G-invariant norms and bicircular projections

MAJA FOŠNER, DIJANA ILIŠEVIĆ, CHI-KWONG LI
FL, University of Maribor, Mariborska cesta 2, 3000 Celje, Slovenia

Department of Mathematics, University of Zagreb,
Bijenička 30, P.O. Box 335, 10002 Zagreb, Croatia

Department of Mathematics, College of William and Mary,
Williamsburg, VA 23187-8795, USA.

maja.fosner@uni-mb.si, ilisevic@math.hr, ckli@math.wm.edu

Abstract

It is shown that for many finite dimensional normed vector spaces $\mathbf{V}$ over $\mathbb{C}$, a linear projection $P : \mathbf{V} \to \mathbf{V}$ will have nice structure if $P + \lambda(I - P)$ is an isometry for some complex unit not equal to one. From these results, one can readily determine the structure of bicircular projections, i.e., those linear projections P such that $P + \mu(I - P)$ is a an isometry for every complex unit μ. The key ingredient in the proofs is the knowledge of the isometry group of the given norm. The proof techniques also apply to real vector spaces. In such cases, characterizations are given to linear projections P such that $P - (I - P) = 2P - I$ is an isometry.

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1 Introduction

Let $(X, \|\cdot\|)$ be a complex Banach space and let $P : X \to X$ be a linear projection. Denote by $\overline{P} = I - P$ its complementary projection. The projection P is called bicircular if the mapping $e^{i\alpha}P + e^{i\beta}\overline{P}$ is an isometry for all $\alpha, \beta \in \mathbb{R}$. Obviously, this is equivalent to the fact that the mapping $P + e^{i\varphi}\overline{P}$ is an isometry for all $\varphi \in \mathbb{R}$.

Let $B(\mathcal{H})$ be the algebra of all bounded linear operators acting on a complex Hilbert space $\mathcal{H}$. Bicircular projections on $B(\mathcal{H})$ and some subspaces of $B(\mathcal{H})$, with respect to the spectral norm, have nice structures as shown in [14]:

1. Let $P : B(\mathcal{H}) \to B(\mathcal{H})$ be a bicircular projection. Then there exists a selfadjoint projection $p \in B(\mathcal{H})$ such that either $P(x) = px$ for all $x \in B(\mathcal{H})$, or $P(x) = xp$ for all $x \in B(\mathcal{H})$.

2. Let $S(\mathcal{H}) = \{a \in B(\mathcal{H}) : a^t = a\}$. Let $P : S(\mathcal{H}) \to S(\mathcal{H})$ be a bicircular projection. Then either $P = 0$ or $P = I$.

3. Let $K(\mathcal{H}) = \{a \in B(\mathcal{H}) : a^t = -a\}$. Let $P : K(\mathcal{H}) \to K(\mathcal{H})$ be a bicircular projection. Then there exists a unit vector $\alpha \in \mathcal{H}$ such that either $P(x) = px + xp^t$ for all $x \in K(\mathcal{H})$, or $\overline{P}(x) = px + xp^t$ for all $x \in K(\mathcal{H})$, where $p = \alpha \otimes \alpha$. In the second case we can also write $P(x) = qxq^t$, where $q = 1 - p$.

In this paper, we show that for many finite dimensional normed vector spaces $\mathbf{V}$ over $\mathbb{C}$, a linear projection $P : \mathbf{V} \to \mathbf{V}$ will have nice structure if $P + \lambda\overline{P}$ is an isometry for some $\lambda \in \{\mu \in \mathbb{F} : |\mu| = 1, \mu \neq 1\}$. From these results, we can readily determine the structure of bicircular projections. As we will see, the key ingredient in our proofs is the knowledge of the isometries of the given norm. Our proof techniques also apply to real vector spaces. In such cases, we determine the structure of linear projections $P : \mathbf{V} \to \mathbf{V}$ such that $P - \overline{P} = 2P - I$ is an isometry.

We present our results for different classes of norms in sections 2-5. Additional results and remarks are given in section 6.

In our discussion, we always assume that $\mathbf{V}$ is a finite dimensional vector space over $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$ equipped with a fixed inner product $\langle \cdot\, , \cdot \rangle$. For a pair of column vectors or a pair of matrices x and y, we have the usual inner product $\langle x, y\rangle = \mathrm{tr}\,(xy^*) = \mathrm{tr}\,(y^*x)$. Denote by $U(\mathbf{V})$ the group of unitary (if $\mathbb{F} = \mathbb{C}$) or orthogonal (if $\mathbb{F} = \mathbb{R}$) operators on $\mathbf{V}$. Suppose G is a closed subgroup of $U(\mathbf{V})$. A norm $\|\cdot\|$ on $\mathbf{V}$ is said to be G-invariant if

$$\|g(v)\| = \|v\| \qquad \text{for all } g \in G,\ v \in \mathbf{V}.$$

Denote by $\mathbb{T} = \{\mu \in \mathbb{F} : |\mu| = 1\}$. To avoid trivial consideration, we always consider non-trivial linear projections, i.e., projections not equal to zero or the identity map. The following lemma is useful in our discussion.

Lemma 1.1. *Let $\|\cdot\|$ be a norm on $\mathbf{V}$ with isometry group $\mathcal{K}$, and let $\lambda \in \mathbb{T} \setminus \{1\}$. Suppose $P : \mathbf{V} \to \mathbf{V}$ is a non-trivial projection. The following conditions are mutually equivalent:*

(i) $P + \lambda\overline{P} \in \mathcal{K}$,

(ii) $P = (T - \lambda I)/(1 - \lambda)$ *for some* $T \in \mathcal{K}$ *such that* $(T - I)(T - \lambda I) = 0$.

Proof. (i) $\Rightarrow$ (ii) If we define $T = P + \lambda\overline{P}$, then $T \in \mathcal{K}$ and $P = (T - \lambda I)/(1 - \lambda)$. Since $P^2 = P$, we get $(T - I)(T - \lambda I) = 0$.

(ii) $\Rightarrow$ (i) From $P = (T - \lambda I)/(1 - \lambda)$ we get $P + \lambda\overline{P} = T \in \mathcal{K}$. Since $(T - I)(T - \lambda I) = 0$, we have $P^2 = P$. ∎

Remark 1.2. Suppose condition (ii) in Lemma 1.1 holds. If $\lambda = -1$ then $T^2 = I$; if $\lambda^2 \neq 1$ then $T = (T^2 + \lambda I)/(1 + \lambda)$, and hence $P = (T - \lambda I)/(1 - \lambda) = (T^2 - \lambda^2 I)/(1 - \lambda^2)$.

2 Inner product norms

Suppose a norm on $\mathbf{V}$ is a multiple of the inner product norm, i.e., the norm induced by the inner product on $\mathbf{V}$. Then the isometry group $\mathcal{K}$ is just $U(\mathbf{V})$. Our problem has a very simple answer.

Proposition 2.1. *Let $\mathbf{V}$ be an n-dimensional inner product space, and let $\|\cdot\|$ be a multiple of the norm induced by the inner product. Suppose $P : \mathbf{V} \to \mathbf{V}$ is a non-trivial linear projection and $\lambda \in \mathbb{T} \setminus \{1\}$. The following conditions are mutually equivalent.*

(i) *$P + \lambda\overline{P}$ is an isometry,*

(ii) *P is an orthogonal projection, i.e., there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ for $\mathbf{V}$ such that $P(e_j) = \lambda_j e_j$ where $\lambda_j \in \{0, 1\}$ for all $j = 1, \ldots, n$.*

Clearly, using the complex case of the above proposition, we see that P is a bicircular projection if and only if condition (ii) holds.

Proof. Note that the isometry group of $\|\cdot\|$ is $U(\mathbf{V})$. Furthermore, there is an orthonormal basis $\mathcal{B}$ of $\mathbf{V}$ such that P and $P + \lambda\overline{P}$ have matrix representations

$$\begin{pmatrix} I_k & X \\ 0 & 0_{n-k} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I_k & (1-\lambda)X \\ 0 & \lambda I_{n-k} \end{pmatrix}.$$

Evidently, there is $\lambda \in \mathbb{T} \setminus \{1\}$ such that $P + \lambda\overline{P} \in U(\mathbf{V})$ if and only if $X = 0$. ■

3 Symmetric norms

Let $\mathbf{V} = \mathbb{F}^n$ and let G be $GP(n)$, the group of generalized permutation matrices (matrices of the form DP, where D is a diagonal matrix in $U(\mathbb{F}^n)$ and P is a permutation matrix). Then G-invariant norms are also known as symmetric norms (symmetric gauge functions). We will study our problem for symmetric norms in the following. It is useful to have the following information about the isometries for symmetric norms.

Since G is irreducible, the isometry group of a given symmetric norm is a subgroup of $U(\mathbf{V})$. A characterization of all the possible isometry groups of a G-invariant norm can be found in e.g. [3, 7]. Assume that a G-invariant norm $\|\cdot\|$ is not a multiple of the inner product norm on $\mathbf{V}$. Let

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \qquad B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

If $\mathbb{F}^n \neq \mathbb{R}^2$ and $\mathbb{F}^n \neq \mathbb{R}^4$, then $\mathcal{K} = G$. If $\mathbb{F}^n = \mathbb{R}^4$, then one of the following holds: (1) $\mathcal{K} = G$, (2) $\mathcal{K} = \langle G, A\rangle$, the group generated by A and the elements in G, (3) $\mathcal{K} = \langle G, B\rangle$, the group generated by B and the elements in G. If $\mathbb{F}^n = \mathbb{R}^2$, then $\mathcal{K} = G$ or $\mathcal{K}$ is the dihedral group with $8k$ elements.

Proposition 3.1. *Let $\|\cdot\|$ be a symmetric norm on $\mathbb{F}^n$ not equal to a multiple of the norm induced by the inner product norm $\langle x, x\rangle^{1/2}$, and let $\mathcal{K}$ be the isometry group of $\|\cdot\|$. Suppose $P : \mathbb{F}^n \to \mathbb{F}^n$ is a non-trivial linear projection and $\lambda \in \mathbb{T} \setminus \{1\}$. Then $P + \lambda\overline{P}$ is an isometry for $\|\cdot\|$ if and only if one of the following holds.*

(a) $P = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ *with* $\lambda_j \in \{0, 1\}$ *for all* $j = 1, \ldots, n$.

(b) $\lambda = -1$, *there is* $k \geq 1$ *and* $m = n - 2k$ *such that P is permutationally similar to* $P = P_1 \oplus \cdots \oplus P_k \oplus \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$, *where* $\lambda_j \in \{0, 1\}$ *for all* $j = 1, \ldots, m$, *and*

$$P_i = \frac{1}{2}\begin{pmatrix} 1 & p_i \\ \bar{p}_i & 1 \end{pmatrix} \qquad \textit{with} \quad |p_i| = 1, \quad i = 1, \ldots, k.$$

(c) $(\mathbb{F}^n, \lambda) = (\mathbb{R}^4, -1)$, $\mathcal{K} \in \{\langle G, A\rangle, \langle G, B\rangle\}$, *and there is* $T \in \mathcal{K}$ *with* $T = T^t$ *such that* $P = (I + T)/2$.

(d) $(\mathbb{F}^n, \lambda) = (\mathbb{R}^2, -1)$, *$\mathcal{K}$ is a dihedral group, and there is* $T \in \mathcal{K}$ *with* $T = T^t$ *such that* $P = (I + T)/2$.

We remark that conditions (a) – (d) can be summarized into one single condition, namely, there is $T \in \mathcal{K}$ with $(T - \lambda I)(T - I) = 0$ such that $P = (T - \lambda I)/(1 - \lambda)$. Our conditions give a concrete description of the structure of P. In particular, in condition (c) and (d) one can actually enumerate all isometries T such that $T = T^t$ if so desired.

Using the complex case of the above proposition, we see that P is a bicircular projection if and only if condition (a) holds,

Proof. ($\Rightarrow$) Suppose $T = P + \lambda\overline{P} = \lambda I + (1 - \lambda)P \in \mathcal{K}$. We consider three cases.

Case 1. Suppose $\mathcal{K} = G$. Let $T = DR$ be such that D is a diagonal matrix and R is a permutation matrix. Then R is permutationally similar to $R_1 \oplus R_2 \oplus \cdots \oplus R_k \oplus I_m$ such that each R_j is a permutation matrix with ones in the $(1, 2), (2, 3), \ldots, (n_{j-1}, n_j), (n_j, 1)$ positions, where $n_j > 1$ is the order of the matrix R_j. Suppose D is permutationally similar to $D_1 \oplus \cdots \oplus D_k \oplus D_0$ accordingly. Then the spectrum of $P = (T - \lambda I)/(1 - \lambda)$ is $\{1, 0\}$, which is a union of those of $P_1, \ldots, P_k, P_0$, where $P_i = (D_i R_i - \lambda I_{n_i})/(1 - \lambda)$ for $i = 1, \ldots, k$, and $P_0 = (D_0 - \lambda I_m)/(1 - \lambda)$. Note that for $i = 1, \ldots, k$, P_i has n_i distinct eigenvalues.

If $k = 0$, then condition (a) holds. If $k > 0$, then $n_1 = \cdots = n_k = 2$, and each P_i has eigenvalues $1, 0$. Hence, P_i has trace 1 and determinant 0. Thus, $\lambda = -1$ and P_i has the form described in (b). Evidently, $P_0 = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$ also has the form described in (b). So, condition (b) follows.

Case 2. Suppose $\mathbb{F}^n = \mathbb{R}^4$, and $\mathcal{K} \in \{\langle G, A\rangle, \langle G, B\rangle\}$. Then $\lambda = -1$. By Remark 1.2, we have $T^2 = I_4$. Since $T^tT = I_4$, we have $T = T^{-1} = T^t$. So, condition (c) holds.

Case 3. Suppose $\mathbb{F}^n = \mathbb{R}^2$ and $\mathcal{K}$ is dihedral group. Using an argument similar to those in Case 2, we get condition (d).

The converse of the result is clear. ∎

4 Unitarily invariant norms

Let $\mathbf{V} = M_{m,n}(\mathbb{F})$ and let G be the group of all linear operators of the form $A \mapsto UAV$ for some fixed unitary (orthogonal) $U \in M_m(\mathbb{F})$ and $V \in M_n(\mathbb{F})$. Then G-invariant norms are called unitarily invariant norms. In the following we determine the structure of those linear projections $P : M_{m,n}(\mathbb{F}) \to M_{m,n}(\mathbb{F})$ such that $P + \lambda\overline{P}$ is an isometry for a unitarily invariant norm. The result is covered by Proposition 2.1 if the norm is a multiple of the Frobenius norm. So, we exclude this case in our result. We first describe the isometries for unitarily invariant norms.

Let $\mathcal{K}$ be the isometry group of a unitarily invariant norm $\|\cdot\|$ on $\mathbf{V}$ and let τ be the transposition operator on $M_n(\mathbb{F})$, i.e., $\tau(A) = A^t$. Let $\varphi : M_4(\mathbb{R}) \to M_4(\mathbb{R})$ be the linear operator defined by

$$\varphi(A) = (A + B_1AC_1 + B_2AC_2 + B_3AC_3)/2,$$

where

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; C_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$B_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; C_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$B_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; C_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Suppose $\|\cdot\|$ is a G-invariant norm which is not a multiple of the Frobenius norm, i.e., the norm induced by the inner product $\langle X, Y\rangle = \operatorname{tr} XY^*$ on $M_{m,n}(\mathbb{F})$. Then the following holds (see for example [2, 7, 10]): if $m \neq n$, the isometry group of $\|\cdot\|$ is G; if $\mathbf{V} = M_n(\mathbb{F}) \neq M_4(\mathbb{R})$, the isometry group of $\|\cdot\|$ is $\langle G, \tau\rangle$; if $\mathbf{V} = M_4(\mathbb{R})$, the isometry group of $\|\cdot\|$ is $\langle G, \tau\rangle$ or $\langle G, \tau, \varphi\rangle$.

Proposition 4.1. *Let $\|\cdot\|$ be a unitarily invariant norm on $M_{m,n}(\mathbb{F})$ not equal to a multiple of the Frobenius norm, and let $\mathcal{K}$ be the isometry group of $\|\cdot\|$. Suppose $P : M_{m,n}(\mathbb{F}) \to M_{m,n}(\mathbb{F})$ is a non-trivial linear projection and $\lambda \in \mathbb{T} \setminus \{1\}$. Then $P + \lambda\overline{P} \in \mathcal{K}$ if and only if one of the following holds.*

(a) *There exist $R = R^* = R^2$ in $M_m(\mathbb{F})$ and $S = S^* = S^2$ in $M_n(\mathbb{F})$ such that P has the form $A \mapsto RA$ or $A \mapsto AS$.*

(b) $\lambda = -1$, *and there exist* $R = R^* = R^2$ *in* $M_m(\mathbb{F})$ *and* $S = S^* = S^2$ *in* $M_n(\mathbb{F})$ *such that* P *has the form* $A \mapsto RAS + (I_m - R)A(I_n - S)$.

(c) $m = n$, $\lambda = -1$, *and there is* $U \in U(\mathbb{F}^n)$ *such that* P *or* $\overline{P}$ *has the form* $A \mapsto (A + UA^t\overline{U})/2$.

(d) $M_{m,n}(\mathbb{F}) = M_4(\mathbb{R})$, *and there is* $T \in \mathcal{K}$ *with* $T^2 = I$ *such that* $P = (I + T)/2$.

Using the complex case of the above proposition, we see that P is a bicircular projection if and only if condition (a) holds. Also, it is possible to enumerate all $T \in \mathcal{K}$ satisfying condition (d); in particular, since every T in $\mathcal{K}$ is an orthogonal operator on $M_4(\mathbb{R})$, we see that $T^2 = I$ if and only if T is self-adjoint, equivalently, $(T(E_{pq}), E_{rs}) = (E_{pq}, T(E_{rs}))$ for all $p, q, r, s \in \{1, \ldots, 4\}$.

Proof. Suppose $T = P + \lambda\overline{P} = \lambda I + (1 - \lambda)P \in \mathcal{K}$. We consider three cases.

Case 1. Suppose T has the form $A \mapsto UAV$ for some $U \in U(\mathbb{F}^m)$ and $V \in U(\mathbb{F}^n)$. By Lemma 1.1, T has spectrum $\{1, \lambda\}$. Assume that U has eigenvalues $u_1, \ldots, u_m$ and V has eigenvalues $v_1, \ldots, v_n$. Then $T = U \otimes V^t$ has eigenvalues $u_1v_1, u_1v_2, \ldots, u_mv_n$. We may assume that $u_1v_1 = 1$. We may further assume that $u_1 = 1 = v_1$. Otherwise, replace (U, V) by $(U/u_1, V/v_1)$. Because $u_1v_j \in \{1, \lambda\}$ for all j, we see that V has spectrum $\{1, \lambda\}$. Similarly, we can show that U has spectrum $\{1, \lambda\}$. If $\lambda \neq -1$ then $U = I_m$ or $V = I_n$; otherwise, $T = U \otimes V$ has spectrum $\{1, \lambda, \lambda^2\} \neq \{1, \lambda\}$. Thus, condition (a) holds. If $\lambda = -1$, then both U and V may have spectrum $\{1, -1\}$. So, $U \in U(\mathbb{F}^m)$ and $V \in U(\mathbb{F}^n)$ satisfy $U = U^*$ and $V = V^*$. Moreover, the range space of P is spanned by $\{X \in M_{m,n}(\mathbb{F}) : UXV = X\}$. Thus, condition (b) holds.

Case 2. Suppose $m = n$ and T has the form $A \mapsto UA^tV$ for some $U \in U(\mathbb{F}^m)$ and $V \in U(\mathbb{F}^n)$. Since T has spectrum $\{1, \lambda\}$, we see that T^2 has the form $A \mapsto UV^tAU^tV$ and has spectrum $\{1, \lambda^2\}$. Let $X = UV^t$ and $Y = U^tV$. Then X and Y have the same eigenvalues, say, $\mu_1, \ldots, \mu_n$. Further, T^2 is diagonalizable and has eigenvalues $\mu_i\mu_j$ for $1 \leq i, j \leq n$.

If $\lambda^2 \notin \{1, -1\}$, then X cannot have spectrum $\{1, \lambda^2\}$; otherwise, T^2 will have spectrum $\{1, \lambda^2, \lambda^4\} \neq \{1, \lambda^2\}$. So, X is a scalar matrix and so is Y. It follows that T^2 is a scalar operator, which contradicts the fact that T^2 has spectrum $\{1, \lambda^2\}$.

If $\lambda^2 = -1$, then $\mathbb{F} = \mathbb{C}$, $\lambda = \pm i$, $X^2 = I$, and $V = X^t\overline{U}$. Suppose $W_1 \in U(\mathbb{C}^n)$ is such that $X = W_1^*DW_1$ with $D = I_k \oplus -I_{n-k}$ for some $k \in \{1, \ldots, n-1\}$. Then T^2 has the form

$$A \mapsto XAU^tX^t\overline{U} = W_1^*DW_1AW_2^*DW_2$$

with $W_2 = \overline{W}_1\overline{U}$. Thus, $T^2(A) = A$ if and only if $W_1AW_2^* = A_1 \oplus A_2$ for some $A_1 \in M_k(\mathbb{C})$ and $A_2 \in M_{n-k}(\mathbb{C})$. Since T is a diagonalizable operator on $M_n(\mathbb{C})$ with spectrum $\{1, \lambda\}$ where $\lambda = \pm i$, we see that $T(A) = A$ if and only if $T^2(A) = A$. Thus, for any A of the form $W_1^*(A_1 \oplus A_2)W_2$, we have

$$\begin{aligned} & W_1^*(A_1 \oplus A_2)W_2 = A = T(A) = UA^tX^t\overline{U} = UW_2^t(A_1^t \oplus A_2^t)\overline{W}_1X^t\overline{U} \\ = \; & U(U^*W_1^*)(A_1^t \oplus A_2^t)\overline{W}_1(W_1^tD\overline{W}_1)\overline{U} = W_1^*(A_1^t \oplus A_2^t)DW_2, \end{aligned}$$

which is impossible.

If $\lambda^2 = 1$, then $T^2 = I$ and $\lambda = -1$. We may assume that all the eigenvalues of X equal $\mu \in \{1, -1\}$. Thus, $UV^t = I$ or $UV^t = -I$. Hence, T has the form $A \mapsto \pm UA^t\overline{U}$. By Lemma 1.1, $P = (T+I)/2$ has the form $A \mapsto (\pm UA^t\overline{U} + A)/2$. Thus, condition (c) holds.

Case 3. Suppose $M_{m,n}(\mathbb{F}) = M_4(\mathbb{R})$ and $\mathcal{K} = \langle G, \tau, \varphi \rangle$. Then $\lambda = -1$. By Lemma 1.1, there is $T \in \mathcal{K}$ with $T^2 = I$ such that $P = (I+T)/2$. Thus, condition (d) holds.

The converse is clear. ∎

5 Unitary congruence invariant norms

In this section we consider $\mathbf{V}$ to be one of the following matrix spaces: $S_n(\mathbb{C})$ is the linear space of all $n \times n$ symmetric matrices over $\mathbb{C}$, and $K_n(\mathbb{F})$ is the linear space of all $n \times n$ skew-symmetric matrices over $\mathbb{F}$. Let G be the group of all linear operators of the form $A \mapsto U^tAU$ for some fixed unitary (orthogonal) $U \in M_n(\mathbb{F})$. Then G-invariant norms are called unitary congruence invariant norms. Of course, if U is unitary and $T : \mathbf{V} \to \mathbf{V}$ is defined by $T(A) = UAU^t$, then T is a unitary operator on $\mathbf{V}$ and preserves any unitary congruence invariant norm. Moreover, suppose $U \in U(\mathbb{C}^n)$ has eigenvalues $\mu_1, \ldots, \mu_n$ with orthonormal eigenvectors $u_1, \ldots, u_n$. Then for $\mathbf{V} = S_n(\mathbb{C})$, T has eigenvalues $\mu_i\mu_j$ with eigenvector $u_iu_j^t + u_ju_i^t$ for $1 \le i, j \le n$; for $\mathbf{V} = K_n(\mathbb{C})$, T has eigenvalues $\mu_i\mu_j$ with eigenvector $u_iu_j^t - u_ju_i^t$ for $1 \le i < j \le n$. For $K_n(\mathbb{R})$, we can extend T to $K_n(\mathbb{C})$ and conclude that T has eigenvalues $\mu_i\mu_j$ for $1 \le i < j \le n$ as well. This observation will be used in our discussion.

If $\mathcal{K}$ is the isometry group of a unitary congruence invariant norm on $S_n(\mathbb{C})$, which is not a multiple of the Frobenius norm, then $\mathcal{K} = G$ (see [5, 7]). If $\mathcal{K}$ is the isometry group of a unitary congruence invariant norm on $K_n(\mathbb{F})$, which is not a multiple of the Frobenius norm, then one of the following holds (see [5, 7]):

1. $\mathcal{K} = G$ if $\mathbb{F} = \mathbb{C}$, or $\mathcal{K} = \langle G, \tau \rangle$ if $\mathbb{F} = \mathbb{R}$.
2. $n = 4$ and $\mathcal{K} = \langle G, \psi \rangle$ if $\mathbb{F} = \mathbb{C}$, or $\mathcal{K} = \langle G, \tau, \psi \rangle$ if $\mathbb{F} = \mathbb{R}$, where $\psi(A)$ is obtained from A by interchanging its $(1,4)$ and $(2,3)$ entries, and interchanging its $(4,1)$ and $(3,2)$ entries accordingly.

Note that the mapping on $K_4(\mathbb{C})$ defined by $A \mapsto \psi(UAU^t)$ can be written as $A \mapsto \det(U)W\psi(A)W^t$ with $W = R\overline{U}R$, where $R = E_{14} - E_{23} + E_{32} - E_{41}$. (For instance, one can verify the equality of the two mappings for $A = E_{ij} - E_{ji}$ with $1 \le i < j \le 4$.) As a result, for each mapping T in $\langle G, \psi \rangle \setminus G$ there are $X, Y \in U(\mathbb{C}^4)$ such that $T(A) = \psi(XAX^t) = Y\psi(A)Y^t$ for all $A \in K_4(\mathbb{C})$. Similarly, for each $U \in U(\mathbb{R}^4)$, a mapping defined by $A \mapsto \psi(UAU^t)$ can be written as $A \mapsto \det(U)RUR\psi(A)RU^tR$, where R is defined as above. Consequently, for each mapping T in $\langle G, \tau, \psi \rangle \setminus G$ there are $X, Y \in U(\mathbb{R}^4)$ such that $T(A) = \pm\psi(XAX^t) = \pm\det(X)Y\psi(A)Y^t$ for all $A \in K_4(\mathbb{R})$.

Proposition 5.1. *Let $\|\cdot\|$ be a unitary congruence invariant norm on $S_n(\mathbb{C})$, which is not a multiple of the Frobenius norm, and let $\mathcal{K}$ be the isometry group of $\|\cdot\|$. Suppose $P : S_n(\mathbb{C}) \to S_n(\mathbb{C})$ is a non-trivial linear projection and $\lambda \in \mathbb{T} \setminus \{1\}$. Then $P + \lambda\overline{P} \in \mathcal{K}$ if and only if $\lambda = -1$ and there exists $R = R^* = R^2$ in $M_n(\mathbb{C})$ such that P or $\overline{P}$ has the form $A \mapsto R^tAR + (I - R^t)A(I - R)$.*

By the above result, one sees that there are no non-trivial bicircular projections on $S_n(\mathbb{C})$.

Proof. Let $T = P + \lambda\overline{P} = \lambda I + (1-\lambda)P \in \mathcal{K}$. Then there exists $U \in U(\mathbb{C}^n)$ such that $T(A) = U^tAU$ for all $A \in S_n(\mathbb{C})$. By Lemma 1.1, T has spectrum $\{1, \lambda\}$. Suppose U has eigenvalues $\mu_1, \dots, \mu_n$. Then T has eigenvalues $\mu_j\mu_k$, $1 \le j, k \le n$. If U has at least 3 distinct eigenvalues, say, μ_1, μ_2, μ_3, then $\mu_1\mu_2, \mu_2\mu_3, \mu_1\mu_3$ are distinct eigenvalues of T, which is impossible. Hence U has only two distinct eigenvalues, say, μ_1, μ_2. Thus, $\{\mu_1^2, \mu_2^2, \mu_1\mu_2\} = \{1, \lambda\}$. Hence, two of the numbers $\mu_1^2, \mu_2^2, \mu_1\mu_2$ are equal, and we have $\mu_1 = -\mu_2$. As a result, $\mu_1^2 = \mu_2^2 = -\mu_1\mu_2$. Hence, $\lambda = -1$ and either

(1) $\mu_1^2 = \mu_2^2 = 1$ with $\mu_1\mu_2 = -1$ so that $\{\mu_1, \mu_2\} = \{1, -1\}$, or
(2) $\mu_1^2 = \mu_2^2 = -1$ with $\mu_1\mu_2 = 1$ so that $\{\mu_1, \mu_2\} = \{i, -i\}$.

If (1) holds, then $U = U^*$; if (2) holds, then $U = -U^*$. Since T has the form $A \mapsto U^tAU$, Lemma 1.1 implies that P has the form $A \mapsto (A + U^tAU)/2$. If $U = U^*$, then P has the form $A \mapsto R^tAR + (I - R^t)A(I - R)$, with $R = (I - U)/2$. If $U = -U^*$, then $\overline{P}$ has the form $A \mapsto R^tAR + (I - R^t)A(I - R)$, with $R = (I - iU)/2$. In both cases $R = R^* = R^2$.

The converse can be easily verified. ■

Next, we consider $K_n(\mathbb{F})$. Since $K_2(\mathbb{F})$ is one dimensional, we assume that $n \ge 3$ to avoid trivial consideration.

Proposition 5.2. *Let $n \ge 3$ and $\|\cdot\|$ be a unitary congruence invariant norm on $K_n(\mathbb{F})$, which is not a multiple of the Frobenius norm. Let $\mathcal{K}$ be the isometry group of $\|\cdot\|$. Suppose $P : K_n(\mathbb{F}) \to K_n(\mathbb{F})$ is a non-trivial linear projection and $\lambda \in \mathbb{T} \setminus \{1\}$. Then $P + \lambda\overline{P} \in \mathcal{K}$ if and only if one of the following holds.*

(a) *$\mathbb{F} = \mathbb{C}$, there exists $R = vv^*$ for a unit vector $v \in \mathbb{C}^n$ such that P has the form $A \mapsto R^tA + AR$ or $A \mapsto (I - R^t)A(I - R)$. (In the second case $\overline{P}$ is of the form $A \mapsto R^tA + AR$.)*

(b) *$\lambda = -1$, $\mathcal{K} = G$ if $\mathbb{F} = \mathbb{C}$ or $\mathcal{K} = \langle G, \tau\rangle$ if $\mathbb{F} = \mathbb{R}$, and there exists $R = R^* = R^2$ in $M_n(\mathbb{F})$ such that P or $\overline{P}$ has the form $A \mapsto R^tAR + (I - R^t)A(I - R)$.*

(c) *$(\lambda, n) = (-1, 4)$, $\psi \in \mathcal{K}$ and there is $U \in U(\mathbb{F}^4)$, satisfying $\psi(U^tAU) = \overline{U}\psi(A)U^*$ for all $A \in K_4(\mathbb{F})$, such that P or $\overline{P}$ has the form $A \mapsto (A+\psi(U^tAU))/2 = (A+\overline{U}\psi(A)U^*)/2$.*

By the above result, we see that P is a bicircular projection if and only if condition (a) holds. Note that most of $U \in U(\mathbb{F}^4)$ do not satisfy the condition required in (c). Here,

$$U = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \in U(\mathbb{F}^4)$$

is an example of a matrix satisfying condition (c). One sees that condition (c) is equivalent to the condition that $(2P - I)^2 = I$.

Proof. ($\Leftarrow$) Suppose that (a) holds. For all $A \in K_n(\mathbb{C})$ we have $(v^tAv)^t = -v^tAv$. Since $v^tAv \in M_1(\mathbb{C})$ we conclude $v^tAv = 0$ and thus $R^tAR = 0$. Furthermore, $R = R^* = R^2$.

If P is of the form $A \mapsto R^tA + AR$, then $P + \lambda\overline{P}$ is of the form $A \mapsto \lambda A + (1-\lambda)(R^tA + AR)$. If we define $U = R/\sqrt{\lambda} + \sqrt{\lambda}(I - R)$, then U is unitary and $U^tAU = (P + \lambda\overline{P})(A)$ for all $A \in K_n(\mathbb{C})$. Hence, $P + \lambda\overline{P} \in G \subseteq \mathcal{K}$.

If P is of the form $A \mapsto (I - R^t)A(I - R)$, then $P + \lambda\overline{P}$ has the form

$$A \mapsto A + (\lambda - 1)(R^tA + AR) = U^tAU$$

for the unitary $U = \lambda R + (I - R)$. Thus $P + \lambda\overline{P} \in G \subseteq \mathcal{K}$.

If (b) or (c) holds, then one can easily verify that $2P - I \in \mathcal{K}$.

($\Rightarrow$) Let $T = P + \lambda\overline{P} = \lambda I + (1 - \lambda)P \in \mathcal{K}$. We consider two cases.

Case 1. Suppose $\lambda = -1$. Then T^2 is the identity operator.

Assume that there is $U \in U(\mathbb{F}^n)$ such that T has the form $A \mapsto U^tAU$ or $A \mapsto -U^tAU$. In both cases, T^2 has the form $A \mapsto X^tAX$, where $X = U^2$. Since $A = T^2(A) = X^tAX$ for all $A \in K_n(\mathbb{F})$, it follows that $X = I$ or $-I$. Hence $U = U^*$ or $U = -U^*$. Note that the second case cannot occur if $\mathbb{F} = \mathbb{R}$. If $\mathbb{F} = \mathbb{R}$, then $U = U^*$ and T has the form $A \mapsto U^tAU$ or $A \mapsto -U^tAU$. Lemma 1.1 implies that P has the form $A \mapsto (A + U^tAU)/2$, or $\overline{P}$ has the form $A \mapsto (A + U^tAU)/2$. If we define $R = (I - U)/2$, then $R = R^* = R^2$, and P or $\overline{P}$ has the form $A \mapsto R^tAR + (I - R^t)A(I - R)$. If $\mathbb{F} = \mathbb{C}$, then $U = U^*$ or $U = -U^*$, and T has the form $A \mapsto U^tAU$. If $U = U^*$, then P has the form $A \mapsto R^tAR + (I - R^t)A(I - R)$, with $R = (I - U)/2$. If $U = -U^*$, then $\overline{P}$ has the form $A \mapsto R^tAR + (I - R^t)A(I - R)$, with $R = (I - iU)/2$ satisfying $R = R^* = R^2$. Thus we get (b).

Suppose $n = 4$ and there is $U \in U(\mathbb{F}^4)$ such that T has the form $A \mapsto \psi(U^tAU)$ or $A \mapsto -\psi(U^tAU)$. By the remark before Proposition 5.1, $T^2(A) = \det(U)X^tAX$ with $X = UR\overline{U}R$, where $R = E_{14} - E_{23} + E_{32} - E_{41}$. Since T^2 is the identity operator, there is $\xi \in \mathbb{C}$ with $\xi^2 = \det(U)$ such that $I/\xi = X = UR\overline{U}R$. If $\mathbb{F} = \mathbb{R}$, then X is a real matrix, and thus $\det(U) = 1$. If $\mathbb{F} = \mathbb{C}$, then we see that $U^* = \xi R\overline{U}R$. In both cases, we have

$$\psi(U^tAU) = \overline{U}\psi(A)U^*$$

for all $A \in K_4(\mathbb{F})$. We get condition (c).

Case 2. Suppose $\lambda \neq -1$. Then $\mathbb{F} = \mathbb{C}$.

Assume that there is $U \in U(\mathbb{C}^n)$ such that T has the form $A \mapsto U^tAU$. If U has eigenvalues $\mu_1, \ldots, \mu_n$, then T has eigenvalues $\mu_i\mu_j$ with $1 \leq i < j \leq n$. If U has 3 distinct eigenvalues, say, μ_1, μ_2, μ_3, then $\mu_1\mu_2, \mu_1\mu_3, \mu_2\mu_3$ are distinct eigenvalues of T, which is impossible. So, U has two distinct eigenvalues, say, μ_1 and μ_2. If each of them has multiplicities at least 2, then $\{\mu_1^2, \mu_2^2, \mu_1\mu_2\} = \{1, \lambda\}$. Thus, two of the three numbers $\mu_1^2, \mu_2^2, \mu_1\mu_2$ are equal. It follows that $\mu_1 = -\mu_2$, and $\mu_1^2 = \mu_2^2 = -\mu_1\mu_2$. Hence, $\lambda = -1$, which is a contradiction. As a result, the eigenvalues of U have the form $\mu_1, \cdots, \mu_1, \mu_2$ such that either

(1) $\mu_1^2 = \lambda$ and $\mu_1\mu_2 = 1$, or (2) $\mu_1^2 = 1$ and $\mu_1\mu_2 = \lambda$.

Consequently, the eigenvalues of U have one of the following patterns.

(i) $\sqrt{\lambda}, \cdots, \sqrt{\lambda}, 1/\sqrt{\lambda}$, (ii) $-\sqrt{\lambda}, \cdots, -\sqrt{\lambda}, -1/\sqrt{\lambda}$, (iii) $1, \cdots, 1, \lambda$, (iv) $-1, \cdots, -1, -\lambda$.

If (i) or (ii) holds, then $U = \pm W^*((1/\sqrt{\lambda})E_{11} + \sqrt{\lambda}(I - E_{11}))W$. Since $E_{11}AE_{11} = 0$ for all $A \in K_n(\mathbb{C})$, we have

$$T(W^tAW) = W^t(\lambda A + (1-\lambda)(E_{11}A + AE_{11}))W$$

for all $A \in K_n(\mathbb{C})$. If we put $R = W^*E_{11}W$, then

$$T(W^tAW) = \lambda W^tAW + (1-\lambda)(R^tW^tAW + W^tAWR)$$

for all $A \in K_n(\mathbb{C})$. By Lemma 1.1, P has the form $A \mapsto R^tA + AR$.

Now suppose (iii) or (iv) holds. Thus $U = \pm W^*(\lambda E_{11} + (I - E_{11}))W$. For all $A \in K_n(\mathbb{C})$, we have

$$T(W^tAW) = W^tAW - (1-\lambda)(R^tW^tAW + W^tAWR)$$

with $R = W^*E_{11}W$. By Lemma 1.1, $\overline{P}$ has the form $A \mapsto R^tA + AR$.

Note that $R = W^*E_{11}W = vv^*$ for the unit vector $v = W^*e_1 \in \mathbb{C}^n$. Furthermore, $R^tAR = W^tE_{11}(\overline{W}AW^*)E_{11}W = 0$ since $\overline{W}AW^* \in K_n(\mathbb{C})$. Therefore, if $\overline{P}$ has the form $A \mapsto R^tA + AR$, we can also write $P(A) = (I - R^t)A(I - R)$. Hence we get (a).

Next, suppose $n = 4$ and there is $U \in U(\mathbb{C}^4)$ such that T has the form $A \mapsto \psi(U^tAU)$. By the remark before Proposition 5.1, T^2 has the form $A \mapsto \det(U)X^tAX$, where $X = UR\overline{U}R$ with $R = E_{14} - E_{23} + E_{32} - E_{41}$. If X has eigenvalues $\mu_1, \mu_2, \mu_3, \mu_4$, then T^2 has spectrum $\{\det(U)\mu_i\mu_j : 1 \leq i < j \leq 4\} = \{1, \lambda^2\}$.

Note that X cannot have 3 distinct eigenvalues, say, μ_1, μ_2, μ_3. Otherwise, $T^2/\det(U)$ will have distinct eigenvalues $\mu_1\mu_2, \mu_1\mu_3, \mu_2\mu_3$.

If $X = \xi I$, then T^2 has spectrum $\{\det(U)\xi^2\}$. Then $\lambda^2 = 1$, i.e., $\lambda = -1$, which contradicts our assumption.

So, X has two distinct eigenvalues, say, μ_1, μ_2.

Let $Y = \overline{U}R$ and $Z = UR$. Then $\overline{X} = YZ$ and $X = ZY$ have the same eigenvalues. Thus, the non-real eigenvalues of X occurs in complex conjugate pairs. Moreover, $\det(X) = \det(U)\det(\overline{U}) = 1$.

(1) If μ_1, μ_2 are real numbers, then X has eigenvalues $1, 1, -1, -1$, and $T^2/\det(U)$ has spectrum $\{1, -1\}$. Hence, $\det(U) = \pm 1$ and $\lambda^2 = -1$.

(2) If μ_1 or μ_2 is non-real, then $\mu_1 = \overline{\mu}_2$ and $T^2/\det(U)$ has spectrum $\{1, \mu_1^2, \overline{\mu}_1^2\}$. It follows that $\mu_1 = \pm i$. Thus, T^2 has spectrum $\{\det(U), -\det(U)\}$. So, $\det(U) = \pm 1$ and $\lambda^2 = -1$.

Let $X = \mu W(I_2 \oplus -I_2)W^*$ with $\mu = \{1, \lambda\}$. Then

$$T^2(A) = \xi \overline{W}(I_2 \oplus -I_2)W^t A W(I_2 \oplus -I_2)W^*$$

with $\xi = \pm 1$. Suppose $\xi = 1$. Since $T = (T^2 + \lambda I)/(1 + \lambda)$, for all $A \in K_4(\mathbb{C})$ of the form

$$\overline{W} \begin{pmatrix} A_1 & B \\ -B^t & A_2 \end{pmatrix} W^*, \qquad A_1, A_2 \in K_2(\mathbb{C}),$$

we have

$$T(A) = \overline{W} \begin{pmatrix} A_1 & \lambda B \\ -\lambda B^t & A_2 \end{pmatrix} W^*.$$

Now, the mapping L defined by

$$L(A) = (\lambda I_2 \oplus I_2)W^t T(\overline{W} A W^*)W(\lambda I_2 \oplus I_2)$$

is an isometry for $K_4(\mathbb{C})$, and for any $A \in K_4(\mathbb{C})$ of the form

$$\begin{pmatrix} A_1 & B \\ -B^t & A_2 \end{pmatrix}, \qquad A_1, A_2 \in K_2(\mathbb{C}),$$

we have

$$L(A) = \begin{pmatrix} -A_1 & -B \\ B^t & A_2 \end{pmatrix}.$$

Since $\| \cdot \|$ is not a multiple of the inner product norm, there is X, Y with singular values $1, 1, 0, 0$ and $\cos t, \cos t, \sin t, \sin t$ for some $t \in (0, \pi/4]$ such that $\|X\| \neq \|Y\|$. Let

$$Z = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \cos t & \sin t & 0 \\ -\cos t & 0 & 0 & -\cos t \\ -\sin t & 0 & 0 & -\sin t \\ 0 & \cos t & \sin t & 0 \end{pmatrix}.$$

Then Z has singular values $1, 1, 0, 0$, and $L(Z)$ has singular values $\cos t, \cos t, \sin t, \sin t$. Hence, Z is unitarily congruent to X and $L(Z)$ is unitarily congruent to Y. However,

$$\|Z\| = \|X\| \neq \|Y\| = \|L(Z)\|,$$

which is a contradiction.

If $\xi = -1$, then we observe $\overline{P}$ instead of P. Applying the above arguments to $\overline{P}$ we can derive a contradiction. ■

6 Additional results and remarks

We can use the same proof strategy to obtain results on other normed vector spaces equipped with G-invariant norms. We mention a few more examples and some remarks in this section.

Let $\mathbf{V} = H_n(\mathbb{C})$ be the real space of all $n \times n$ complex hermitian matrices. Let G be the group of linear operators on $\mathbf{V}$ of the form $A \mapsto U^*AU$ for some $U \in U(\mathbb{C}^n)$. Then G-invariant norms are called unitary similarity invariant norms. Let $\mathcal{K}$ be the isometry group of a unitary similarity invariant norm $\|\cdot\|$ on $\mathbf{V}$. Then $\mathcal{K}$ must be of one of the following forms (see [5, 7, 11, 12]):

(a) $\mathcal{K} = SU(\mathbf{V})S^{-1}$ for some $S \in \Gamma$, where Γ is the group of invertible operators of the form $A \mapsto \alpha A + (\beta - \alpha)(\operatorname{tr} A)I_n/n$ for some positive $\alpha, \beta \in \mathbb{R}$,

(b) $\mathcal{K} = U'(\mathbf{V}) = \{U \in U(\mathbf{V}) \mid U(I_n) = \pm I_n\}$,

(c) $\mathcal{K} = \langle G, \tau, T_0 \rangle$, where T_0 is defined by $T_0(A) = A - 2(\operatorname{tr} A)I_n/n$,

(d) $\mathcal{K} = \langle G, \tau \rangle$.

Note that if $U \in U(\mathbb{C}^n)$ has eigenvalues $\mu_1, \dots, \mu_n$, then the operator $A \mapsto UAU^*$ on $H_n(\mathbb{C})$ has the same eigenvalues as the operator acting on $M_n(\mathbb{C}) = H_n(\mathbb{C}) + iH_n(\mathbb{C})$, namely, $\mu_i \overline{\mu}_j$ with $1 \le i, j \le n$.

Proposition 6.1. *Let $\|\cdot\|$ be a unitary similarity invariant norm on $\mathbf{V} = H_n(\mathbb{C})$, and let $\mathcal{K}$ be the isometry group of $\|\cdot\|$. Suppose $P : \mathbf{V} \to \mathbf{V}$ is a non-trivial linear projection. Then $P - \overline{P} \in \mathcal{K}$ if and only if one of the following holds.*

(a) *$\mathcal{K} = SU(\mathbf{V})S^{-1}$ for some $S \in \Gamma$, and there exists $R : \mathbf{V} \to \mathbf{V}$, satisfying $R = R^* = R^2$, such that $P = SRS^{-1}$.*

(b) *$\mathcal{K} = U'(\mathbf{V})$, $P = P^*$, and $P(I_n) = 0$ or $P(I_n) = I_n$.*

(c) *$\mathcal{K} = \langle G, \tau \rangle$ or $\mathcal{K} = \langle G, \tau, T_0 \rangle$,*

(c.1) *there exists $R = R^2 \in H_n(\mathbb{C})$ such that P or $\overline{P}$ has the form*

$$A \mapsto RAR + (I - R)A(I - R),$$

or

(c.2) *there exists $U \in U(\mathbb{C}^n)$ with $U = \pm U^t$ such that P or $\overline{P}$ has the form*

$$A \mapsto (A + \overline{U}A^tU)/2.$$

(d) $\mathcal{K} = \langle G, \tau, T_0 \rangle$,

(d.1) *there exists* $R = R^2 \in H_n(\mathbb{C})$ *such that* P *or* $\overline{P}$ *has the form*

$$A \mapsto RAR + (I - R)A(I - R) - (\operatorname{tr} A)I_n/n,$$

or

(d.2) *there exists* $U \in U(\mathbb{C}^n)$ *with* $U = \pm U^t$ *such that* P *or* $\overline{P}$ *has the form*

$$A \mapsto (A + \overline{U}A^tU)/2 - (\operatorname{tr} A)I_n/n.$$

Observe that case (a) happens if and only if the norm is induced by the inner product $(X, Y) = (SX, SY)$. Thus, the result also follows from Proposition 2.1.

Proof. Let us define $T = P - \overline{P} = 2P - I \in \mathcal{K}$. We consider three cases.

Case 1. Let $\mathcal{K} = SU(\mathbf{V})S^{-1}$ for some $S \in \Gamma$. Then $T = SUS^{-1}$ for some fixed $U \in U(\mathbf{V})$. If we define $R = (U + I)/2$, then

$$P = (T + I)/2 = S(U + I)S^{-1}/2 = SRS^{-1}.$$

Furthermore, R is normal and its spectrum is $\{0, 1\}$, thus $R^2 = R = R^*$. Hence we get (a).

Case 2. Let $\mathcal{K} = U'(\mathbf{V})$. Then T is unitary, and $T(I_n) = I_n$ or $T(I_n) = -I_n$. Since $T^2 = I$, we have $T = T^*$. Then $P = (T + I)/2$ implies $P^* = P$. Since $T(I_n)$ equals $-I_n$ or I_n, we get $P(I_n)$ equals 0 or I_n. Thus condition (b) follows.

Case 3. Suppose T has one of the following forms:

(i) $A \mapsto aU^*AU + (b - a)(\operatorname{tr} A)I_n/n$, or (ii) $A \mapsto aU^*A^tU + (b - a)(\operatorname{tr} A)I_n/n$,

for some $U \in U(\mathbb{C}^n)$ and $a, b \in \{-1, 1\}$. Then T^2 has the form $A \mapsto X^*AX$, with $X = U^2$ if T has the first form, and $X = \overline{U}U$ if T has the second form. Since T has spectrum $\{1, -1\}$, T^2 is the identity operator. Since X is unitary, it has modulus one eigenvalues, say, $\mu_1, \ldots, \mu_n$. Then T^2 has eigenvalues $\overline{\mu_i}\mu_j$ for $1 \le i, j \le n$. Hence $\overline{\mu_i}\mu_j = 1$ and thus $\mu_j = \mu_i$ for $1 \le i, j \le n$. Therefore, X is a (unitary) scalar matrix.

Assume that T has the form (i) and $U^2 = \mu I$ for some modulus one $\mu \in \mathbb{C}$. Let $R = (I - (1/\sqrt{\mu})U)/2$. Then $R = R^* = R^2$ and P or $\overline{P}$ has the form $A \mapsto RAR + (I - R)A(I - R)$ or $A \mapsto RAR + (I - R)A(I - R) - (\operatorname{tr} A)I_n/n$. So, we get (c.1) or (d.1). Now assume that T has the form (ii) and $\overline{U}U = \mu I$. Then $U = \mu U^t = \mu(\mu U^t)^t = \mu^2 U$, so $\mu^2 = 1$. Thus U is symmetric or skew-symmetric. In this case P or $\overline{P}$ has the form $A \mapsto (A + aU^*A^tU + (b - a)(\operatorname{tr} A)I_n/n)/2$. We get (c.2) or (d.2).

The converse is easy to verify. ∎

One may consider orthogonal similarity invariant norms on the real space $S_n(\mathbb{R})$ of $n \times n$ real symmetric matrices, i.e., those norms $\|\cdot\|$ on $S_n(\mathbb{R})$ such that $\|UAU^t\| = \|A\|$ for all $A \in S_n(\mathbb{R})$ and $U \in U(\mathbb{R}^n)$. The results on isometries of such norms are very similar to those of unitary similarity invariant norms on $H_n(\mathbb{C})$. Accordingly, one can obtain a result similar to Proposition 6.1 for orthogonal similarity invariant norms on $S_n(\mathbb{R})$. In that case (c.2) and (d.2) reduce to (c.1) and (d.1) respectively.

One can consider unitary congruence invariant norms on $M_n(\mathbb{F})$. We may focus on those norms which are not unitarily invariant. Otherwise, we are back to section 4. In many cases, the isometry group is reducible and act on the subspaces $S_n(\mathbb{F})$ and $K_n(\mathbb{F})$ independently; see [5, 7]. One can deduce the results on $M_n(\mathbb{F})$ using those on $S_n(\mathbb{F})$ and $K_n(\mathbb{F})$.

Similarly, one can consider unitary similarity invariant norms on $M_n(\mathbb{C})$. Again, we should assume that the norms are not unitarily invariant. In many cases (see [5]), the isometry group would leave $H_n(\mathbb{C})$ invariant (up to a unit multiple). Then, we can apply the result on $H_n(\mathbb{C})$ to obtain the result on $M_n(\mathbb{C})$.

Our proof techniques can also be used to study other matrix spaces equipped with G-invariant norms; [5, 9, 12]. It would be interesting to extend our techniques and results to infinite dimensional normed spaces; [1, 4, 6, 8, 13].

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References

[1] J.T. Chan, C.K. Li and N.C. Tu, A class of unitarily invariant norms on $B(H)$, Proc. Amer. Math. Soc. 129 (2001), 1065-1076.

[2] D.Ž. Đoković and C.K. Li, Overgroups of some classical linear groups with applications to linear preserver problems, Linear Algebra Appl. 197/198 (1994), 31-61.

[3] D.Ž. Đoković, C.K. Li and L. Rodman, Isometries of symmetric gauge functions, Linear Multilinear Algebra 30 (1991), 81-92.

[4] M. Fošner and D. Ilišević, On a class of projections on $*$-rings, Comm. Algebra 33 (2005), 3293–3310.

[5] R.M. Guralnick and C.K. Li, Invertible preservers and algebraic groups. III. Preservers of unitary similarity (congruence) invariants and overgroups of some unitary subgroups, Linear Multilinear Algebra 43 (1997), 257-282.

[6] R.V. Kadison, Isometries of operator algebras, Ann. of Math. (2) 54 (1951), 325-338.

[7] C.K. Li, Some aspects of the theory of norms, Linear Algebra Appl. 212/213 (1994), 71-100.

[8] C.K. Li and B. Randrianantoanina, Isometries of direct sums of sequence spaces, Asian J. Math. 2 (1998), 157-180.

[9] C.K. Li, P. Šemrl, and A.R. Sourour, Isometries for Ky-Fan norms on block triangular matrix algebras, Arch. Math. (Basel) 81 (2003), 175–181.

[10] C.K. Li and N.K. Tsing, Linear operators preserving unitarily invariant norms of matrices, Linear Multilinear Algebra 26 (1990), 119-132.

[11] C.K. Li and N.K. Tsing, Linear operators preserving unitary similarity invariant norms, Linear Multilinear Algebra 27 (1990), 213-224.

[12] V.P. Platonov and D.Ž. Đoković, Subgroups of $GL(n^2, \mathbb{C})$ containing $PSU(n)$, Trans. Amer. Math. Soc. 348 (1996), 141-152.

[13] A.R. Sourour, Isometries of norm ideals of compact operators, J. Funct. Anal. 43 (1981), 69-77.

[14] L.L. Stachó and B. Zalar, Bicircular projections on some matrix and operator spaces, Linear Algebra Appl. 384 (2004), 9-20.