Elementary proofs for some results on the circular symmetry of the numerical range

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Dedicated to Professor Yik-Hoi Au-Yeung on the occasion of his 75th birthday.

Abstract

Let $W(A)$ be the numerical range of an $n \times n$ complex matrix A. Using algebraic geometry technique and a result of Kippenhahn, Anderson showed that if $W(A)$ is contained in a circular disk, and $W(A)$ contains more than n boundary points of the circular disk, then $W(A)$ equals to the circular disk, and the center of the circular disk will be an eigenvalue of the matrix A. Many researchers have reproved and refined the result of Anderson. Very recently, Wu unified these results and proved the following statements for a matrix A of the form $\begin{bmatrix} B & 0 \\ C & \alpha I_{n-m} \end{bmatrix}$.

1. If $W(A)$ is contained in a circular disk $\mathcal{D}$ centered at α, and $W(A)$ contains at least $m+1$ boundary points of $\mathcal{D}$, then $W(A) = \mathcal{D}$.
2. If $W(A)$ contains a circular disk $\mathcal{D}$ centered at α, and the boundary of $W(A)$ contains at least $m+1$ boundary points of $\mathcal{D}$, then the boundary of $W(A)$ contains a circular arc which is the boundary of $\mathcal{D}$.
3. If the boundary of $W(A)$ contains at least $2m+1$ boundary points of a circular disk $\mathcal{D}$ centered at α, then $\mathcal{D} \subseteq W(A)$.

Moreover, under any one of the three conditions, α is an eigenvalue of A with algebraic multiplicity larger than its geometric multiplicity. The proofs of Wu utilized the Bézout's theorem and Riesz-Fejér theorem, etc. In this note, short and elementary proofs using only simple properties of polynomials and continuous functions are given to Wu's results. Furthermore, the results are extended to the higher numerical ranges of matrices.

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1 Introduction

Let M_n be the set of $n \times n$ complex matrices. The *numerical range* of $A \in M_n$ is defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, \; x^*x = 1\},$$

which has been studied extensively; see [5, 6, 7]. In 1970s, J. Anderson used a result of Kippenhahn on numerical range based on some theory of algebra geometry to prove the following.

Theorem 1.1 *Suppose $A \in M_n$ such that $W(A)$ is a subset of a circular disk $\mathcal{D}$, the boundary of $W(A)$ has more than n boundary point of $\mathcal{D}$, then $W(A) = \mathcal{D}$ and the center of $\mathcal{D}$ is an eigenvalue of the matrix A.*

The result was unpublished. Nevertheless, it generated a lot of interest in studying matrices A such that $W(A)$ is a circular disk; for example see [1, 3, 4, 9, 13, 14]. Using advanced techniques such as algebraic geometry, Bézout's theorem, Riesz-Fejér theorem, etc., researchers reproved and refined the result of Anderson. In particular, if the boundary of $W(A)$ contains "a large number of" boundary points of a circular disk, then A has nice algebraic properties. To date, the most general result in this line of research was obtained in [14]; see the statement of Theorem 2.2. In Section 2, we give a short and elementary proof using only simple properties of polynomials and continuous functions to Wu's result. We then apply a simple affine transform to the Hermitian part and skew-Hermitian part of a complex matrix to obtain relations between matrices A such that the boundary of $W(A)$ contains "a larger number of" boundary points of an elliptical disk. Because of the simplicity of our proofs, we show that one can readily extend the results to the k-numerical ranges of matrices for $k \in \{1, \ldots, n-1\}$ in Section 3.

2 Classical numerical range

We begin with the following simple observation.

Lemma 2.1 *Let $A \in M_n$. Then λ is an eigenvalue of $\frac{1}{2}(e^{i\theta}A + e^{-i\theta}A^*)$ if and only if $x = e^{i\theta}$ is a solution of the polynomial equation* $\det(x^2 A + A^* - 2\lambda x I) = 0$.

We are now ready to give short elementary proofs of the results of Wu [14].

Theorem 2.2 *Let $A = \begin{bmatrix} B & 0 \\ C & \alpha I_{n-m} \end{bmatrix} \in M_n$ with $1 \le m \le n$, and let $\mathcal{D}$ be a circular disk centered at α in case $m < n$.*

1. *If $W(A) \subseteq \mathcal{D}$ and the boundary of $W(A)$ contains $m+1$ boundary points of $\mathcal{D}$, then $W(A) = \mathcal{D}$.*

2. *If $\mathcal{D} \subseteq W(A)$ and the boundary of $W(A)$ contains $m+1$ boundary points of $\mathcal{D}$, then the boundary of $W(A)$ contains a circular arc which is part of the boundary of $\mathcal{D}$.*

3. *If the boundary of $W(A)$ contains $2m+1$ boundary points of $\mathcal{D}$, then $W(A)$ contains $\mathcal{D}$.*

Under any of these three conditions the center of $\mathcal{D}$ is an eigenvalue of B.

Proof. We may replace A by $\xi(A - \alpha I)$ for some nonzero $\xi \in \mathbb{C}$ and assume that $\mathcal{D}$ is centered at $\alpha = 0$ and $\mathcal{D} = \{\mu \in \mathbb{C} : |\mu| \le 1\}$.

Since $A = \begin{bmatrix} B & 0 \\ C & O_{n-m} \end{bmatrix}$, we have

$$\det(x^2 A + A^* - 2\lambda x I) = x^{n-m} \det \begin{bmatrix} x^2 B - 2\lambda x I_m + B^* & C^* \\ xC & -2\lambda I_{n-m} \end{bmatrix} = x^{n-m} p_{A,\lambda}(x),$$

where $p_{A,\lambda}(x)$ has degree at most $2m$.

Let $f(\theta)$ be the largest eigenvalue of $(e^{i\theta}A + e^{-i\theta}A^*)/2$. Then f is continuous on $[0, 2\pi]$.

To prove (1) and (2), suppose $u_0, \ldots, u_m$ are such that $0 \le u_0 < \cdots < u_m < 2\pi$

$$f(u_0) = f(u_1) = \cdots = f(u_m) = 1$$

lies on the boundary of $\mathcal{D}$. Set $u_{m+1} = u_0 + 2\pi$.

(1) Suppose $W(A) \subseteq \mathcal{D}$. Let

$$r_j = \min\{f(\theta) : \theta \in [u_{j-1}, u_j]\} \quad \text{for } j = 1, \ldots, m+1.$$

If $r = \max\{r_j : 1 \le j \le m+1\} < 1$. Then for any $\lambda \in (r, 1)$, there will be two distinct values $\theta_{j1}, \theta_{j2} \in (u_{j-1}, u_j)$ such that $f(\theta_{j1}) = f(\theta_{j2}) = \lambda$ for $j = 1, \ldots, m+1$. Hence, $p_{A,\lambda}(x) = 0$ for all x. But then, letting $\lambda \to 1$, we see that $p_{A,1}(x) = 0$ for all x, contradicting $f(\theta_{j1}) = \lambda < 1$. As a result, there is $j \in \{1, \ldots, m+1\}$ such that $r_j = 1$, then $f(\theta) = 1$ for all $\theta \in [u_{j-1}, u_j]$. Again $p_{A,1}(x)$ is the zero polynomial. Thus $f(\theta) = 1$ for all θ, and $W(A)$ is the circular disk.

(2) Suppose $\mathcal{D} \subseteq W(A)$. Let

$$r_j = \max\{f(\theta) : \theta \in [u_{j-1}, u_j]\} \quad \text{for } j = 1, \ldots, m+1.$$

If $r = \min\{r_j : 1 \le j \le m+1\} > 1$. Then for any $\lambda \in (1, r)$, there will be two distinct values $\theta_{j1}, \theta_{j2} \in (u_{j-1}, u_j)$ such that $f(\theta_{j1}) = f(\theta_{j2}) = \lambda$ for $j = 1, \ldots, m+1$. Hence, there will be $2(m+1)$ values θ in (u_0, u_{m+1}) such that $p_{A,\lambda}(e^{i\theta}) = 0$. Hence, $p_{A,\lambda}(x) = 0$ for all x. By Lemma 2.1 again, λ is an eigenvalue for $(e^{i\theta}A + e^{-i\theta}A^*)/2$ for all θ, contradicting the fact that $f(u_j) = 1$. As a result, there is $j \in \{1, \ldots, m+1\}$ such that $r_j = 1$, and the corresponding circular arc $\{e^{i\theta} : \theta \in [u_{j-1}, u_j]\}$ will be part of the boundary of $W(A)$. Note $f(\theta) = 1$ for all $\theta \in [u_{j-1}, u_j]$, therefore $p_{A,1}(e^{i\theta}) = 0$ for all $\theta \in [u_{j-1}, u_j]$, implying that $p_{A,1}(x)$ is the zero polynomial.

(3) Let $v_0, \ldots, v_{2m}$ be such that $v_0 < \cdots < v_{2m} < v_0 + 2\pi$ and $e^{iv_0}, \ldots, e^{iv_{2m}}$ are boundary points of $W(A)$.We claim there exists $w_0, \ldots, w_{2m}$ satisfying $f(w_j) = 1$. If $f(v_j) = 1$ then we let $w_j = v_j$. If $f(v_j) \ne 1$, then e^{iv_j} could not be a sharp point and we let θ_j corresponds to the angle of the normal direction of the tangent line touching e^{iv_j}, we have $f(\theta_j) \le 1 \le f(v_j)$, and hence there exists w_j between θ_j and v_j such that $f(w_j) = 1$. Note that each $[v_j, \theta_j]$ or $[\theta_j, v_j]$ is disjoint from the other intervals, so all w_j's are distinct. By Lemma 2.1, $x = e^{iw_j}$, is a solution of the polynomial equation $p_{A,1}(x) = 0$ for all $j = 0, 2, \ldots, 2m$. Since $p_{A,1}(x)$ has degree at most $2m$, we see that $p_{A,1}(x)$ is the zero polynomial. Therefore $p_{A,1}(x) = 0$ for all x.

By Lemma 2.1 again, 1 is an eigenvalue for $(e^{i\theta}A + e^{-i\theta}A^*)/2$ for all θ. Thus $W(A)$ contains the unit disk.

In all the three cases, we have shown that $0 = p_{A,1}(0) = -2^m \det B^*$. Thus, 0 is an eigenvalue of B. ■

In Theorem 2.2 (1) and (2), the number $m+1$ in the hypothesis is optimal. For example, if

$$A = \operatorname{diag}(1, w^2, \ldots, w^{m-1}) \oplus O_{n-m} \quad \text{for } w = e^{i2\pi/m}, \tag{2.1}$$

then

$$W(A) = \text{conv}\,\{1, w^2, \ldots, w^{m-1}\} \subseteq \{\mu \in \mathbb{C} : |\mu| \le 1\},$$

but the boundary of $W(A)$ does not contain any circular arc.

In Theorem 2.2 (3), the value $2m+1$ is optimal in the hypothesis. For example, if A is defined as in (2.1), then $W(A)$ will have $2m$ boundary points of the circle $\{(1-d)e^{i\theta} : \theta \in [0, 2\pi)\}$ for a sufficiently small $d > 0$, but $W(A)$ does not contain the circle.

Corollary 2.3 *Suppose $A \in M_n$ and $\mathcal{E}$ is a non-degenerate elliptical disk not equal to a circular disk.*

1. *If $W(A) \subseteq \mathcal{E}$ and the boundary of $W(A)$ contains $n+1$ boundary points of $\mathcal{E}$, then $W(A) = \mathcal{E}$.*
2. *If $\mathcal{E} \subseteq W(A)$ and the boundary of $W(A)$ contains $n+1$ boundary points of $\mathcal{E}$, then the boundary of $W(A)$ contains an elliptical arc which is part of the boundary of $\mathcal{E}$.*
3. *If the boundary of $W(A)$ contains $2n+1$ boundary points of $\mathcal{E}$, then $\mathcal{E} \subseteq W(A)$.*

Under any of these three conditions, the two foci of $\mathcal{E}$ are eigenvalues of A.

Proof. Since $\mathcal{E}$ is non-degenerate and not equal to a circular disk, we may apply an invertible (real affine) transform $\mu \mapsto \xi(\mu - \alpha)$ to $\mathcal{E}$ and assume that

$$\mathcal{E} = \{x + iy : x^2 + y^2/b^2 \le 1\} \quad \text{with } b \in (0,1). \tag{2.2}$$

We may replace A and $W(A)$ by $\xi(A - \alpha I)$ and the set $\{\xi(\mu - \alpha) : \mu \in W(A)\}$ accordingly so that we need only verify the results for the elliptical disk $\mathcal{E}$ in the form (2.2).

Now, suppose $\mathcal{E}$ has the form in (2.2), and $A = H + iG$, where $H = H^*$ and $G = G^*$. Then

$$W(H + iG/b) = \{x + iy/b : x + iy \in W(H + iG)\},$$

and

$$\mathcal{D} = \{x + iy/b : x + iy \in \mathcal{E}\} = \{\mu \in \mathbb{C} : |\mu| \le 1\}.$$

Moreover,

(a) $W(H + iG/b) \subseteq \mathcal{D}$ if and only if $W(H + iG) \subseteq \mathcal{E}$;

(b) $W(H + iG/b) \supseteq \mathcal{D}$ is and only if $W(H + iG) \supseteq \mathcal{E}$;

(c) the boundary of $W(H + iG)$ has k boundary points of $\mathcal{E}$ if and only if the boundary of $W(H + iG/b)$ has k boundary points of $\mathcal{D}$.

By Theorem 2.1, we see that the three conditions of the theorem hold.

Note that the two foci of the elliptical disk $\mathcal{E}$ are $\pm\sqrt{1-b^2}$. Following the proof of Theorem 2.2, we see that $0 = \det(x^2\hat{A} + \hat{A}^* - 2xI)$ for any $x \in \mathbb{C}$ with $\hat{A} = H + iG/b$. Thus, we have

$$0 = \det(x^2(H + iG/b) + (H - iG/b) - 2xI) = \det((x^2+1)H + ((x^2-1)/b)iG - 2xI). \tag{2.3}$$

Let $x = \pm\sqrt{\frac{1+b}{1-b}}$. Then $x^2+1 = \frac{x^2-1}{b}$ and $\frac{2x}{x^2+1} = \pm\sqrt{1-b^2}$. Furthermore,

$$(x^2+1)H + ((x^2-1)/b)iG - 2xI = (x^2+1)(H+iG) - 2xI = (x^2+1)(A \pm \sqrt{1-b^2}I).$$

Hence, equation (2.3) becomes

$$0 = \det((x^2+1)(A \pm \sqrt{1-b^2}I)).$$

As a result, $\pm\sqrt{1-b^2}$ are eigenvalues of A as asserted. ■

Corollary 2.4 *Suppose $A \in M_n$, and the boundary of $W(A)$ contains an elliptical arc Γ. Then $W(A)$ contains an elliptical disk $\mathcal{E}$ with Γ as part of its boundary and the foci of $\mathcal{E}$ are eigenvalues of the matrix A. If the elliptical arc is circular, then the center of the disk is an eigenvalue of the matrix with algebraic multiplicity strictly larger than its geometric multiplicity.*

Proof. Apply the same technique as in Corollary 2.3 to transform the ellipse into a circle. The result will then follow from Theorem 2.2. ■

Corollary 2.5 *If $A \in M_n$ is diagonalizable, then the boundary of its numerical range cannot contain a circular arc.*

Proof. If $A \in M_n$ is diagonalizable, then the geometric multiplicity and the algebraic multiplicity of its eigenvalues are always the same. By Corollary 2.4, we see that the boundary of $W(A)$ cannot contain a circular arc. ■

3 Higher numerical range

For $A \in M_n$ and $k \in \{1, \ldots, n-1\}$, define the *k-numerical range* of A by

$$W_k(A) = \left\{ \sum_{j=1}^{k} x_j^* A x_j : \{x_1, \ldots, x_k\} \text{ is an orthonormal set} \right\}.$$

One may see [6, 8, 10], [7, Chapter 2], and their references, for the basic and useful properties of the k-numerical range. Denote by $\lambda_1(H) \geq \cdots \geq \lambda_n(H)$ the eigenvalues of a Hermitian matrix $H \in M_n$. It is known that

$$W_k(A) = \bigcap_{t \in [0,2\pi)} \left\{ \mu \in \mathbb{C} : e^{it}\mu + e^{-it}\bar{\mu} \leq \sum_{j=1}^{k} \lambda_j(e^{it}A + e^{-it}A^*) \right\}.$$

Let $D_k(A)$ be the kth additive compound of A defined as the linear term in the expansion of the kth compound matrix

$$C_k(tA + I) = t^k C_k(A) + \cdots + tD_k(A) + I_{\binom{n}{k}}.$$

If is known that (see [11, Chapter 19]) if A has eigenvalues $\alpha_1, \ldots, \alpha_n$, then $D_k(A)$ has eigenvalues

$$\alpha_{j_1} + \cdots + \alpha_{j_k}, \qquad 1 \le j_1 < \cdots < j_k \le n.$$

By the linearity of $D_k(\cdot)$, we have

$$\sum_{j=1}^{k} \lambda_j(e^{it}A + e^{-it}A^*) = \lambda_1(e^{it}D_k(A) + e^{-it}D_k(A)^*).$$

Using these concepts and properties, one can readily extend the results in Section 2 to the following.

Theorem 3.1 *Suppose $A \in M_n$, $1 \le k < n$, and $\mathcal{E}$ is an elliptical disk.*

1. *Suppose $W_k(A) \subseteq \mathcal{E}$ and the boundary of $W_k(A)$ contains $\binom{n}{k}+1$ boundary points of $\mathcal{E}$. Then $W_k(A) = \mathcal{E}$.*
2. *Suppose $\mathcal{E} \subseteq W_k(A)$ and the boundary of $W_k(A)$ contains $\binom{n}{k}+1$ boundary points of $\mathcal{E}$. Then the boundary of $W_k(A)$ contains an elliptical arc which is part of the boundary of $\mathcal{E}$.*
3. *Suppose the boundary of $W_k(A)$ contains $2\binom{n}{k}+1$ boundary points of $\mathcal{E}$. Then $\mathcal{E} \subseteq W_k(A)$.*

Under any of these three conditions the foci of $\mathcal{E}$ are eigenvalues of $D_k(A)$; in the case that $\mathcal{E}$ is a circular disk the center will be and eigenvalue of $D_k(A)$ with algebraic multiplicity larger than its geometric multiplicity.

Proof. Apply the proofs in Section 2 to the matrix $D_k(A)$ instead of A. ■

Corollary 3.2 *Let $A \in M_n$ and $1 \le k < n$. Suppose the boundary of $W_k(A)$ contains an elliptical arc which is the boundary of the elliptical disk $\mathcal{E}$. Then $\mathcal{E} \subseteq W_k(A)$ and the foci of $\mathcal{E}$ are eigenvalues of $D_k(A)$; in the case that $\mathcal{E}$ is a circular disk the center will be and eigenvalue of $D_k(A)$ with algebraic multiplicity larger than its geometric multiplicity.*

Corollary 3.3 *Let $A \in M_n$ and $1 \le k < n$. If A is diagonalizable, then the boundary of $W_k(A)$ cannot contain a circular arc.*

Proof. If $A \in M_n$ is diagonalizable, then so is $D_k(A)$. Applying Corollary 2.5 with A replaced by $D_k(A)$, we get the conclusion. ■

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