LINEAR PRESERVERS AND QUANTUM INFORMATION SCIENCE

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Dedicated to Professor Pjek-Hwee Lee on the occasion of his retirement.

Abstract. In this paper, a brief survey of recent results on linear preserver problems and quantum information science is given. In addition, characterization is obtained for linear operators ϕ on $mn \times mn$ Hermitian matrices such that $\phi(A \otimes B)$ and $A \otimes B$ have the same spectrum for any $m \times m$ Hermitian A and $n \times n$ Hermitian B. Such a map has the form $A \otimes B \mapsto U(\varphi_1(A) \otimes \varphi_2(B))U^*$ for $mn \times mn$ Hermitian matrices in tensor form $A \otimes B$, where U is a unitary matrix, and for $j \in \{1, 2\}$, φ_j is the identity map $X \mapsto X$ or the transposition map $X \mapsto X^t$. The structure of linear maps leaving invariant the spectral radius of matrices in tensor form $A \otimes B$ is also obtained. The results are connected bipartite (quantum) systems and are extended to multipartite systems.

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1. Introduction

The study of linear preserver problems has a long history. It concerns the characterization of linear maps on matrices or operators with special properties. For example, Frobenius [7] showed that a linear operator $\phi : M_n \to M_n$ satisfies

$$\det(\phi(A)) = \det(A) \qquad \text{for all } A \in M_n$$

if and only if there are $M, N \in M_n$ with $\det(MN) = 1$ such that ϕ has the form

$$A \mapsto MAN \quad \text{or} \quad A \mapsto MA^tN, \tag{1}$$

where M_n denotes the set of $n \times n$ complex matrices. Clearly, a map of the form (1) is linear and leaves the determinant function invariant. It is interesting that a linear map preserving the determinant function must be of this form. In [4] Dieudonné showed that an invertible linear operator $\phi : M_n \to M_n$ maps the set of singular matrices into itself if and only if there are invertible $M, N \in M_n$ such that ϕ has the form (1). One may see [16] and its references for results on linear preserver problems. There are many new directions and active research on preserver problems motivated by theory and applications; see [1, 27, 34].

In this paper, we focus on linear preserver problems related to quantum information science. In Section 2, we briefly survey some recent results on such research, and motivate our study in Section 3, in which we characterize linear preservers of the spectral radius or the spectrum of the tensor product of two Hermitian matrices, and discuss the implications of the result to bipartite quantum systems. The results are extended to the tensor product of m Hermitian matrices with $m > 2$ corresponding to the multipartite quantum systems. Additional remarks, results and open problems are also presented.

2. Quantum information science and preservers

Let H_n be the set of Hermitian matrices in M_n. In quantum physics, *quantum states* of a system with n physical states are represented as density matrices A in H_n, i.e., A is positive semi-definite with trace one. Rank one orthogonal projections are pure states.

The classical Wigner's theorem in quantum mechanics asserts that a bijective map ϕ on the set of pure states satisfying $\mathrm{tr}(AB) = \mathrm{tr}(\phi(A)\phi(B))$ must be of the form

$$A \mapsto UAU^* \qquad \text{or} \qquad A \mapsto UA^tU^* \tag{2}$$

for some unitary operator U. Uhlhorn [33] showed that a bijective map ϕ on the set of pure states also has the form (2) under the weaker assumption that $\mathrm{tr}(AB) = 0$ if and only if $\mathrm{tr}\phi(A)\phi(B)) = 0$. The result was extended to Hilbert modules over matrix algebras, prime C*-algebras, and indefinite inner product spaces; see [22, 25]. In [17], the authors extended Uhlhorn's result to Hermitian matrices, symmetric matrices, the set of orthogonal projections, the set of rank one orthogonal projections, and the set of effect algebra, and studied bijective maps on these matrix sets such that

$$\mathrm{tr}(AB) = c \qquad \text{if and only if} \qquad \mathrm{tr}(\phi(A)\phi(B)) = c$$

for a given $c > 0$.

In a series of interesting papers [23, 24, 25, 26, 28], Molnár and his collaborators characterized bijective maps on the set of complex matrices, Hermitian matrices, bounded observables, effect algebra, etc. preserving special subsets or relations. In many cases, the map has the form (2). One may see also [27] for additional results along this direction.

Suppose $A \in H_m$ and $B \in H_n$ are the states of two quantum systems. Then the *tensor (Kronecker) state* $A \otimes B \in H_{mn}$ describes the joint (bipartite) system. A density matrix $C \in H_{mn}$ is *separable* if it is the convex combination of tensor states, i.e., $C = \sum_{j=1}^{r} t_j A_j \otimes B_j$ for some positive numbers $t_1, \ldots, t_r$ summing up to one, and tensor states $A_1 \otimes B_1, \ldots, A_r \otimes B_r$. Otherwise, C is *entangled*. Identifying separable states in H_{mn} is an NP-hard problem; see [8]. Nevertheless, there is of interest in finding easy ways to check necessary or sufficient conditions of separability of states. In particular, it is interesting to find transformations which will simplify a given state so that it is easier to determine whether it is separable or not. Evidently, the transformations used should not change the set of separable states. This leads to the study of linear operators leaving invariant the set of separable states (entangled states). Similar definitions and questions can be considered for multipartite systems. The following result was proved in [6].

Theorem 2.1. *Let $n_1, \ldots, n_m \in \{2, 3, \ldots\}$ and $N = \prod_{j=1}^{m} n_j$. Suppose $\mathcal{S}$ is one of the following.*

(a) *The set of tensor product (of pure) states $A_1 \otimes \cdots \otimes A_m$, where $A_j \in H_{n_j}$ is a (pure) state for each $j \in \{1, \ldots, m\}$.*

(b) *The set of separable states in H_N, viz, the convex hull of the set of tensor product (of pure) states.*

Then a linear map $\phi : H_N \to H_N$ satisfies $\phi(\mathcal{S}) = \mathcal{S}$ if and only if there is a permutation $(p_1, \ldots, p_m)$ of $(1, \ldots, m)$ such that

$$A_1 \otimes \cdots \otimes A_m \mapsto \psi_1(A_{p_1}) \otimes \cdots \otimes \psi_m(A_{p_m}),$$

where for each $j \in \{1, \ldots, m\}$, $n_j = n_{p_j}$ *and* $\psi_j : M_{n_j} \to M_{n_j}$ *is a linear map of the form*

$$X \mapsto U_j X U_j^* \qquad \text{or} \qquad X \mapsto U_j X^t U_j^*$$

for a unitary $U_j \in M_{n_j}$.

The result was generalized in three directions by researchers. First, Hou and his associates [9] extended the result to the infinite dimensional setting and characterized bounded invertible linear maps leaving invariant the set of tensor product of rank one orthogonal projections acting on infinite dimensional Hilbert spaces, or its convex hull, i.e., the set of separable states. Second, Lim [19] characterized linear map $\phi : H_{n_1} \otimes \cdots \otimes H_{n_m} \to H_{\tilde{n}_1} \otimes \cdots \otimes H_{\tilde{n}_m}$ such that ϕ maps the set of tensor (separable) states in the domain into the set of tensor (separable) states in the codomain. Third, the authors in [18] characterize linear map $\phi : H_{n_1} \otimes \cdots \otimes H_{n_m} \to H_{n_1} \otimes \cdots \otimes H_{n_m}$ such that $\phi(\mathcal{S}_1) = \mathcal{S}_2$, where

$$\mathcal{S}_1 = \{X_1 \otimes \cdots \otimes X_m : X_j \in \mathcal{U}(C_j),\ j = 1, \ldots, m\}$$

and

$$\mathcal{S}_2 = \{Y_1 \otimes \cdots \otimes Y_m : Y_j \in \mathcal{U}(D_j),\ j = 1, \ldots, m\}$$

for given states $C_j, D_j \in H_{n_j}$ with $j = 1, \ldots, m$ and

$$\mathcal{U}(X) = \{U^* X U : U \text{ unitary}\}$$

is the unitary (similarity) orbit of X. When C_i and D_i are pure states, the study reduces to the problem treated in [6], and reveals the fact that there are linear transformations converting a unitary orbit to a different unitary orbit.

In [11], the author showed a number of interesting linear preserver results related to quantum information science. A vector state of a quantum system with m measurable physical states can be represented as a unit vector u in $\mathbb{C}^m$. A product state of two vector states $u \in \mathbb{C}^m$ and $v \in \mathbb{C}^n$ is the tensor product $u \otimes v \in \mathbb{C}^{mn}$, and unit vectors in $\mathbb{C}^{mn}$ can be viewed as vector states in the bipartite system with $\mathbb{C}^m$ and $\mathbb{C}^n$ as components. Every vector $w \in \mathbb{C}^{mn}$ can be identified with an $m \times n$ matrix $[w]$ by putting the first n entries in the first row, the next n entries in the second row, etc. In particular, $u \otimes v$ can be identified with the matrix uv^t. The singular value decomposition of the matrix $[w] = \sum_{j=1}^k s_j u_j v_j^t$ corresponds to the Schmidt decomposition $w = \sum_{j=1}^k s_j u_j \otimes v_j$. The Schmidt rank of a vector (state) w is the rank of the matrix $[w]$. Clearly, the linear span of product states $u \otimes v$ will generate all the vectors in $\mathbb{C}^{mn}$, and a linear map L on $\mathbb{C}^{mn}$ is completely determined once we know $L(u \otimes v)$ for all (or mn linearly independent) product states $u \otimes v$. In [11], the author used some classical results on linear preservers to study maps preserving $\mathcal{P}_k$, the set of all states with Schmidt rank at most k for a given $k \leq \min\{m, n\}$. In particular, it was shown that an invertible linear map $L : \mathbb{C}^{mn} \to \mathbb{C}^{mn}$ satisfies $L(\mathcal{P}_k) \subseteq \mathcal{P}_k$ if and only if there are unitary matrices $P \in M_m$ and $Q \in M_n$ such that one of the following holds.

(a) $L(u \otimes v) = Pu \otimes Qv$ for all $(u, v) \in \mathbb{C}^m \times \mathbb{C}^n$.
(b) $m = n$ and $L(u \otimes v) = Qv \otimes Pu$ for all $(u, v) \in \mathbb{C}^m \times \mathbb{C}^n$.

Notice that this result can also be obtained by a result of Djoković [5], which gave a characterization of surjective maps on tensor products preserving elements of bouned rank. Suppose $\mathcal{S}_k$ is the set of all vectors $w \in \mathbb{C}^{mn}$ with Schmidt rank at most k. Then an invertible linear map $L : \mathbb{C}^{mn} \to \mathbb{C}^{mn}$ satisfies $L(\mathcal{S}_k) \subseteq \mathcal{S}_k$ if and only if there are invertible matrices $P \in M_m$ and $Q \in M_n$ such that (a) or (b) holds.

Another result in [11] asserts that an invertible linear map $\Phi : M_{mn} \to M_{mn}$ satisfies $\Phi(\mathcal{S}) \subseteq \mathcal{S}$, where $\mathcal{S}$ is the set of rank one matrices of the form uv^t such that u and v have Schmidt rank at most k if and only if Φ is a composition of one or more of the following maps.

(1) The transpose map $X \mapsto X^t$.
(2) $X \mapsto (P_1 \otimes Q_1)X(P_2 \otimes Q_2)$ for some invertible matrices $P_i \in M_m$ and $Q_i \in M_n$ for $i = 1, 2$.
(3) $k = 1$, the partial transpose map $[X_{ij}]_{1\le i,j\le m} \mapsto [X_{ij}^t]_{1\le i,j\le m}$, where $X_{ij} \in M_n$.

Furthermore, Johnston considered the norm on $\mathbb{C}^{mn}$ defined by

$$\|u\|_k = \max\{|v^*u| : v \in \mathbb{C}^{mn},\ v^*v = 1,\ \operatorname{rank}([v]) \le k\} = \left\{\sum_{j=1}^{k} s_j^2\right\}^{1/2},$$

where $s_1 \ge s_2 \ge \cdots$ are the singular values of $[u]$, for any $k \le \min\{m, n\}$. He also considered the norm on M_{mn} defined by

$$|||C|||_k = \max\{|u^*Cv| : u, v \in \mathbb{C}^{mn},\ u^*u = v^*v = 1,\ \operatorname{rank}([u]) \le k,\ \operatorname{rank}([v]) \le k\}.$$

These norms have recently been studied in [3, 12, 13, 14, 29] and were shown to be related to the problem of characterizing k-positive linear maps and detecting bound entangled non-positive partial transpose states.

In connection to the preserver problems, it was shown that a linear map $L : \mathbb{C}^{mn} \to \mathbb{C}^{mn}$ satisfies

$$\|L(u)\|_k = \|u\|_k \qquad \text{for all } u \in \mathbb{C}^{mn}$$

if and only if there are unitary $P \in M_m$ and $Q \in M_n$ such that condition (a) or (b) mentioned above holds.

If $k = \min\{m, n\}$ one sees that $|||C|||_k$ is just the operator norm. It is known that a linear preserver on M_{mn} of the operator norm has the form

$$X \mapsto UXV \quad \text{or} \quad X \mapsto UX^tV$$

for some unitary $U, V \in M_{mn}$. For $k < \min\{m, n\}$, Johnston showed that a linear map $\Phi : M_{mn} \to M_{mn}$ satisfies

$$|||\Phi(X)|||_k = |||X|||_k \quad \text{for all } X \in M_{mn}$$

if and only if Φ is a composition of one or more of the maps described in (1), (2) or (3) above with the additional restriction that P and Q in (2) are unitary.

Many of the above results are extended to multi-partite system, e.g., [6, 11, 18, 19].

Next, we consider another line of research in preserver problems. There has been considerable interest in studying spectrum preserving maps (see [2, 10, 20] etc). On Hermitian matrices, it is known that a linear map on H_n that leaves invariant the spectrum has the form

$$A \mapsto UAU^* \quad \text{or} \quad A \mapsto UA^tU^*$$

for some unitary $U \in M_n$. If one gives up the Hermitian preserving property and considers a (complex) linear operator $\phi : M_n \to M_n$ that leaves invariant the eigenvalues of Hermitian matrices, then ϕ has the form

$$A \mapsto SAS^{-1} \quad \text{or} \quad A \mapsto SA^tS^{-1} \tag{3}$$

for some invertible $S \in M_n$.

In [31, 32], the authors studied non-classical correlation in a bipartite systems and showed that for any spectrum preserving linear map $\phi : H_n \to M_n$, either

$$\sigma((\mathrm{Id}_m \otimes \phi)(C)) = \sigma(C) \quad \text{for all} \quad C \in H_m \otimes H_n,$$

or

$$\sigma((\mathrm{Id}_m \otimes \phi)(C) = \sigma(\mathrm{PT}_2(C)) \quad \text{for all} \quad C \in H_m \otimes H_n,$$

where $\mathrm{PT}_2(A \otimes B) = A \otimes B^t$ is the partial transpose map for the second component and Id_m is the identity map on $m \times m$ matrices.

Following this line of study, we consider linear operators leaving invariant the spectrum of tensor states and related problems in the next section. It turns out that even if one assumes only that a linear operator ϕ leaves invariant the spectrum of matrices in tensor form $A \otimes B \in H_m \otimes H_n$, the operator ϕ has a nice structure, namely, up to a unitary similarity, ϕ has the form $A \otimes B \mapsto \psi_1(A) \otimes \psi_2(B)$ for all tensor states $A \otimes B$, where ψ_j is the identity map $X \mapsto X$ or the transposition map $X \mapsto X^t$. Moreover, if $\sigma(C) = \sigma(\phi(C))$ for a carefully chosen $C \in H_{mn}$, then ϕ will actually preserve the spectrum of every matrix in H_{mn}, and will be of the form $X \mapsto VXV^*$ or $X \mapsto VX^tV^*$ on H_{mn} for some unitary matrix $V \in H_{mn}$. Similar results are obtained for linear maps leaving invariant the spectral radius of tensor states $A \otimes B$ in $H_m \otimes H_n$.

3. Preservers of spectral radius or spectrum

Suppose $A \in H_m$ has eigenvalues $a_1 \geq \cdots \geq a_m$ associated with orthonormal eigenvectors $x_1, \ldots, x_m$, and $B \in H_n$ has eigenvalues $b_1 \geq \cdots \geq b_n$ associated with orthonormal eigenvectors $y_1, \ldots, y_n$, then $A \otimes B$ has eigenvalues $a_r b_s$ associated with eigenvectors $x_r \otimes y_s$ for $(r, s) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$. Denote by $\sigma(X)$ and $r(X)$ the spectrum and spectral radius of a matrix $X \in M_n$. In Subsection 3.1, we show that a linear map $\phi : H_m \otimes H_n \to H_m \otimes H_n$ satisfies

$$\sigma(\phi(A \otimes B)) = \sigma(A \otimes B)$$

for all $A \otimes B \in H_m \otimes H_n$ if and only if there is a unitary $U \in M_{mn}$ such that

$$A \otimes B \mapsto U(\varphi_1(A) \otimes \varphi_2(B))U^*, \tag{4}$$

where φ_j, $j = 1, 2$, is either the identity map or the transposition map $X \mapsto X^t$ (see Theorem 3.2). Furthermore, we will also show that a linear map on H_{mn} leaving the spectral radius of tensor states invariant, i.e.,

$$r(\phi(A \otimes B)) = r(A \otimes B)$$

for all $A \otimes B \in H_m \otimes H_n$, is ± 1 multiple of a map of the standard form (4) (see Theorem 3.3). In Subsection 3.2, we will extend the results to multipartite systems (Theorem 3.4 and Theorem 3.5). Additional remarks, results, and open problems will be presented in Subsection 3.3.

3.1. **Bipartite system.** Throughout this paper, we denote by $E_{ij}, 1 \leq i, j \leq n$ the standard basis of M_n. We need the following lemma.

Lemma 3.1. *Let $m > n$ and $A \in H_m$ with $\sigma(A) = \{a_1, \ldots, a_n, 0, \ldots, 0\}$. If*

$$\sigma(A + t(I_n \oplus 0_{m-n})) = \{a_1 + t, \ldots, a_n + t, 0, \ldots, 0\} \text{ for all } t \in \mathbb{R},$$

then $A = B \oplus 0_{m-n}$ for some $B \in H_n$.

Proof. Choose a sufficient large $s \in \mathbb{R}$ so that $C = A + s(I_n \oplus 0_{m-n})$ is positive semi-definite with eigenvalues $c_1, \ldots, c_n, 0, \ldots, 0$ where $c_j = a_j + s$, $j = 1, \ldots, n$. Then

$$\sigma(C + t(I_n \oplus 0_{m-n})) = \sigma(A + (s+t)(I_n \oplus 0_{m-n})) = \{c_1 + t, \ldots, c_n + t, 0, \ldots, 0\}.$$

Denote by $\{e_1, \ldots, e_m\}$ the standard basis of $\mathbb{C}^m$. Then for any unit vector $v \in \text{span}\,\{e_{n+1}, \ldots, e_m\}$,

$$v^*Cv = v^*(C + t(I_n \oplus 0_{m-n}))v \in \text{conv}\,\{c_1 + t, \ldots, c_n + t, 0\} \quad \text{for all} \quad t \in \mathbb{R},$$

where $\text{conv}\, S$ denotes the convex hull of the set S. Since this holds for all t in $\mathbb{R}$, this is possible only when $v^*Cv = 0$. As C is positive semi-definite, v is an eigenvector of C with eigenvalue 0. As v is arbitrary in $\text{span}\,\{e_{n+1}, \ldots, e_m\}$, C must have the form $C_1 \oplus 0_{m-n}$. Hence, $A = B \oplus 0_{m-n}$ with $B = C_1 - sI_n$. □

Theorem 3.2. *A linear map $\phi : H_{mn} \to H_{mn}$ satisfies*

$$\sigma(\phi(A \otimes B)) = \sigma(A \otimes B)$$

for all $A \otimes B \in H_m \otimes H_n$ if and only if there is a unitary $U \in M_{mn}$ such that

$$\phi(A \otimes B) = U(\varphi_1(A) \otimes \varphi_2(B))U^*,$$

where φ_j is the identity map or the transposition map $X \mapsto X^t$ for $j \in \{1, 2\}$.

Proof. The sufficiency part is clear. We consider the necessity part. Since $\sigma(\phi(I_m \otimes I_n)) = \sigma(I_m \otimes I_n) = \{1\}$, we see that $\phi(I_m \otimes I_n) = I_m \otimes I_n$. Consider any distinct pairs (j, k) and (r, s) for $j, r \in \{1, \ldots, m\}$, $k, s \in \{1, \ldots, n\}$. Then $\phi(E_{jj} \otimes E_{kk})$ and $\phi(E_{rr} \otimes E_{ss})$ are nonzero orthogonal projections. Now, $I_{mn} = \phi(I_{mn}) = \sum_{j,k} \phi(E_{jj} \otimes E_{kk})$ has trace mn. It follows that each $\phi(E_{jj} \otimes E_{kk})$ has rank one. Moreover, $\phi(E_{jj} \otimes E_{kk})$ and $\phi(E_{rr} \otimes E_{ss})$ have disjoint range spaces for any distinct pairs (j, k) and (r, s). Hence, there exists a unitary $W \in M_{mn}$ such that

$$\phi(E_{jj} \otimes E_{kk}) = W(E_{jj} \otimes E_{kk})W^*$$

for all $1 \leq j \leq m$ and $1 \leq k \leq n$.

For any $B \in H_n$, $t \in \mathbb{R}$, and $1 \leq j \leq m$, we have

$$\begin{aligned} & \sigma\left(\phi(E_{jj} \otimes B) + t\phi(E_{jj} \otimes I_n)\right) = \sigma\left(\phi(E_{jj} \otimes (B + tI_n))\right) \\ = \;& \sigma\left(E_{jj} \otimes (B + tI_n)\right) = \{b + t : b \in \sigma(B)\} \cup \{0\}. \end{aligned}$$

Since $\phi(E_{jj} \otimes I_n) = W(E_{jj} \otimes I_n)W^*$, applying Lemma 3.1 and using permutation similarity if necessary, we have

$$\phi(E_{jj} \otimes B) = W(E_{jj} \otimes \psi_j(B))W^*$$

for some $\psi_j(B) \in H_n$. Furthermore, B and $\psi_j(B)$ have the same spectrum. So ψ_j has the form

$$B \mapsto U_j B U_j^* \quad \text{or} \quad B \mapsto U_j B^t U_j^*$$

for some unitary U_j. Replace W with $W(U_1 \oplus \cdots \oplus U_m)$. Then

$$\phi(E_{jj} \otimes B) = W(E_{jj} \otimes \varphi_j(B))W^* \tag{5}$$

for all $1 \le j \le m$ and $B \in H_n$, where each map φ_j is the identity map or the transposition map $X \mapsto X^t$.

Repeating the same argument, one can show that for any unitary $U \in M_m$,

$$\phi(UE_{jj}U^* \otimes B) = W_U(E_{jj} \otimes \varphi_{j,U}(B))W_U^*$$

for all $1 \le j \le m$ and $B \in H_n$, where $W_U \in M_{mn}$ is a unitary matrix, depending on U, and $\varphi_{j,U}$ is either the identity map or the transposition map, depending on j and U. Replacing ϕ by the map $A \mapsto W_{I_{mn}}^* \phi(A) W_{I_{mn}}$, we may assume that

$$W_{I_{mn}} = I_{mn} \quad \text{and} \quad \phi(E_{jj} \otimes E_{kk}) = E_{jj} \otimes E_{kk}$$

for all $1 \le j \le m$ and $1 \le k \le n$. Now, for any real symmetric $S \in H_n$ and unitary $U \in M_m$, we have $\varphi_{j,U}(S) = S$ for all $j = 1, \ldots, m$, and, hence,

$$\phi(I_m \otimes S) = \phi\left(\sum_{j=1}^m UE_{jj}U^* \otimes S\right) = W_U\left(\sum_{j=1}^m E_{jj} \otimes S\right)W_U^* = W_U\left(I_m \otimes S\right)W_U^*$$

for some unitary $W_U \in M_{mn}$. In particular, when $U = I_m$, $\phi(I_m \otimes S) = I_m \otimes S$. Thus, $W_U\left(I_m \otimes S\right)W_U^* = I_m \otimes S$. It follows that W_U commutes with $I_m \otimes S$ for all real symmetric S. Hence, W_U has the form $V_U \otimes I_n$ for some $V_U \in M_m$ and

$$\phi\left(UE_{jj}U^* \otimes B\right) = (V_U E_{jj} V_U^*) \otimes \varphi_{j,U}(B) \tag{6}$$

for $1 \le j \le m$ and $B \in M_m$. Consider the linear maps $\mathrm{tr}_1 : H_{mn} \to H_n$ and $\Phi : H_{mn} \to H_n$ defined by

$$\mathrm{tr}_1(A \otimes B) = (\mathrm{tr}A)B \qquad \text{and} \qquad \Phi(A \otimes B) = \mathrm{tr}_1\left(\phi(A \otimes B)\right)$$

for any $A \otimes B \in H_m \otimes H_n$. Then

$$\Phi\left(UE_{jj}U^* \otimes B\right) = \varphi_{j,U}(B).$$

Recall that a continuous image of a connected space is still connected. Since Φ is linear and continuous, $\{xx^* \in M_m : x^*x = 1\}$ is connected, and $\varphi_{j,U}$ is either the identity map or the transposition map, all the maps $\varphi_{j,U}$ have to be the same. Replacing ϕ by the map $A \otimes B \mapsto \phi(A \otimes B^t)$, if necessary, we may assume that this common map is the identity map. Next, by linearity, one can conclude that for every $A \in H_m$ and $B \in H_n$ we have

$$\phi\left(A \otimes B\right) = \varphi_1(A) \otimes B$$

for some $\varphi_1(A) \in H_m$, where $\varphi_1(A)$ depends on A only. Note that $\varphi_1 : H_m \to H_m$ is a linear map and $\sigma(\varphi_1(A)) = \sigma(A)$ for all $A \in H_m$. Hence, by [20], a map φ_1 has the form $A \mapsto VAV^*$ or $A \mapsto VA^tV^*$. The proof is completed. □

In the following, we consider linear maps on H_{mn} leaving the spectral radius of the tensor product of two Hermitian matrices invariant.

Theorem 3.3. *A linear map $\phi : H_{mn} \to H_{mn}$ satisfies*

$$r(\phi(A \otimes B)) = r(A \otimes B)$$

for all $A \otimes B \in H_m \otimes H_n$ if and only if there is a unitary $U \in M_{mn}$ and $\lambda \in \{-1, 1\}$ such that

$$\phi(A \otimes B) = \lambda U(\varphi_1(A) \otimes \varphi_2(B))U^*,$$

where φ_j is the identity map or the transposition map $X \mapsto X^t$ for $j \in \{1, 2\}$.

Proof. The sufficiency part is clear. For the converse, suppose that a linear map $\phi : H_{mn} \to H_{mn}$ preserves the spectral radius of tensor states and let $1 \le j \le m$, $1 \le k \le n$. Then $\phi(E_{jj} \otimes E_{kk})$ has an eigenvalue in $\{1, -1\}$. For $t \ne k$, we have $r(\phi(E_{jj} \otimes (E_{kk} \pm E_{tt}))) = 1$. This yields that every eigenvector of $\phi(E_{jj} \otimes E_{kk})$ corresponding to the eigenvalue 1 or -1 lies in the kernel of $\phi(E_{jj} \otimes E_{tt})$. Since this is true for any pair of k and t, for any orthogonal diagonal matrix $D \in M_n$ at least n eigenvalues of $\phi(E_{jj} \otimes D)$ lie in $\{1, -1\}$. Since $r(\phi((E_{jj} \pm E_{ss}) \otimes D)) = 1$ for any $j \ne s$, $1 \le j, s \le m$, and any diagonal orthogonal matrix $D \in H_n$, $\phi(E_{jj} \otimes D)$ and $\phi(E_{ss} \otimes D)$ have disjoint support and, hence, $\phi(E_{jj} \otimes D)$ has rank n. It follows that all $\phi(E_{jj} \otimes E_{kk})$ must be rank one and $\phi(E_{jj} \otimes E_{kk})$ and $\phi(E_{ss} \otimes E_{tt})$ have disjoint support for any distinct (j, k) and (s, t). Therefore, there is a unitary $W \in M_{mn}$ and $\mu_{jk} \in \{1, -1\}$ such that

$$\phi(E_{jj} \otimes E_{kk}) = \mu_{jk} W(E_{jj} \otimes E_{kk})W^* \quad \text{for all} \quad 1 \le j \le m, 1 \le k \le n.$$

For the sake of the simplicity, suppose that $W = I_{mn}$ and $\phi(E_{jj} \otimes I_n) = E_{jj} \otimes P_j$, where $P_1, \dots, P_m \in H_n$ are diagonal orthogonal matrices.

For any unitary $V \in M_n$, applying the same arguments to $E_{jj} \otimes V E_{kk} V^*$, $1 \le j \le m$, $1 \le k \le n$, we see that $\phi(E_{jj} \otimes V E_{kk} V^*)$ has rank one with spectral radius 1. If $t > 0$, we have

$$r(\phi(E_{jj} \otimes (V E_{kk} V^* + tI_n))) = 1 + t.$$

Thus, the eigenspace of the nonzero eigenvalue of $\phi(E_{jj} \otimes V E_{kk} V^*)$ must lie in the eigenspace of $\phi(E_{jj} \otimes I_n) = E_{jj} \otimes P_j$. Consequently, we see that $\phi(E_{jj} \otimes B) = E_{jj} \otimes \varphi_j(B)$ for any $B \in H_n$. Clearly, φ_j preserves spectral radius on H_n and, hence, by [15] it has the form

$$B \mapsto \xi Y B Y^* \quad \text{or} \quad B \mapsto \xi Y B^t Y^*$$

for some $\xi \in \{1, -1\}$ and unitary $Y \in M_n$. In particular, $\varphi_j(I_n) \in \{I_n, -I_n\}$. So, $\phi(I_{mn}) = D \otimes I_n$ for some diagonal orthogonal matrix $D \in M_m$.

By considering $U E_{jj} U^* \otimes E_{kk}$ for unitary $U \in M_m$ and using the same arugment as in the last paragraph, one can show that $\phi(I_{mn}) = I_m \otimes \tilde{D}$ for some diagonal orthogonal matrix $\tilde{D} \in M_n$. Since $\phi(I_{mn}) = I_m \otimes \tilde{D} = D \otimes I_n$, we conclude that $\phi(I_{mn}) = \pm I_{mn}$. Without loss of generality, we may assume that $\phi(I_{mn}) = I_{mn}$. Thus, all μ_{jk} are equal to 1, i.e.,

$$\phi(E_{jj} \otimes E_{kk}) = E_{jj} \otimes E_{kk} \quad \text{for all} \quad 1 \le j \le m, 1 \le k \le n.$$

For any $A \otimes B \in H_m \otimes H_n$, there are unitary $U \in M_m$ and $V \in M_n$ such that UAU^* and VBV^* are diagonal matrices. Without loss of generality, we assume that $A = \text{Diag}\,(a_1, \dots, a_m)$ and $B = \text{Diag}\,(b_1, \dots, b_n)$. Then

$$\phi(A \otimes B) = \phi\left(\left(\sum_{j=1}^{m} a_j E_{jj}\right) \otimes \left(\sum_{k=1}^{n} b_k E_{kk}\right)\right) = A \otimes B.$$

Thus, $\sigma(\phi(A\otimes B))=\sigma(A\otimes B)$ and the result is followed by Theorem 3.2. $\square$

3.2. **Multipartite systems.** In this section we will extend Theorem 3.2 and Theorem 3.3 to multipartite system $H_{n_1\cdots n_m}=H_{n_1}\otimes\cdots\otimes H_{n_m}$, $m\geq 2$.

Theorem 3.4. *A linear map $\phi: H_{n_1\cdots n_m}\to H_{n_1\cdots n_m}$ satisfies*

$$\sigma(\phi(A_1\otimes\cdots\otimes A_m))=\sigma(A_1\otimes\cdots\otimes A_m)$$

for all $A_1\otimes\cdots\otimes A_m\in H_{n_1\cdots n_m}$ if and only if there is a unitary $U\in M_{n_1\cdots n_m}$ such that

$$\phi(A_1\otimes\cdots\otimes A_m)=U(\varphi_1(A_1)\otimes\cdots\otimes\varphi_m(A_m))U^*, \tag{7}$$

where φ_j is the identity map or the transposition map $X\mapsto X^t$ for $j\in\{1,\ldots,m\}$.

Proof. The sufficiency part is clear. To prove the necessity part, we use induction on m. By Theorem 3.2, we already know that the statement of Theorem 3.4 is true for bipartite systems. So, assume that $m\geq 3$ and that the result holds for all $(m-1)$-partite systems. We would like to prove that the same is true for m-partite systems.

As in the proof of Theorem 3.2, we can show that there exists a unitary $W\in M_{n_1\cdots n_m}$ such that

$$\phi(E_{j_1j_1}\otimes\cdots\otimes E_{j_mj_m})=W(E_{j_1j_1}\otimes\cdots\otimes E_{j_mj_m})W^*$$

for all $1\leq j_p\leq n_p$ with $1\leq p\leq m$. Moreover, for any $B\in H_{n_1}$ and $1\leq j_p\leq n_p$ with $2\leq p\leq m$, we have

$$\phi(B\otimes E_{j_2j_2}\otimes\cdots\otimes E_{j_mj_m})=W(\psi_{j_2,\ldots,j_m}(B)\otimes E_{j_2j_2}\otimes\cdots\otimes E_{j_mj_m})W^*$$

for some $\varphi_{j_2,\ldots,j_m}(B)\in H_{n_1}$. Then B and $\varphi_{j_2,\ldots,j_m}(B)$ have the same spectrum. By the fact that $\varphi_{j_2,\ldots,j_m}(E_{kk})=E_{kk}$ for all $1\leq k\leq n_1$, the map $\varphi_{j_2,\ldots,j_m}$ can be assumed either the identity map or the transposition map. By a similar argument, we can show that

$$\phi\left(B\otimes\left(\bigotimes_{p=2}^{m}U_pE_{j_pj_p}U_p^*\right)\right)=W_{U_2,\ldots,U_m}\left(\varphi^{U_2,\ldots,U_m}_{j_2,\ldots,j_m}(B)\otimes E_{j_2j_2}\otimes\cdots\otimes E_{j_mj_m}\right)W^*_{U_2,\ldots,U_m}$$

for all $B\in H_{n_1}$ and $1\leq j_p\leq n_p$ with $2\leq p\leq m$, where $W_{U_2,\ldots,U_m}\in M_{n_1\cdots n_m}$ is a unitary matrix depending on $U_2,\ldots,U_m$ only and $\varphi^{U_2,\ldots,U_m}_{j_2,\ldots,j_m}$ is either the identity map or the transposition map, depending on $j_2,\ldots,j_m$ and $U_2,\ldots,U_m$. Replacing ϕ by the map $A\mapsto W^*_{I_{n_2},\ldots,I_{n_m}}\phi(A)W_{I_{n_2},\ldots,I_{n_m}}$, we may assume that

$$W_{I_{n_2},\ldots,I_{n_m}}=I_{n_1\cdots n_m}\quad\text{and}\quad\phi\left(E_{j_1j_1}\otimes\cdots\otimes E_{j_mj_m}\right)=E_{j_1j_1}\otimes\cdots\otimes E_{j_mj_m}$$

for all $1\leq j_p\leq n_p$ with $1\leq p\leq m$. Again, considering all symmetric $S\in H_{n_1}$ as in the proof of Theorem 3.2, we can show that there exists $V_{U_2,\ldots,U_m}\in M_{n_2\cdots n_m}$ such that

$$\phi\left(B\otimes\left(\bigotimes_{p=2}^{m}U_pE_{j_pj_p}U_p^*\right)\right)=\varphi^{U_2,\ldots,U_m}_{j_2,\ldots,j_m}(B)\otimes V_{U_2,\ldots,U_m}\left(E_{j_2j_2}\otimes\cdots\otimes E_{j_mj_m}\right)V^*_{U_2,\ldots,U_m}.$$

Using the trace function, we see that all the maps $\varphi^{U_2,\ldots,U_m}_{j_2,\ldots,j_m}$ have to be the same. Assume that this common map is equal to φ, which is either the identity map or the transposition map. By linearity, one can conclude that for any $A = A_2 \otimes \cdots \otimes A_m \in H_{n_2 \cdots n_m}$ and $B \in H_{n_1}$,

$$\phi(B \otimes A_2 \otimes \cdots \otimes A_m) = \varphi(B) \otimes \psi(A_2 \otimes \cdots \otimes A_m)$$

for some $\psi(A) = \psi_1(A_2 \otimes \cdots \otimes A_m) \in H_{n_2 \cdots n_m}$, where $\psi(A)$ depends on A only. Note that $\psi : H_{n_2 \cdots n_m} \to H_{n_2 \cdots n_m}$ is a linear map and $\sigma(\psi(A)) = \sigma(A)$ for all $A \in H_{n_2 \cdots n_m}$. Hence, by induction hypothesis, ϕ has the form (7), as desired. The proof is completed. □

Theorem 3.5. *A linear map $\phi : H_{n_1 \cdots n_m} \to H_{n_1 \cdots n_m}$ satisfies*

$$r(\phi(A_1 \otimes \cdots \otimes A_m)) = r(A_1 \otimes \cdots \otimes A_m)$$

for all $A_1 \otimes \cdots \otimes A_m \in H_{n_1 \cdots n_m}$ if and only if there is a unitary $U \in M_{n_1 \cdots n_m}$ and $\lambda \in \{-1, 1\}$ such that

$$\phi(A_1 \otimes \cdots \otimes A_m) = \lambda U(\varphi_1(A_1) \otimes \cdots \otimes \varphi_m(A_m))U^*, \tag{8}$$

where φ_j is the identity map or the transposition map $X \mapsto X^t$ for $j \in \{1, \ldots, m\}$.

Proof. The sufficiency part is clear. To prove the converse, by a similar argument as in Theorem 3.3, we can show that $\phi(E_{j_1 j_1} \otimes \cdots \otimes E_{j_m j_m})$ has an eigenvalue in $\{1, -1\}$ for any index set $(j_1, \ldots, j_m)$, where $1 \le j_p \le n_p$ with $1 \le p \le m$. Next, one can show that for any orthogonal diagonal matrix $D_1 \in H_{n_1}$, $\phi(D_1 \otimes E_{j_2 j_2} \otimes \cdots \otimes E_{j_m j_m})$ has at least n_1 eigenvalues lying in $\{1, -1\}$. Furthermore, for any orthogonal diagonal matrices $D_1 \in H_{n_1}$ and $D_2 \in H_{n_2}$, $\phi(D_1 \otimes D_2 \otimes E_{j_3 j_3} \otimes \cdots \otimes E_{j_m j_m})$ has at least $n_1 n_2$ eigenvalues lying in $\{1, -1\}$. Recurrently, one can show that for any orthogonal diagonal $D_p \in H_{n_p}$ with $1 \le p \le m$, $\phi(D_1 \otimes D_2 \otimes \cdots \otimes D_m)$ has $n_1 n_2 \cdots n_m$ eigenvalues lying in $\{1, -1\}$. This is possible only when $\phi(E_{j_1 j_1} \otimes \cdots \otimes E_{j_m j_m})$ is rank one and for any distinct index sets $(j_1, \ldots, j_m)$ and $(k_1, \ldots, k_m)$, $\phi(E_{j_1 j_1} \otimes \cdots \otimes E_{j_m j_m})$ and $\phi(E_{k_1 k_1} \otimes \cdots \otimes E_{k_m k_m})$ have disjoint support. Therefore, there is a unitary matrix $W \in M_{n_1 \cdots n_m}$ and $\mu_{j_1, \ldots, j_m} \in \{1, -1\}$ such that

$$\phi(E_{j_1 j_1} \otimes \cdots \otimes E_{j_m j_m}) = \mu_{j_1, \ldots, j_m} W(E_{j_1 j_1} \otimes \cdots \otimes E_{j_m j_m})W^*.$$

Suppose $P_{j_2, \ldots, j_m}$ are diagonal orthogonal matrices such that

$$\phi(I_{n_1} \otimes E_{j_2 j_2} \otimes \cdots \otimes E_{j_m j_m}) = W(P_{j_2, \ldots, j_m} \otimes E_{j_2 j_2} \otimes \cdots \otimes E_{j_m j_m})W^*.$$

Since every rank one matrix $R \in H_{n_1}$ can be expressed as $UE_{11}U^*$ for some unitary $U \in M_{n_1}$, using the same argument as above, one can show that $\phi(R \otimes E_{j_2 j_2} \otimes \cdots \otimes E_{j_m j_m})$ has rank one with spectral radius 1 for all $1 \le j_p \le n_p$ with $2 \le p \le m$. By considering

$$r(\phi((R + tI_{n_1}) \otimes E_{j_2 j_2} \otimes \cdots \otimes E_{j_m j_m})) = 1 + t \quad \text{for all } t > 0,$$

one can conclude that $\phi(R \otimes E_{j_2 j_2} \otimes \cdots \otimes E_{j_m j_m}) = W(\psi_{j_2, \ldots, j_m}(R) \otimes E_{j_2 j_2} \otimes \cdots \otimes E_{j_m j_m})W^*$ and hence for any $B \in H_{n_1}$,

$$\phi(B \otimes E_{j_2 j_2} \otimes \cdots \otimes E_{j_m j_m}) = W(\psi_{j_2, \ldots, j_m}(B) \otimes E_{j_2 j_2} \otimes \cdots \otimes E_{j_m j_m})W^*.$$

Clearly, $\psi_{j_2, \ldots, j_m}$ preserves spectral radius on H_{n_1} and, hence, has the form

$$B \mapsto \xi YBY^* \quad \text{or} \quad B \mapsto \xi YB^tY^*$$

for some $\xi \in \{1, -1\}$ and unitary $Y \in M_{n_1}$. Then, one can see that the scalar $\mu_{j_1, \ldots, j_m}$ has to be independent of the first index j_1, i.e., $\mu_{j_1, j_2 \ldots, j_m} = \mu_{j_1', j_2, \ldots, j_m}$ for any $1 \le j_1, j_1' \le n_1$. Applying the same argument on the pth subsystem for $p = 2, \ldots, m$, one can deduce that $\mu_{j_1, \ldots, j_m}$

is independent of the pth index j_p. Therefore, $\mu_{j_1,\ldots,j_m} = \mu_{k_1,\ldots,k_m}$ for any the index sets $(j_1,\ldots,j_m)$ and $(k_1,\ldots,k_m)$ and hence $\mu_{j_1,\ldots,j_m} = \mu$ is a constant. So

$$\phi(E_{j_1j_1}\otimes\cdots\otimes E_{j_mj_m}) = \mu W(E_{j_1j_1}\otimes\cdots\otimes E_{j_mj_m})W^* \quad \text{for all} \quad 1\le j_p\le m \text{ with } 1\le p\le m.$$

By the same argument, one can show that for any unitary $U_p\in M_{n_p}$ with $1\le p\le m$,

$$\phi(U_1E_{j_1j_1}U_1^*\otimes\cdots\otimes U_mE_{j_mj_m}U_m^*) = \mu_{U_1,\ldots,U_m}W_{U_1,\ldots,U_m}(E_{j_1j_1}\otimes\cdots\otimes E_{j_mj_m})W_{U_1,\ldots,U_m}^*$$

for all $1\le j_p\le n_p$ with $1\le p\le m$. Here the scalar $\mu_{U_1,\ldots,U_m}\in\{1,-1\}$ and the unitary matrix $W_{U_1,\ldots,U_m}\in M_{n_1\cdots n_m}$ depend on $U_1,\ldots,U_m$ only. Furthermore, summing up for all the indices $j_1,\ldots,j_m$ yields $\phi(I_{n_1\cdots n_m}) = \mu_{U_1,\ldots,U_m}I_{n_1\cdots n_m}$. So $\mu_{U_1,\ldots,U_m} = \mu_{I_{n_1},\ldots,I_{n_m}} = \mu$ is independent of the choice of $U_1,\ldots,U_m$. Without loss of generality, we may assume that $\mu = 1$. Then by linearity, $\sigma(\phi(A_1\otimes\cdots\otimes A_m)) = \sigma(A_1\otimes\cdots\otimes A_m)$ for all $A_1\otimes\cdots\otimes A_m\in H_{n_1}\otimes\cdots\otimes H_{n_m}$, and the result follows from Theorem 3.4. □

3.3. **Additional remarks and results.** Several remarks concerning our results in the last two subsections are in order.

First, in all previous study of linear preservers involving tensor product spaces, one always imposed the assumption that the preservers send tensor states to tensor states. As a result, the structure of the preservers have the form

$$A\otimes B\mapsto \psi_1(A)\otimes\psi_2(B) \quad \text{or} \quad A\otimes B\mapsto\psi_2(B)\otimes\psi_1(A). \tag{9}$$

In our case, we do not assume that the preservers send tensor states to tensor states. Nevertheless, our results show that up to a unitary similarity, we still have the form (9).

Second, we characterize linear operators ϕ such that $A\otimes B$ and $\phi(A\otimes B)$ have the same spectrum (respectively, spectral radius). The resulting map may not preserve the spectrum (respectively, spectral radius) of a general matrix $C\in H_{mn}$. For example, if $C = E_{11}\otimes E_{11}+E_{22}\otimes E_{22}+E_{12}\otimes E_{12}+E_{21}\otimes E_{21}$, then the map ϕ of the form $A\otimes B\mapsto A\otimes B^t$ for tensor states will preserve the spectral radius (and spectrum) of tensor states, but $\phi(C)$ and C will not have the same spectral radius (and spectrum). One can easily extend the above observation to the following.

Theorem 3.6. *Suppose $\phi: H_{n_1\cdots n_m}\to H_{n_1\cdots n_m}$ is linear such that $r(\phi(C)) = r(C)$ (respectively, $\sigma(\phi(C)) = \sigma(C)$) for all $C = A_1\otimes\cdots\otimes A_m$ with $A_j\in H_{n_j}$, $j = 1,\ldots,m$, and for C obtained from $I_{n_1}\otimes\cdots\otimes I_{n_m}$ by replacing $I_{n_i}\otimes I_{n_{i+1}}$ with $E_{11}\otimes E_{11}+E_{22}\otimes E_{22}+E_{12}\otimes E_{12}+E_{21}\otimes E_{21}$, $i = 1,\ldots,m-1$. Then there are a unitary U and $\xi\in\{1,-1\}$ (respectively, $\xi = 1$) such that ϕ has the form*

$$X\mapsto\xi UXU^* \quad \text{or} \quad X\mapsto\xi UX^tU^*.$$

Third, one may consider affine maps ψ on the set of density matrices in $H_N = H_{n_1}\otimes\cdots\otimes H_{n_m}$ instead of linear maps on H_N. One may extend an affine map on density matrices in H_N in the standard way, namely, define for any positive semi-definite matrix C, $\phi(tC) = t\phi(C)$, and $\phi(C) = \psi(C)$ if $\mathrm{tr}C = 1$. Then use the fact that every $X\in H_N$ is a difference of two positive semi-definite C_1 and C_2, and that $\phi(C_1)-\phi(C_2) = \phi(D_1)-\phi(D_2)$ if $C_1-C_2 = D_1-D_2$.

Finally, it is interesting to study (real or complex) linear maps $\phi: M_m\otimes M_n\to M_m\otimes M_n$ such that $A\otimes B$ and $\phi(A\otimes B)$ always have the same spectrum (respectively, spectral radius).

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