_{ll}$. Since $b_{jk} \neq 0$, by Lemma 3.4, $b_{jj}b_{kk}$ is not an end point of the segment $W_2^*(B[j,k])$ and thus z is not an end point of the segment $(\prod_{p\neq j,k} b_{pp})W_2^*(B[j,k]) \subseteq W_n^*(A)$. Since $W_n^*(A)$ is convex and z is on the boundary, the segment $(\prod_{p\neq j,k} b_{pp})W_2^*(B[j,k])$ is on the boundary of $W_n^*(A)$. Now choose a 2×2 unitary matrix U such that $\tilde{B} = (U \oplus (1))^* B[j,k,l](U \oplus (1))$ has distinct diagonal entries and its $(1,2)$ entry is non-zero. Suppose the three diagonal entries are $b_{jj} + \delta$, $b_{kk} - \delta$ and b_{ll}. Since

$$w = (\prod_{p\neq j,k} b_{pp})(b_{jj} + \delta)(b_{kk} - \delta) \in (\prod_{p\neq j,k} b_{pp})W_2^*(B[j,k]),$$

we see that w is on the boundary of $W_n^*(A)$. However, if the $(1,3)$ or the $(2,3)$ entry of $\tilde{B}$ is nonzero then, by Lemma 3.5(b), $(b_{jj} + \delta)(b_{kk} - \delta)b_{ll}$ is an interior point of $W_3^*(\tilde{B})$. By Lemma 3.7, we deduce that w is an interior point of $W_n^*(A)$. Also, if the $(1,3)$ and $(2,3)$ entries of $\tilde{B}$ are zero, then we can check that $\tilde{B} - iI$ has distinct eigenvalues and $b_{ll} - i$ is the second one. By Lemma 3.5(b) again, $(b_{jj} + \delta)(b_{kk} - \delta)b_{ll}$ is an interior point of $W_3^*(\tilde{B})$ and consequently w is an interior point of $W_n^*(A)$. In both case, we get a contradiction. Thus our claim is proved.

We now continue with the proof of the Assertion. Suppose $r = 2$. Then, by the claim above, $B = B_{11} \oplus B_{22}$. By Lemma 3.6, all the diagonal entries of B are $\frac{1}{n}\sum_{j=1}^{n}\mu_j + i$ and again we have a contradiction. The result follows.

Suppose $r \geq 3$. If $B = \oplus_{j=1}^{r} B_{jj}$ then, by Lemma 3.6, all the diagonal entries of B are $\frac{1}{n}\sum_{j=1}^{n}\mu_j + i$ and we are through. Suppose there is only one 2×2 principal submatrix such that the two diagonal entries are distinct and the off-diagonal entries are nonzero. Without loss of generality, using the above claim, we may assume the matrix is $B[1,2]$. Then $B = B[1,2] \oplus B(1,2)$ where $B(1,2)$ is the principal submatrix complement to $B[1,2]$, and $B(1,2)$ is a direct sum of matrices of constant diagonal entries. By Lemma 3.6, every diagonal entry of $B(1,2)$ is $\frac{1}{n}\sum_{j=1}^{n}\mu_j + i$. Consider the submatrix $B[1,2,3] = B[1,2] \oplus (b_{33})$. Because every diagonal entries of $B(1,2)$ is $\frac{1}{n}\sum_{j=1}^{n}\mu_j + i$, we easily deduce that $B[1,2,3] - iI$ has three distinct eigenvalues and that $b_{33} - i$ is the second one. Then, by Lemma 3.5(b),

$b_{11}b_{22}b_{33}$ is an interior point of $W_3^*(B[1,2,3])$ and consequently z is an interior point of $W_n^*(A)$. This contradicts the fact that z is a boundary point of $W_n^*(A)$

Now suppose there exist two 2×2 principal submatrices, say $B[1,2]$ and $B[p,q]$, with $p < q$, such that b_{12} and b_{pq} are nonzero. If these two submatrices overlap, they actually come from the 3×3 principal submatrix $B[1,2,q]$. Then two of the off-diagonal entries b_{12}, b_{1q} and b_{2q} are nonzero. Similarly, we can deduce that z in an interior point of $W_n^*(A)$. This gives a contradiction.

Now suppose $B[1,2]$ and $B[p,q]$ do not overlap and so we have the 4×4 submatrix $B[1,2,p,q]$. By the above claim again, all the four diagonal entries are distinct. As b_{12} and b_{pq} are non-zero, by Lemma 3.4, we have $b_{11}b_{22} \in W_2^*(B[1,2])$ but not the end points, $b_{pp}b_{qq} \in W_2^*(B[p,q])$ but not the end points. From Lemma 3.4, $W_2^*(B[1,2])$ and $W_2^*(B[p,q])$ are two horizontal segments. So, unless both of them lie on the real axis, it is obvious that $b_{11}b_{22}b_{pp}b_{qq}$ is in the interior of $W_2^*(B[1,2]) \cdot W_2^*(B[p,q])$, where

$$\begin{aligned} W_2^*(B[1,2]) \cdot W_2^*(B[p,q]) &= \{z_1z_2 : z_1 \in W_2^*(B[1,2]), z_2 \in W_2^*(B[p,q])\} \\ &\subseteq W_4^*(B[1,2,p,q]). \end{aligned}$$

By Lemma 3.7, we are done except for that particular case.

Suppose now $W_2^*(B[1,2])$ and $W_2^*(B[p,q])$ are two horizontal segments lying on the real axis. From Lemma 3.4, we know that $(b_{11} - i) + (b_{22} - i) = 0$ and $(b_{pp} - i) + (b_{qq} - i) = 0$. Thus $b_{11} - i$, $b_{22} - i$, $b_{pp} - i$ and $b_{qq} - i$ are nonzero because b_{11}, b_{22}, b_{pp} and b_{qq} are distinct. Without loss of generality, suppose $|b_{11} - i| > |b_{pp} - i|$. Consider the 3×3 submatrix $B[1,2,p]$. If b_{1p} or b_{2p} is nonzero, then by Lemma 3.5(b), $b_{11}b_{22}b_{pp}$ is an interior point of $W_3^*(B[1,2,p])$. Otherwise, we can deduce that $B[1,2,p] - iI$ has distinct eigenvalues and $b_{pp} - i$ is the second eigenvalue. In both cases, as before, we can deduce that z is an interior point of $W_n^*(A)$ and this gives a contradiction. The proof is now complete. □

We conclude our paper with the following remark concerning another possible approach to Conjecture 1.1.

Let λ be an indeterminate and A an $n \times n$ normal matrix with eigenvalues $\alpha_1, \ldots, \alpha_n$. It follows from definition that every polynomial in $W_m^*(\lambda I - A)$ has all its roots in $W(A)$, the numerical range of A. Let

$$P(m,A) = \left\{ \prod_{j=1}^{m} (\lambda - \alpha_{i_j}) : 1 \leq i_1 < \cdots < i_m \leq n \right\} \subseteq W_m^*(\lambda I - A).$$

It is known [4] that if $2 \leq m \leq n-2$ and the eigenvalues of A are not collinear, then $\mathrm{conv} P(m,A)$ contains a polynomial which has a root $z_0 \notin W(A)$. Consequently, $0 \notin W_m^*(z_0 I - A)$ but $0 \in \mathrm{conv}\left\{\prod_{j=1}^m (z_0 - \alpha_{i_j}) : 1 \leq i_1 < \cdots < i_m \leq n\right\}$. Thus, we have

Theorem 3.8 *Suppose $2 \leq m \leq n-2$. If the eigenvalues of a normal matrix A are not collinear, then there exists a complex number z_0 such that $W_m^*(z_0 I - A)$ is not convex.*

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