Generalized Doubly Stochastic Matrices and Linear Preservers

Hanley Chiang[1] and Chi-Kwong Li[2]

Department of Mathematics, College of William and Mary,
P.O.Box 8795, Williamsburg, VA 23187-8795
E-mail: hchiang@fas.harvard.edu   ckli@math.wm.edu

Abstract

A real or complex $n \times n$ matrix is generalized doubly stochastic if all of its row sums and column sums equal one. Denote by $\mathcal{V}$ the linear space spanned by such matrices. We study the reducibility of $\mathcal{V}$ under the group Γ of linear operators of the form $A \mapsto PAQ$, where P and Q are $n \times n$ permutation matrices. Using this result, we show that every linear operator $\phi : \mathcal{V} \to \mathcal{V}$ mapping the set of generalized doubly stochastic matrices into itself is a linear combination of the operators in Γ followed by a translation of a fixed matrix in $\mathcal{V}$. We compare our results with those from related studies by Sinkhorn and Benson. We also consider similar problems for the generalized symmetric doubly stochastic matrices.

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1 Introduction

There has been considerable interest in studying linear transformations on matrix spaces leaving a certain subset invariant, i.e., mapping the subset onto itself; see [4]. For example, if $\mathcal{S}$ is the set of $n \times n$ doubly (sub-)stochastic matrices or the set of $n \times n$ (sub-)permutation matrices, and $\mathcal{V} = \operatorname{span} \mathcal{S}$ is the linear span of $\mathcal{S}$, then a linear transformation $\phi : \mathcal{V} \to \mathcal{V}$ satisfying $\phi(\mathcal{S}) = \mathcal{S}$ must be of the form

$$X \mapsto PXQ \qquad \text{or} \qquad X \mapsto PX^tQ \tag{1}$$

for some $n \times n$ permutation matrices P and Q. In [2], the authors of this paper solved the open problem in [4] concerning linear transformations on $\operatorname{span} \mathcal{S}$ satisfying $\phi(\mathcal{S}) = \mathcal{S}$, where $\mathcal{S}$ is the set of $n \times n$ even permutation matrices, i.e., permutation matrices with determinant one. It was shown that for $n \geq 5$, the transformation has the form (1) for some permutation matrices P, Q such that $\det(PQ) = 1$. When $n \leq 4$, the transformation can have other forms, which were also characterized.

In [3] (see also [5]), the authors studied linear transformations ϕ satisfying $\phi(\mathcal{S}) = \mathcal{S}$, where $\mathcal{S}$ is the set of symmetric doubly stochastic matrices. It was shown that such transformations have the expected form

$$X \mapsto P^tXP \tag{2}$$

for some permutation matrix P.

[1]Current address: GSAS Mail Center, Child Hall 311, 26 Everett Street, Cambridge, MA 02138.
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In some earlier studies (see [6] and [1]), researchers considered $n \times n$ complex generalized doubly stochastic matrices, i.e., $n \times n$ complex matrices with all row sums and column sums equal to one, and characterized linear transformations mapping the set of such matrices into itself. In particular, they showed that such a transformation can be written as the sum of simple transformations of the form $X \mapsto MXN$ for some generalized doubly stochastic matrices M and N, together with a few other simple transformations. In this paper, we show that one can actually write such a transformation as a linear combination of transformations of the form $X \mapsto PXQ$ for some permutation matrices P and Q, together with a few other simple transformations. Our proofs work for both real and complex matrices (and matrices over many other fields). Moreover, similar problems for the generalized symmetric doubly stochastic matrices are also considered.

The following notations will be used in our discussion.
$\mathbf{M}_n$: the set of $n \times n$ real or complex matrices.
$\mathbf{GDS}_n$: the set of $n \times n$ generalized doubly stochastic matrices in $\mathbf{M}_n$.
$\mathbf{GSDS}_n$: the set of $n \times n$ generalized symmetric doubly stochastic matrices in $\mathbf{M}_n$.
$\mathbf{P}_n$: the set of $n \times n$ permutation matrices.
$\mathbf{V}_n$: the set of matrices in $\mathbf{M}_n$ with equal row sums and column sums.
$\mathbf{SV}_n$: the set of symmetric matrices in $\mathbf{M}_n$ with equal row sums and column sums.
I_n: the identity matrix in $\mathbf{M}_n$.
J_n: the matrix in $\mathbf{M}_n$ with all entries equal to one.
$\widetilde{J}_n = J_n - I_n$.
$\{e_1, \ldots, e_n\}$: the standard basis for $\mathbb{F}^n$.
$e = e_1 + \cdots + e_n$.
$E_{ij} = e_i {e_j}^t$.

2 Generalized Doubly Stochastic Matrices

It is easy to verify that the linear span of $\mathbf{GDS}_n$ is the space $\mathbf{V}_n$ of $n \times n$ matrices with equal row sums and column sums. Suppose Γ is the group of operators acting on $\mathbf{V}_n$ of the form $X \mapsto PXQ$ for some permutation matrices P and Q. Recall that a nonzero subspace $\mathbf{W}$ of $\mathbf{V}_n$ is *invariant* under Γ if $\phi(\mathbf{W}) \subseteq \mathbf{W}$ for all operators ϕ in Γ, and $\mathbf{W}$ is *irreducible* under Γ if $\mathbf{W}$ does not contain a proper invariant subspace under Γ. We have the following result.

Theorem 2.1 *Let $\mathbf{S}_1 = \langle J_n \rangle$ and $\mathbf{S}_2 = \{X \in \mathbf{V}_n : XJ_n = J_nX = 0\}$. Then $\mathbf{S}_1$ and $\mathbf{S}_2$ are irreducible invariant subspaces of $\mathbf{V}_n$ under the group Γ, and $\mathbf{V}_n = \mathbf{S}_1 \oplus \mathbf{S}_2$.*

Proof. Evidently, $\mathbf{S}_1 \cap \mathbf{S}_2 = \{0\}$. For any $X = (x_{ij}) \in \mathbf{V}_n$, let $\lambda = (e^tXe)/n^2$ and let $\widetilde{X} = X - \lambda J_n \in \mathbf{S}_2$. Then $X = \lambda J_n + \widetilde{X}$. Thus, $\mathbf{V}_n = \mathbf{S}_1 \oplus \mathbf{S}_2$.

It is clear that $\mathbf{S}_1$ and $\mathbf{S}_2$ are invariant subspaces under Γ and $\mathbf{S}_1$ is irreducible under Γ. To show that $\mathbf{S}_2$ is irreducible, suppose $\mathbf{W}$ is an invariant subspace of $\mathbf{S}_2$ under Γ containing a nonzero matrix $A = (a_{ij})$. We show that $\mathbf{S}_2 = \text{span}\{PAQ : P, Q \in \mathbf{P}_n\} \subseteq \mathbf{W}$. We may assume that $a_{11} \neq 0$; otherwise, we may replace A by PAQ for some suitable

$P, Q \in \mathbf{P}_n$ and assume that A has a nonzero entry in the $(1,1)$ position. Let $\widetilde{P} \in \mathbf{P}_n$ be the permutation matrix with 1 at the $(1,1), (2,n)$, and $(k, k-1)$ positions for $3 \le k \le n$. Suppose $B_1 = (\widetilde{P}A + \widetilde{P}^2 A + \cdots + \widetilde{P}^{n-1}A)/((n-1)a_{11}) \in \mathbf{W}$. By our construction and the fact that B_1 has column sums equal to zero, we see that each of the second through the nth rows of B_1 is equal to $-\mathbf{b}/(n-1)$, where $\mathbf{b} = (1, b_2, \ldots, b_n)$ is the first row of B_1. Let $B_2 = B_1\widetilde{P} + B_1\widetilde{P}^2 + \cdots + B_1\widetilde{P}^{n-1} \in \mathbf{W}$. Then

$$B_2 = \begin{pmatrix} n-1 & -1 \cdots -1 \\ -1 & \\ \vdots & (n-1)^{-1} J_{n-1} \\ -1 & \end{pmatrix}.$$

Next, let $\widetilde{Q} \in \mathbf{P}_n$ correspond to the permutation that interchanges 1 and 2. We construct $B_3, B_4 \in \mathbf{W}$ such that $B_3 = B_2 - \widetilde{Q}B_2$ and $B_4 = B_3 - B_3\widetilde{Q}$. Then

$$B_4 = [n + 1 + 1/(n-1)] \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \oplus 0_{n-2} \in \mathbf{W},$$

and hence

$$B_5 = (n + 1 + 1/(n-1))^{-1} B_4 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \oplus 0_{n-2} \in \mathbf{W}.$$

Finally, we show that span $\{PB_5Q : P, Q \in \mathbf{P}_n\} = \mathbf{S}_2$. To this end, let $X = (x_{ij}) \in \mathbf{S}_2$. Then X is uniquely determined by x_{ij} with $1 \le i, j \le n-1$. For $1 \le i, j \le n-1$, let $C_{ij} \in \mathbf{S}_2$ be the matrix with 1 at the (i,j) and (n,n) positions, with -1 at the (i,n) and (n,j) positions, and with zero elsewhere. Evidently, for each C_{ij}, we may find $P, Q \in \mathbf{P}_n$ such that $PB_5Q = C_{ij}$. Now, $X = \sum_{1 \le i,j \le n-1} x_{ij} C_{ij}$. It follows that

$$\text{span}\,\{PAQ : P, Q \in \mathbf{P}_n\} = \text{span}\,\{PB_5Q : P, Q \in \mathbf{P}_n\} = \mathbf{S}_2,$$

and hence $\mathbf{S}_2$ is irreducible under Γ. ■

Theorem 2.2 *Define $\mathbf{S}_1$ and $\mathbf{S}_2$ as in Theorem 2.1. There exist $P_1, \ldots, P_m, Q_1, \ldots, Q_m \in \mathbf{P}_n$ with $m = (n-1)^4$ such that every linear map $\psi : \mathbf{S}_2 \to \mathbf{S}_2$ has the form*

$$X \mapsto \sum_{j=1}^{m} \alpha_j P_j X Q_j$$

for some $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$. Consequently, if $\phi : \mathbf{V}_n \to \mathbf{V}_n$ is linear and satisfies $\phi(\mathbf{GDS}_n) \subseteq \mathbf{GDS}_n$, then ϕ has the form

$$X \mapsto \sum_{j=1}^{m} \gamma_j P_j X Q_j + \frac{e^t X e}{n} Z_0$$

where $\gamma_1, \ldots, \gamma_m \in \mathbb{F}$, and $Z_0 \in \mathbf{V}_n$ has row sums and column sums equal to $(1 - \sum_{j=1}^{m} \gamma_j)$.

Proof. Since $\mathbf{S}_2$ is an irreducible invariant subspace of $\mathbf{V}_n$ under Γ, by Wedderburn's theorem (see [7, Section 13.11]), span Γ equals $\mathrm{End}(\mathbf{S}_2)$ – the algebra of linear transformations on $\mathbf{S}_2$ in the complex case. It follows that Γ contains a basis for the vector space $\mathrm{End}(\mathbf{S}_2)$ over $\mathbb{C}$, which will also be a basis for $\mathrm{End}(\mathbf{S}_2)$ over $\mathbb{R}$; thus, the span of Γ equals $\mathrm{End}(\mathbf{S}_2)$ over $\mathbb{R}$ as well. As a result, in both the real and complex cases, there exist $P_1, \ldots, P_m, Q_1, \ldots, Q_m \in \mathbf{P}_n$ with $m = (n-1)^4$, which is the dimension of $\mathrm{End}(\mathbf{S}_2)$, such that every linear map $\psi : \mathbf{S}_2 \to \mathbf{S}_2$ has the form

$$X \mapsto \sum_{j=1}^{m} \alpha_j P_j X Q_j$$

for some $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$.

Now, suppose $\phi : \mathbf{V}_n \to \mathbf{V}_n$ is linear and satisfies $\phi(\mathbf{GDS}_n) \subseteq \mathbf{GDS}_n$. Let $\phi(J_n/n) = Y_0 \in \mathbf{GDS}_n$. For any $X \in \mathbf{V}_n$, we have

$$\phi\left(X - \left(\frac{e^tXe}{n} - 1\right)\frac{J_n}{n}\right) = \phi(X) - \left(\frac{e^tXe}{n} - 1\right) Y_0 \in \mathbf{GDS}_n,$$

implying that $\phi(X)$ has row sums and column sums equal to $(e^tXe)/n$. Thus, ϕ preserves row sums and column sums, and, in particular, $\phi(\mathbf{S}_2) \subseteq \mathbf{S}_2$. For any $X \in \mathbf{S}_2$, let $\psi : \mathbf{S}_2 \to \mathbf{S}_2$ be defined by $\psi(X) = \phi(X)$. Then

$$\psi(X) = \sum_{j=1}^{m} \gamma_j P_j X Q_j$$

for some $\gamma_1, \ldots, \gamma_m \in \mathbb{F}$ and $P_1, \ldots, P_m, Q_1, \ldots, Q_m \in \mathbf{P}_n$. Thus, for any $X \in \mathbf{V}_n$, we have

$$\begin{aligned} \phi(X) &= \psi\left(X - \left(\frac{e^tXe}{n}\right)\frac{J_n}{n}\right) + \phi\left(\left(\frac{e^tXe}{n}\right)\frac{J_n}{n}\right) \\ &= \sum_{j=1}^{m} \gamma_j P_j \left[X - \left(\frac{e^tXe}{n}\right)\frac{J_n}{n}\right] Q_j + \frac{e^tXe}{n} Y_0 \\ &= \sum_{j=1}^{m} \gamma_j P_j X Q_j + \frac{e^tXe}{n} Y_0 - \left(\sum_{j=1}^{m} \gamma_j\right)\left(\frac{e^tXe}{n}\right)\frac{J_n}{n}. \end{aligned}$$

Let $Z_0 = Y_0 - (\sum_{j=1}^{m} \gamma_j) J_n/n$. Then ϕ has the asserted form. ∎

In the above theorem, we use the mappings in Γ as the basic building blocks of linear transformations from $\mathbf{V}_n$ to $\mathbf{V}_n$ that preserve $\mathbf{GDS}_n$. One may also consider using operators of the form $X \mapsto RXS$ with $R, S \in \mathbf{GDS}_n$ as the basic building blocks. In such case, we can assume that the linear coefficients $\gamma_1 = \cdots = \gamma_m = 1$.

Corollary 2.3 *Suppose $\phi : \mathbf{V}_n \to \mathbf{V}_n$ is linear and satisfies $\phi(\mathbf{GDS}_n) \subseteq \mathbf{GDS}_n$. Then ϕ has the form*

$$X \mapsto \sum_{j=1}^{k} R_j X S_j + \frac{e^t X e}{n} Z_0 \tag{3}$$

where $k \leq 2(n-1)^4$, $Z_0 \in \mathbf{V}_n$ has row sums and column sums equal to $(1-k)$, and $R_1, \ldots, R_k, S_1, \ldots, S_k \in \mathbf{GDS}_n$.

Proof. By Theorem 2.2, ϕ has the form

$$X \mapsto \sum_{j=1}^{m} \gamma_j P_j X Q_j + \frac{e^t X e}{n} \left\{ Y_0 - \left(\sum_{j=1}^{m} \gamma_j \right) \frac{J_n}{n} \right\} \tag{4}$$

where $m = (n-1)^4$, $Y_0 \in \mathbf{GDS}_n$, $P_1, \ldots, P_m, Q_1, \ldots, Q_m \in \mathbf{P}_n$, and $\gamma_1, \ldots, \gamma_m \in \mathbb{F}$. Let $R_j = \gamma_j P_j + (1-\gamma_j) J_n/n$ and $S_j = Q_j$ for $j = 1, \ldots, m$. Furthermore, let $R_{m+j} = J_n/n$ and $S_{m+j} = (\gamma_j - 1) Q_j + (2-\gamma_j) J_n/n$ for $j = 1, \ldots, m$. Then $R_1, \ldots, R_{2m}, S_1, \ldots, S_{2m} \in \mathbf{GDS}_n$, and

$$\begin{aligned}
\sum_{j=1}^{m} \gamma_j P_j X Q_j &= \sum_{j=1}^{m} R_j X S_j + \sum_{j=1}^{m} (\gamma_j - 1)(J_n/n) X Q_j \\
&= \sum_{j=1}^{2m} R_j X S_j + \sum_{j=1}^{m} (\gamma_j - 2) J_n X J_n / n^2 \\
&= \sum_{j=1}^{2m} R_j X S_j + \left(\sum_{j=1}^{m} \gamma_j - 2m \right) \frac{e^t X e}{n} \frac{J_n}{n}.
\end{aligned}$$

Thus, ϕ has the form

$$X \mapsto \sum_{j=1}^{2m} R_j X S_j + \frac{e^t X e}{n} \left\{ Y_0 - 2m \frac{J_n}{n} \right\}.$$

Let $Z_0 = Y_0 - 2m(J_n/n)$. Then ϕ has the form (3) with $k = 2m$. Note that if some γ_j in (4) equals 1, then $R_{m+j} X S_{m+j} = J_n X J_n / n^2$ and can be absorbed in the matrix Z_0. Thus, ϕ has the form (3) with $k \leq 2m$ in general. ∎

One can further extend the results of Theorem 2.2 and Corollary 2.3 to linear operators $\phi : \mathbf{M}_n \to \mathbf{M}_n$ satisfying $\phi(\mathbf{GDS}_n) \subseteq \mathbf{GDS}_n$. In such case, for every $A \in \mathbf{M}_n$ we have $\phi(A) = \phi_1(A) + \phi_2(A)$, where

$$\phi_1(A) = \phi(A - J_n A/n - A J_n/n) \quad \text{and} \quad \phi_2(A) = \phi(J_n A/n + A J_n/n).$$

One can check that $\phi_1(\mathbf{V}_n) \subseteq \mathbf{V}_n$ and $\phi_1(\mathbf{GDS}_n) \subseteq \mathbf{GDS}_n$. Thus, one can apply Theorem 2.2 or Corollary 2.3 to ϕ_1 and then deal with ϕ_2 separately. In [6], the author assumed that $\phi(\mathbf{GDS}_n) \subseteq \phi(\mathbf{GDS}_n)$ and the adjoint transformation $\phi^* : \mathbf{M}_n \to \mathbf{M}_n$ also satisfies $\phi^*(\mathbf{GDS}_n) \subseteq \phi^*(\mathbf{GDS}_n)$; it follows that $\phi_2(\mathbf{V}_n^\perp) \subseteq \mathbf{V}_n^\perp$ and $\phi(J_n) = J_n$, and ϕ has simple structures. In [1], the author used mappings of the form $X \mapsto RXS$ with R and S from a wider class of matrices to be the basic building blocks of ϕ and obtained other results.

3 Generalized Symmetric Doubly Stochastic Matrices

Recall that $\mathbf{SV}_n$ is the set of symmetric matrices in $\mathbf{M}_n$ with equal row sums and column sums, and $\mathbf{GSDS}_n$ is the set of generalized symmetric doubly stochastic matrices in $\mathbf{M}_n$. From Corollary 2.2 of [3], it can be deduced that $\operatorname{span}\mathbf{GSDS}_n = \mathbf{SV}_n$. Let $\tilde{\Gamma}$ denote the group of operators acting on $\mathbf{SV}_n$ of the form $X \mapsto PXP^t$ with $P \in \mathbf{P}_n$. We study irreducible invariant subspaces of $\mathbf{SV}_n$ under $\tilde{\Gamma}$. It turns out that there are four such subspaces if $n \geq 4$, and the mappings in $\tilde{\Gamma}$ are the basic building blocks of linear operators on these subspaces.

Theorem 3.1 *Suppose $n \geq 4$, and $\mathbf{T}_1 = \langle I_n \rangle$, $\mathbf{T}_2 = \langle \tilde{J}_n \rangle$, $\mathbf{T}_3 = \operatorname{span}\{PR_1P^t : P \in \mathbf{P}_n\}$, and $\mathbf{T}_4 = \operatorname{span}\{PR_2P^t : P \in \mathbf{P}_n\}$, where*

$$R_1 = \begin{pmatrix} n-1 & -1 \cdots -1 \\ -1 & \\ \vdots & \frac{2}{n-2}\tilde{J}_{n-1} - I_{n-1} \\ -1 & \end{pmatrix} \quad \textit{and} \quad R_2 = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} \oplus O_{n-4}.$$

Then $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$, and $\mathbf{T}_4$ are irreducible invariant subspaces of $\mathbf{SV}_n$ under the group $\tilde{\Gamma}$, and $\mathbf{SV}_n = \mathbf{T}_1 \oplus \mathbf{T}_2 \oplus \mathbf{T}_3 \oplus \mathbf{T}_4$.

Proof. It is clear that each of $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$, and $\mathbf{T}_4$ is invariant under $\tilde{\Gamma}$, and $\mathbf{T}_1$ and $\mathbf{T}_2$ are irreducible subspaces under $\tilde{\Gamma}$.

We next show that $\mathbf{T}_3$ has dimension $n-1$ and is irreducible under $\tilde{\Gamma}$. Let $S \in \mathbf{P}_n$ be the permutation matrix with 1 at the $(n,1)$ and $(k,k+1)$ positions for $1 \leq k \leq n-1$, and let $D_j = (S^t)^{j-1}R_1S^{j-1}$ for $j = 1, \ldots, n$. Then the jth row of D_j equals

$$(-1, \ldots, -1, \underbrace{n-1}_{j\text{th position}}, -1, \ldots, -1)$$

and deleting the jth row and column of D_j results in the matrix $2\tilde{J}_{n-1}/(n-2) - I_{n-1}$. Evidently, $\{PR_1P^t : P \in \mathbf{P}_n\} = \{D_1, \ldots, D_n\}$, and $D_1 + \cdots + D_n = 0$. Furthermore, suppose $\sum_{i=1}^{n-1} d_iD_i = 0_n$. Equating the diagonal entries of both sides of the preceding equation, we have

$$(d_1, \ldots, d_{n-1})((n-1)I_{n-1} - \tilde{J}_{n-1}) = 0_{1,n-1}.$$

Since $(n-1)I_{n-1} - \tilde{J}_{n-1}$ is invertible, $d_1 = \cdots = d_{n-1} = 0$. Thus, $\{D_1, \ldots, D_{n-1}\}$ is linearly independent, and $\mathbf{T}_3 = \operatorname{span}\{D_1, \ldots, D_n\}$ has dimension $n-1$.

Suppose $\mathbf{W}$ is an invariant subspace of $\mathbf{T}_3$ under $\tilde{\Gamma}$ containing a nonzero matrix $A = (a_{ij})$. We show that $\mathbf{T}_3 = \operatorname{span}\{PAP^t : P \in \mathbf{P}_n\} \subseteq \mathbf{W}$. Clearly $AJ_n = J_nA = 0$ and $\operatorname{tr} A = 0$. By the arguments in the preceding paragraph, we see that $A = b_1D_1 + \cdots + b_{n-1}D_{n-1}$. Furthermore, A has at least one nonzero diagonal entry; otherwise, all of the diagonal entries of $A = b_1D_1 + \cdots + b_{n-1}D_{n-1}$ are equal to zero, and thus $b_1 = \cdots = b_{n-1} = 0$, which is a contradiction. We may assume that $a_{11} \neq 0$; otherwise, we replace A by PAP^t for some suitable $P \in \mathbf{P}_n$. Now, define

$$\widetilde{B}_A = \sum_{P\in\mathbf{P}_{n-1}} \begin{pmatrix} 1 & 0_{1,n-1} \\ 0_{n-1,1} & P \end{pmatrix} A \begin{pmatrix} 1 & 0_{1,n-1} \\ 0_{n-1,1} & P^t \end{pmatrix} \in \mathbf{W}.$$

Let $\widetilde{B}_A = \begin{pmatrix} \gamma & x^t \\ x & B_A \end{pmatrix}$ where $x \in \mathbb{F}^{n-1}$ and $B_A \in \mathbf{M}_{n-1}$. Then $\gamma = (n-1)!a_{11}$. Also, for any $Q \in \mathbf{P}_{n-1}$, let

$$\widetilde{Q} = \begin{pmatrix} 1 & 0_{1,n-1} \\ 0_{n-1,1} & Q \end{pmatrix}.$$

Then we have

$$\begin{aligned} \begin{pmatrix} \gamma & x^tQ^t \\ Qx & QB_AQ^t \end{pmatrix} &= \widetilde{Q} \begin{pmatrix} \gamma & x^t \\ x & B_A \end{pmatrix} \widetilde{Q}^t \\ &= \widetilde{Q}\Big[\sum_{P\in\mathbf{P}_{n-1}} \begin{pmatrix} 1 & 0_{1,n-1} \\ 0_{n-1,1} & P \end{pmatrix} A \begin{pmatrix} 1 & 0_{1,n-1} \\ 0_{n-1,1} & P^t \end{pmatrix}\Big]\widetilde{Q}^t \\ &= \sum_{P\in\mathbf{P}_{n-1}} \begin{pmatrix} 1 & 0_{1,n-1} \\ 0_{n-1,1} & P \end{pmatrix} A \begin{pmatrix} 1 & 0_{1,n-1} \\ 0_{n-1,1} & P^t \end{pmatrix} \\ &= \begin{pmatrix} \gamma & x^t \\ x & B_A \end{pmatrix}. \end{aligned}$$

Thus, $QB_AQ^t = B_A$ for all $Q \in \mathbf{P}_{n-1}$, implying that $B_A = \alpha I_{n-1} + \beta \widetilde{J}_{n-1}$ for some $\alpha, \beta \in \mathbb{F}$. Moreover, $Qx = x$ for all $Q \in \mathbf{P}_{n-1}$, implying that $x = \delta e$ where $\delta \in \mathbb{F}$ and $e \in \mathbb{F}^{n-1}$ is a vector of ones. By the above construction, and by the fact that the row sums, column sums, and trace of $\widetilde{B}_A \in \mathbf{T}_3$ are all zero, we have

$$\frac{1}{a_{11}(n-2)!}\widetilde{B}_A = \begin{pmatrix} n-1 & -1\cdots-1 \\ -1 & \\ \vdots & \frac{2}{n-2}\tilde{J}_{n-1} - I_{n-1} \\ -1 & \end{pmatrix} = R_1.$$

That is, $R_1 \in \operatorname{span}\{PAP^t : P \in \mathbf{P}_n\}$. It follows that $\operatorname{span}\{PAP^t : P \in \mathbf{P}_n\} = \mathbf{T}_3$, and hence $\mathbf{T}_3$ is irreducible under $\tilde{\Gamma}$.

Denote by $\widetilde{\mathbf{T}}_4$ the set of matrices in $\mathbf{SV}_n$ with row sums, column sums, and diagonal entries equal to zero. We show that $\mathbf{T}_4 = \widetilde{\mathbf{T}}_4$ has dimension $n(n-3)/2$ as follows.

For $1 \le i \le n-3$ and $i+1 \le j \le n-2$, define G_{ij} to be the matrix with 1 at the $(i,j),(j,i),(n-1,n)$, and $(n,n-1)$ positions, with -1 at the $(i,n),(n,i),(j,n-1)$, and $(n-1,j)$ positions, and with zero elsewhere. Moreover, for $1 \le i \le n-3$, define $G_{i,n-1}$ to be the matrix with 1 at the $(i,n-1),(n-1,i),(n-2,n)$, and $(n,n-2)$ positions, with -1 at the $(i,n),(n,i),(n-2,n-1)$, and $(n-1,n-2)$ positions, and with zero elsewhere. For example, for $n=4$, we have

$$G_{12} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \qquad G_{13} = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix};$$

for $n=5$, the three matrices in the first part of the construction are

$$G_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 \end{pmatrix}, \; G_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$G_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \end{pmatrix},$$

and the two matrices in the second part of the construction are

$$G_{14} = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{pmatrix}, \qquad G_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{pmatrix}.$$

Then each G_{ij} is permutationally similar to R_2. Let $\mathbf{G}_n$ be the collection of such G_{ij}. We have

$$\mathbf{G}_n \subseteq \mathbf{T}_4 \subseteq \tilde{\mathbf{T}}_4.$$

Evidently, $\mathbf{G}_n$ is linearly independent. Furthermore, if $X = (x_{ij}) \in \tilde{\mathbf{T}}_4$, then X is fully determined by the entries x_{ij} with $1 \le i \le n-3$ and $i+1 \le j \le n-1$, and we have

$$X = \left(\sum_{i=1}^{n-3} \sum_{j=i+1}^{n-2} x_{ij} G_{ij} \right) + x_{1,n-1} G_{1,n-1} + \sum_{i=2}^{n-3} \left(x_{i,n-1} + \sum_{k=1}^{i-1} x_{ki} \right) G_{i,n-1}.$$

Hence, $\mathbf{G}_n$ with $n(n-3)/2$ elements is a basis for $\widetilde{\mathbf{T}}_4 = \mathbf{T}_4$.

Next, we show that $\mathbf{T}_4$ is irreducible under $\tilde{\Gamma}$. Suppose $\mathbf{W}$ is an invariant subspace of $\mathbf{T}_4$ under $\tilde{\Gamma}$ containing a nonzero matrix $A = (a_{ij})$. We show that $\mathbf{T}_4 = \operatorname{span}\{PAP^t : P \in \mathbf{P}_n\} \subseteq \mathbf{W}$. Denote by P_{ij} the permutation matrix corresponding to the transposition permutation interchanging i and j. Since A is a nonzero matrix, it must have at least one positive and one negative off-diagonal entry. We may assume that $a_{12} > 0$; otherwise, we may replace A by PAP^t for a suitable $P \in \mathbf{P}_n$ and assume that A has a positive entry in the $(1, 2)$ position. Since $a_{12} > 0$, there must exist a negative entry in the second column of A below the diagonal. We may assume that $a_{32} < 0$; otherwise, we have $a_{k2} < 0$ for some $k > 3$, so we may replace A by $P_{3k}AP_{3k}^t$ and assume that A has a negative entry in the $(3, 2)$ position.

Let $A - P_{13}AP_{13}^t = B = (b_{ij})$. Then $B \in \mathbf{W}$, $b_{12} = a_{12} - a_{32} > 0$, and $b_{13} = a_{13} - a_{31} = 0$. Since there must exist a negative entry in the first row of B, we may assume that $b_{14} < 0$; otherwise, we have $b_{1k} < 0$ for some $k > 4$, so we may replace B by $P_{4k}BP_{4k}^t$ and assume that B has a negative entry in the $(1, 4)$ position. Define $C = (b_{12} - b_{14})^{-1}(B - P_{24}BP_{24}^t) \in \mathbf{W}$. Then

$$C = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix} \oplus 0_{n-4}.$$

Since $R_2 = P_{23}CP_{23}^t \in \operatorname{span}\{PAP^t : P \in \mathbf{P}_n\}$, it follows that $\operatorname{span}\{PAP^t : P \in \mathbf{P}_n\} = \mathbf{T}_4$, and hence $\mathbf{T}_4$ is irreducible under $\tilde{\Gamma}$.

It remains to show that $\mathbf{SV}_n = \mathbf{T}_1 \oplus \mathbf{T}_2 \oplus \mathbf{T}_3 \oplus \mathbf{T}_4$. For any $A \in \mathbf{SV}_n$, let $\lambda = (\sum_{i=1}^n a_{ii})/n = (\operatorname{tr} A)/n$, $\delta = (\sum_{i\neq j} a_{ij})/(n(n-1)) = (\operatorname{tr} \widetilde{J}_n X)/(n(n-1))$, and $B = A - \lambda I_n - \delta \widetilde{J}_n$. Then the row sums, column sums, and trace of $B = (b_{ij})$ are equal to zero. Recall that $\{PR_1P^t : P \in \mathbf{P}_n\} = \{D_1, \ldots, D_n\}$ is a spanning set for $\mathbf{T}_3$, where D_i has $n-1$ in the (i, i) position. Let $\widetilde{B} = \sum_{i=1}^n n^{-1} b_{ii} D_i \in \mathbf{T}_3$. Then $C = B - \widetilde{B} = A - \lambda I_n - \delta \widetilde{J}_n - \widetilde{B}$ has row sums, column sums, and diagonal entries all equal to zero, so $C \in \mathbf{T}_4$. Thus, $A = \lambda I_n + \delta \widetilde{J}_n + \widetilde{B} + C$, showing that $\mathbf{SV}_n = \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 + \mathbf{T}_4$.

Now, $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$ and $\mathbf{T}_4$ have dimensions $1, 1, n-1$, and $n(n-3)/2$ respectively, summing up to $1 + n(n-1)/2$, which is the dimension of $\mathbf{SV}_n$. Thus $\mathbf{SV}_n = \mathbf{T}_1 \oplus \mathbf{T}_2 \oplus \mathbf{T}_3 \oplus \mathbf{T}_4$. ■

By Theorem 3.1, we can use arguments similar to those in the proof of Theorem 2.2 to show that any linear map on $\mathbf{T}_4$ has a simple structure, namely, $X \mapsto \sum_i \gamma_i P_i X P_i^t$. One may want to prove results similar to Theorem 2.2 for linear maps $\phi : \mathbf{SV}_n \to \mathbf{SV}_n$ satisfying $\phi(\mathbf{GSDS}_n) \subseteq \mathbf{GSDS}_n$. Such a mapping will have a simple structure if $\phi(\mathbf{T}_4) \subseteq \mathbf{T}_4$; however, this condition is not always satisfied, as shown in the following example.

Example 3.2 For any $X \in \mathbf{SV}_4$, write $X = X_1 + X_2$ with $X_1 \in (\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3)$ and $X_2 = (x_{ij}) \in \mathbf{T}_4$, and define $\phi : \mathbf{SV}_4 \to \mathbf{SV}_4$ by

$$\phi(X) = X_1 + x_{12}(E_{11} + E_{22} - E_{12} - E_{21}).$$

Then ϕ is linear and satisfies $\phi(\mathbf{GSDS}_4) \subseteq \mathbf{GSDS}_4$, but $\phi(\mathbf{T}_4) \not\subseteq \mathbf{T}_4$.

So, we need an extra assumption on ϕ to obtain a simple structure for the transformation.

Theorem 3.3 *Define $\mathbf{T}_1, \ldots, \mathbf{T}_4$ as in Theorem 3.1. There exist $P_1, \ldots, P_m \in \mathbf{P}_n$ with $m = [n(n-3)/2]^2$ such that every linear map $\psi : \mathbf{T}_4 \to \mathbf{T}_4$ has the form*

$$X \mapsto \sum_{j=1}^{m} \alpha_j P_j X P_j^t$$

for some $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$. Suppose $\phi : \mathbf{SV}_n \to \mathbf{SV}_n$ is linear and satisfies $\phi(\mathbf{GSDS}_n) \subseteq \mathbf{GSDS}_n$ and $\phi(\mathbf{H}_n) \subseteq \mathbf{H}_n$, where $\mathbf{H}_n$ is the set of matrices in $\mathbf{SV}_n$ with all diagonal entries equal to zero. Then ϕ has the form

$$X \mapsto \sum_{j=1}^{m} \gamma_j P_j X P_j^t + f_0(X) Y_0 + f_1(X) Y_1 + \sum_{r=1}^{n} g_r(X) Z_r,$$

where $\gamma_1, \ldots, \gamma_m \in \mathbb{F}$, $f_0(X) = \operatorname{tr} X/n$, $f_1(X) = \operatorname{tr}(\tilde{J}_n X)/n$, $g_r(X) = \operatorname{tr}(E_{rr} X)$, $Y_0, Y_1 \in \mathbf{SV}_n$ have row sums and column sums equal to $(1 - \sum_{j=1}^{m} \gamma_j)$, and $Z_1, \ldots, Z_n \in \mathbf{SV}_n$ have row sums and column sums equal to zero with $Z_1 + \cdots + Z_n = 0$.

Proof. The first assertion follows from the facts that $\mathbf{T}_4$ is an irrecucible invariant subspace of $\mathbf{SV}_n$ under $\tilde{\Gamma}$ and $\operatorname{End}(\mathbf{T}_4)$ has dimension $[n(n-3)/2]^2$.

Now, suppose $\phi : \mathbf{SV}_n \to \mathbf{SV}_n$ is linear such that $\phi(\mathbf{GSDS}_n) \subseteq \mathbf{GSDS}_n$ and $\phi(\mathbf{H}_n) \subseteq \mathbf{H}_n$. By arguments analogous to those used in the proof of Theorem 2.2, ϕ preserves row sums and column sums. Since $\phi(\mathbf{H}_n) \subseteq \mathbf{H}_n$, it follows that $\phi(\mathbf{T}_4) \subseteq \mathbf{T}_4$.

Now, define $D_1, \ldots, D_n$ as in the proof of Theorem 3.1. For each $X \in \mathbf{SV}_n$, write

$$\psi(X) = X - f_0(X) I - f_1(X) \tilde{J}/(n-1) - \sum_{r=1}^{n} \tilde{g}_r(X) D_r \in \mathbf{T}_4$$

with $\tilde{g}_r(X) = \operatorname{tr}(E_{rr} X)/n - \operatorname{tr} X/n^2$ for $r = 1, \ldots, n$. Then

$$\phi(X) = \phi(\psi(X)) + f_0(X)\phi(I) + f_1(X)\phi(\tilde{J}/(n-1)) + \sum_{r=1}^{n} \tilde{g}_r(X)\phi(D_r).$$

Since $\psi(\mathbf{SV}_n) \subseteq \mathbf{T}_4$ and $\phi(\mathbf{T}_4) \subseteq \mathbf{T}_4$, there are $\gamma_1, \ldots, \gamma_m \in \mathbb{F}$ such that

$$\phi(\psi(X)) = \sum_{j=1}^{m} \gamma_j P_j \psi(X) P_j^t.$$

Let

$$Y_0 = \phi(I) - \sum_{j=1}^{m} \gamma_j I, \quad Y_1 = \frac{1}{n-1} \left\{ \phi(\tilde{J}) - \sum_{j=1}^{m} \gamma_j \tilde{J} \right\},$$

and

$$Z_r = \frac{1}{n}\left\{\phi(D_r) - \sum_{j=1}^{m} \gamma_j P_j D_r P_j^t\right\}$$

for $r = 1, \ldots, n$. Then ϕ has the asserted form. ■

One may deduce a result similar to that of Corollary 2.3. We omit the discussion. We conclude the paper with the results on the low dimension cases.

For $n = 2$, $\mathbf{SV}_2 = \mathbf{T}_1 \oplus \mathbf{T}_2$, and linear preservers of $\mathbf{GSDS}_2$ have the form

$$(x_{ij}) \mapsto x_{11}Y_1 + x_{12}Y_2$$

for some $Y_1, Y_2 \in \mathbf{GSDS}_2$. For $n = 3$, $\mathbf{SV}_3 = \mathbf{T}_1 \oplus \mathbf{T}_2 \oplus \mathbf{T}_3$, and linear preservers of $\mathbf{GSDS}_3$ have the form

$$X \mapsto (\mathrm{tr}\, X)Y_0/3 + (\mathrm{tr}\, X\tilde{J}_3)Y_1/3 + \sum_{j=1}^{3} \mathrm{tr}\,(E_{jj}X)Z_j,$$

where $Y_0, Y_1 \in \mathbf{GSDS}_3$, and $Z_1, Z_2, Z_3 \in \mathbf{SV}_3$ have row sums and column sums equal to zero with $Z_1 + Z_2 + Z_3 = 0$.

References

[1] D. Benson, A characterization of linear transformations which leave the doubly stochastic matrices invariant, *Linear and Multilinear Algebra* 6 (1978), 65-72.

[2] H. Chiang and C.K. Li, Linear maps leaving the alternating group invariant, *Linear Algebra Appl.* 340 (2002), 69-80.

[3] H. Chiang and C.K. Li, Linear maps leaving invariant subsets of nonnegative symmetric matrices, *Bulletin of Australian Math. Soc.* 68 (2003), 221-231.

[4] C.K. Li, B.S. Tam, N.K. Tsing, Linear maps preserving permutation and stochastic matrices, *Linear Algebra Appl.* 341 (2002), 5-22.

[5] S.H. Lin and B.S. Tam. Strong linear preservers of symmetric doubly stochastic or doubly biostatistic matrices, preprint.

[6] R. Sinkhorn, Linear transformations under which the doubly stochastic matrices are invariant, *Proc. Amer. Math. Soc.* 27 (1971), 213-221.

[7] B.L. van der Waerden, Algebra. Vol. II. Based in part on lectures by E. Artin and E. Noether. Translated from the fifth German edition by John R. Schulenberger. Springer-Verlag, New York, 1991.