C^*-Isomorphisms, Jordan Isomorphisms, and Numerical Range Preserving Maps

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Abstract

Let $\mathbf{V} = B(H)$ or $S(H)$, where $B(H)$ is the algebra of bounded linear operator acting on the Hilbert space H, and $S(H)$ is the set of self-adjoint operators in $B(H)$. Denote the numerical range of $A \in B(H)$ by $W(A) = \{(Ax, x) : x \in H,\ (x, x) = 1\}$. It is shown that a surjective map $\phi : \mathbf{V} \to \mathbf{V}$ satisfies

$$W(AB + BA) = W(\phi(A)\phi(B) + \phi(B)\phi(A)) \qquad \text{for all } A, B \in \mathbf{V}$$

if and only if there is a unitary operator $U \in B(H)$ such that ϕ has the form

$$X \mapsto \pm U^*XU \quad \text{or} \quad X \mapsto \pm U^*X^tU,$$

where X^t is the transpose of X with respect to a fixed orthonormal basis. In other words, the map ϕ or $-\phi$ is a C^*-isomorphism on $B(H)$ and a Jordan isomorphism on $S(H)$. Moreover, if H has finite dimension, then the surjective assumption on ϕ can be removed.

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1 Introduction

Let $B(H)$ be the algebra of bounded linear operator acting on the Hilbert space H, and $S(H)$ be the set of self-adjoint operators in $B(H)$. If H has finite dimension, $B(H)$ is identified with the algebra M_n of $n \times n$ complex matrices, and $S(H)$ is identified with S_n the set of $n \times n$ complex Hermitian matrices. Denote the numerical range of $A \in B(H)$ by

$$W(A) = \{(Ax, x) : x \in H,\ (x, x) = 1\}.$$

Let $U \in B(H)$ be a unitary operator, and define a mapping ϕ on $B(H)$ or $S(H)$ by

$$A \mapsto U^*AU \quad \text{or} \quad A \mapsto U^*A^tU,$$

where A^t is the transpose of A with respect to a fixed orthonormal basis. (We will always use this interpretation of A^t.) Then ϕ is a C^*-isomorphism on the C^*-algebra $B(H)$, and

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a Jordan isomorphism on the Jordan algebra $S(H)$. Evidently, ϕ is bijective linear and preserves the numerical range, i.e., $W(\phi(A)) = W(A)$ for all A. Pellegrini [6] obtained an interesting result on numerical range preserving maps on general C^*-algebra, which implies that a surjective linear map $\phi : B(H) \to B(H)$ preserving the numerical range must be of this form. Furthermore, by the result in [4], we see the conclusion is also valid for linear maps ϕ defined on $S(H)$. In [2], it was shown that a multiplicative map $\phi : M_n \to M_n$ satisfies $W(\phi(A)) = W(A)$ for all A if and only if ϕ has the form $A \mapsto U^*AU$ for some $U \in M_n$. In [3], the authors replaced the condition that "ϕ is multiplicative and preserves the numerical range" on the surjective map $\phi : B(H) \to B(H)$ by the condition that "$W(AB) = W(\phi(A)\phi(B))$ for all A, B", and showed that such a map has the form $A \mapsto \pm U^*AU$ for some unitary operator $U \in B(H)$. They also showed that a surjective map $\phi : B(H) \to B(H)$ satisfies $W(ABA) = W(\phi(A)\phi(B)\phi(A))$ for all $A, B \in B(H)$ if and only if ϕ has the form $A \mapsto \mu U^*AU$ or $A \mapsto \mu U^*A^tU$ for some unitary operator $U \in B(H)$ and $\mu \in \mathbb{C}$ with $\mu^3 = 1$. Similar results for mappings on $S(H)$ were also obtained. It is interesting that under the rather mild assumptions, one can prove that a numerical range preserving map ϕ is a C^*-isomorphism on $B(H)$ or a Jordan isomorphism on $S(H)$ up to a scalar multiple. Following this line of study, we consider the Jordan product $A * B = (AB + BA)/2$ in this paper and prove the following.

Theorem 1 *Let $\mathbf{V} = B(H)$ or $S(H)$. Then a surjective map $\phi : \mathbf{V} \to \mathbf{V}$ satisfies*

$$W(AB + BA) = W(\phi(A)\phi(B) + \phi(B)\phi(A)) \qquad \text{for all } A, B \in \mathbf{V} \tag{1.1}$$

if and only if there is a unitary operator $U \in B(H)$ such that ϕ has the form

$$X \mapsto \pm U^*XU \qquad \text{or} \qquad X \mapsto \pm U^*X^tU,$$

where X^t is the transpose of X with respect to a fixed orthonormal basis. Moreover, if H has finite dimension, then the surjective assumption on ϕ can be removed.

By the above theorem, we see that a mapping $\phi : B(H) \to B(H)$ satisfies (1.1) if and only if ϕ or $-\phi$ is a C^*-isomorphism; a mapping $\phi : S(H) \to S(H)$ satisfies (1.1) if and only if ϕ or $-\phi$ is a Jordan isomorphism.

The proof of the sufficiency part of the theorem is clear. We need only prove the necessity part, which will be done in the next two sections. The following observations will be used.

Lemma 2 *Suppose $\phi : \mathbf{V} \to \mathbf{V}$ satisfies (1.1), where $\mathbf{V} = B(H)$ or $S(H)$.*

1. *For every $A \in \mathbf{V}$, $W(A^2) = W(\phi(A)^2)$.*
2. *For any $A, B \in \mathbf{V}$, $AB + BA = 0$ if and only if $\phi(A)\phi(B) + \phi(B)\phi(A) = 0$.*

For any nonzero vectors $x, y \in H$, xy^* denotes the rank-one operator $(xy^*)z = (z, y)x$ for z in H, and $\operatorname{tr}(Axy^*) = (Ax, y)$ for any $A \in B(H)$.

Proposition 3 *Let* $\mathbf{V} = B(H)$ *or* $S(H)$ *and* $\mathcal{R} = \{xx^* : x \in H,\ (x,x) = 1\}$. *Suppose* $\phi : \mathbf{V} \to \mathbf{V}$ *satisfies* $\phi(\mathcal{R}) = \mathcal{R}$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(\phi(A)\phi(B))$ *for all* $A \in \mathbf{V}$ *and* $B \in \mathcal{R}$. *Then* ϕ *is linear. In the finite dimensional case, one can replace the assumption* $\phi(\mathcal{R}) = \mathcal{R}$ *by* $\phi(\mathcal{R}) \subseteq \mathcal{R}$, *and get the stronger conclusion that* ϕ *is invertible linear.*

Proof. Let $A, B \in \mathbf{V}$. For any $y \in H$ with $(y,y) = 1$, let x be a unit vector in H such that $\phi(xx^*) = yy^*$. Then

$$(\phi(A+B)y, y) = ((A+B)x, x) = (Ax, x) + (Bx, x) = (\phi(A)y, y) + (\phi(B)y, y).$$

Thus, $\phi(A+B) = \phi(A) + \phi(B)$. Similarly, we can prove that $\phi(\mu A) = \mu\phi(A)$ for all $A \in \mathbf{V}$ and scalar μ.

For the finite dimensional case, one can use the argument in the proof of Proposition 1.1 in [1] to get the stronger result. ∎

2 The finite dimensional case

Denote by $\{E_{11}, E_{12}, \ldots, E_{nn}\}$ the standard basis for M_n, $\{e_1, \ldots, e_n\}$ the standard basis for $\mathbb{C}^n$, and $(x,y) = y^*x = \operatorname{tr}(xy^*)$ the inner product on $\mathbb{C}^n$.

2.1 Hermitian matrices

In this subsection, we present a proof of Theorem 1 when $\mathbf{V} = S_n$. Suppose that $\phi : S_n \to S_n$ satisfies (1.1). Note that $\phi(I_n)$ is Hermitian. Since $W(\phi(I_n)^2) = W(I_n^2) = \{1\}$, we see that $\phi(I_n)^2 = I_n$. So, replacing $X \mapsto U\phi(X)U^*$, we may assume that $\phi(I_n) = I_k \oplus -I_{n-k}$. We may assume that $k > 0$. Otherwise, replace ϕ by $-\phi$.

We divide the rest of the proof into three assertions.

Assertion 2.1.1 For each $A = xx^*$ or $xx^* - yy^*$, where $x, y \in \mathbb{C}^n$ are unit vectors with $x^*y = 0$, we have $\phi(A)$ has an eigenvector of 1 in $\operatorname{span}\{e_1, \ldots, e_k\}$ or an eigenvector of -1 in $\operatorname{span}\{e_{k+1}, \ldots, e_n\}$.

Proof. Note that $W(\phi(A)^2) = W(A^2) = [0,1]$. So, $\phi(A)$ is a Hermitian contraction. Let $\phi(A) = \begin{pmatrix} A_1 & A_2 \\ A_2^* & A_3 \end{pmatrix}$ on $\mathbb{C}^k \oplus \mathbb{C}^{n-k}$. Suppose $A = xx^*$. Since $\phi(I_n) = I_k \oplus I_{n-k}$, we have

$$[0,2] = W(2A) = W(\phi(A)\phi(I_n) + \phi(I_n)\phi(A)) = W\left(\begin{pmatrix} 2A_1 & 0 \\ 0 & -2A_3 \end{pmatrix}\right).$$

This shows that A_1 has 1 as an eigenvalue or A_3 has -1 as an eigenvalue. Now, the assertion follows from the fact that $\phi(A)$ is a contraction.

The proof for $A = xx^* - yy^*$ is similar. ∎

Assertion 2.1.2 $\phi(I_n) = I_n$ and $\phi(\mathcal{R}) \subseteq \mathcal{R}$, where $\mathcal{R} = \{xx^* : x \in \mathbb{C}^n,\ x^*x = 1\}$.

Proof. First, we show that $\phi(I_n) = I_n$. Recall that $\phi(I_n) = I_k \oplus -I_{n-d}$. Let

$$\phi(E_{ii}) = \begin{pmatrix} R_i & S_i \\ S_i^* & T_i \end{pmatrix}, \quad i = 1, \dots, n,$$

where R_i is $k \times k$. Since $W(\phi(E_{ii})\phi(I_n) + \phi(I_n)\phi(E_{ii})) = W(2E_{ii}) = [0, 2]$, we see that $\phi(E_{ii})\phi(I_n) + \phi(I_n)\phi(E_{ii}) = 2(R_i \oplus -T_i)$ is positive semi-definite.

Let $J \subseteq \{1, \dots, n\}$ be such that $\phi(E_{ii})$ has 1 as an eigenvalue. We may assume that $J = \{1, \dots, j\}$. Otherwise, replace ϕ be a mapping of the form $X \mapsto \phi(P^t X P)$ for some permutation matrix P. By Assertion 2.1.1, for each $i = 1, \dots, j$, the matrix $\phi(E_{ii})$ has a unit eigenvector $f_i = \hat{f}_i \oplus 0$, corresponding to the eigenvalue 1, where $\hat{f}_i \in \mathbb{C}^k$. We claim that $(f_r, f_s) = (\hat{f}_r, \hat{f}_s) = 0$ whenever $r \neq s$. Without loss of generality, assume $(r, s) = (1, 2)$. Since

$$\begin{aligned} 2\hat{f}_1^* R_2 \hat{f}_1 = 2f_1{}^* \phi(E_{22}) f_1 &= f_1^* \phi(E_{11})\phi(E_{22}) f_1 + f_1^* \phi(E_{22})\phi(E_{11}) f_1 \\ &= f_1^*(\phi(E_{11})\phi(E_{22}) + \phi(E_{22})\phi(E_{11})) f_1 = 0, \end{aligned}$$

we see that $\hat{f}_1$ is an eigenvector of the positive semi-definite matrix R_2 corresponding to the eigenvalue 0. Since $\hat{f}_2$ is an eigenvector of R_2 corresponding to the eigenvalue 1, we see that $(\hat{f}_1, \hat{f}_2) = 0$. Since $\{f_1, \dots, f_j\}$ is an orthonormal set in span$\{e_1, \dots, e_k\}$, we see that $j \leq k$.

By a similar argument, there is an orthonormal set $\{f_{j+1}, \dots, f_n\}$ in span$\{e_{k+1}, \dots, e_n\}$ such that f_i is an eigenvector of $\phi(E_{ii})$ corresponding to the eigenvalue -1. So, we have $n - j \leq n - k$, i.e., $j \geq k$. Combining with the conclusion in the preceding paragraph, we have $j = k$.

Now, suppose $k < n$. Consider $X = E_{1n} + E_{n1}$. By Assertion 2.1.1, $\phi(X)$ is a contraction and has an eigenvalue 1 with an eigenvector in span$\{e_1, \dots, e_k\}$ or an eigenvalue -1 with an eigenvector in span$\{e_{k+1}, \dots, e_n\}$. In the former case, suppose $\phi(X)f = f$ with $f \in$ span$\{e_1, \dots, e_k\}$. Write $f = \hat{f} \oplus 0$ with $\hat{f} \in \mathbb{C}^k$. Then for $i = 2, \dots, k$, we have

$$\begin{aligned} 2\hat{f}^* R_i \hat{f} = 2f^* \phi(E_{ii}) f &= f^* \phi(X)\phi(E_{ii}) f + f^* \phi(E_{ii})\phi(X) f \\ &= f^*(\phi(X)\phi(E_{ii}) + \phi(E_{ii})\phi(X)) f = 0. \end{aligned}$$

Hence, $\hat{f}$ is an eigenvector of R_i corresponding to the eigenvalue 0. So $0 = (\hat{f}, \hat{f}_i) = (f, f_i)$ for $i = 2, \dots, k$, and f is a unit multiple of f_1. But then $f_1^*(\phi(X)\phi(E_{11}) + \phi(E_{11})\phi(X)) f_1 = 2$ contradicting the fact that $W(\phi(X)\phi(E_{11}) + \phi(E_{11})\phi(X)) = W(XE_{11} + E_{11}X) = [-1, 1]$. If the latter case holds, we can obtain a contradiction by a similar argument. Thus $k = n$, that is, $\phi(I_n) = I_n$ and $\phi(E_{ii}) = R_i$ for $i = 1, \dots, n$. Moreover, we have shown that $R_i f_j = 0$ for all $i \neq j$. Let $F = [f_1 \; f_2 \; \dots \; f_n]$. Then $F^* \phi(E_{ii}) F = F^* R_i F = E_{ii}$ and hence $\phi(E_{ii}) = F E_{ii} F^*$ for $i = 1, \dots, n$.

Next, we turn to the condition that $\phi(\mathcal{R}) = \mathcal{R}$. Given a unit vector x, there exists a unitary V such that $Vx = e_1$. Then apply the conclusion of the preceding paragraph to the mapping $A \mapsto \phi(VAV^*)$. We see that $\phi(VE_{11}V^*) = \hat{F} E_{11} \hat{F}^*$ for some unitary matrix $\hat{F}$. Thus $\phi(x_1 x_1^*) = yy^*$ where $y = \hat{F} e_1$. ∎

Assertion 2.1.3 The mapping ϕ has the asserted form.

Proof. Note that for any $A \in S_n$, the mid-point of the line segment $W(AE_{11} + E_{11}A)$ is the trace of AE_{11}. Thus for any $B = xx^*$, where x is a unit vector, the mid-point of the line segment $W(AB + BA)$ is the trace of AB. Similarly, the mid point of the line segment of $W(\phi(A)\phi(B) + \phi(B)\phi(A))$ is the trace of $\phi(A)\phi(B)$. Since $W(AB + BA) = W(\phi(A)\phi(B)+\phi(B)\phi(A))$, we see that $\operatorname{tr}(AB) = \operatorname{tr}\phi(A)\phi(B)$ for all $A \in S_n$. By Proposition 3, ϕ is an invertible linear map. Moreover, we have

$$2W(\phi(A)) = W(\phi(I_n)\phi(A) + \phi(I_n)\phi(A)) = W(I_nA + AI_n) = 2W(A)$$

for all $A \in S_n$. So, ϕ is an invertible linear map preserving the numerical range. The result follows from [4, Theorem 2]. ∎

2.2 Complex matrices

In this subsection, we present a proof of Theorem 1 when $\mathbf{V} = M_n$. Suppose that $\phi : M_n \to M_n$ satisfies (1.1). Again, we divide the proof into several assertions.

Assertion 2.2.1 For any unit vector $x \in \mathbb{C}^n$, $\phi(xx^*)$ has the form $\pm yy^*$.

Proof. Without loss of generalization, assume that $xx^* = E_{11}$. Let $B = [0] \oplus D$ so that $W(B^2)$ is a polygon with n vertices. For example, $B = \operatorname{diag}(b_1, b_2, \ldots, b_n)$ where $b_j = (e^{2(j-1)\pi/n} - 1)^{1/2}$ for $j = 1, 2, \ldots, n$. Since $W(\phi(B)^2) = W(B^2)$, thus $\phi(B)^2$ has n distinct eigenvalues. Then so is $\phi(B)$ and hence $\phi(B) = S^{-1}(0 \oplus F)S$, where S is a nonsingular matrix and F is a diagonal matrix, say, $F = \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$. Since $\alpha_1^2, \alpha_2^2, \ldots, \alpha_{n-1}^2$ are distinct nonzero eigenvalues of $\phi(B)^2$, we obtain that $\alpha_i \neq \alpha_j$ and $\alpha_i \neq -\alpha_j$ for all $1 \leq i \neq j \leq n-1$. Now, let $A = S\phi(E_{11})S^{-1}$. Since

$$0 = \phi(E_{11})\phi(B) + \phi(B)\phi(E_{11}) = S^{-1}(A(0 \oplus F) + (0 \oplus F)A)S,$$

it follows that $A = aE_{11}$ for some scalar $a \in \mathbb{C}$, and then $\phi(E_{11}) = aS^{-1}E_{11}S$. Since $W(\phi(E_{11})^2) = [0, 1]$, we see that $\phi(E_{11})$ has an eigenvalue 1 or -1. It follows that $a = 1$ or $a = -1$. Moreover, since $\phi(E_{11})^2 = S^{-1}E_{11}S$ and $W(\phi(E_{11})^2) = [0, 1]$, $S^{-1}E_{11}S$ is a positive semi-definite rank-one contraction. This implies that $\phi(E_{11}) = \pm yy^*$ for some unit vector $y \in \mathbb{C}^n$. ∎

Assertion 2.2.2 Let x be a unit vector and A in M_n. Then $\operatorname{tr}(Axx^*) = \operatorname{tr}(\phi(A)\phi(xx^*))$.

Proof. Note that the center of the rectangular box from by the vertical and horizontal support lines of $W(xx^*A + Axx^*)$ is equal to $\operatorname{tr}(Axx^*) = (Ax, x)$. Indeed, suppose $\mu_0 \equiv \operatorname{tr}(Axx^*) = (Ax, x) = (A_1x, x) + i(A_2x, x)$, where $A_1 = (A + A^*)/2$ and $A_2 = (A - A^*)/2$. By the proof in Section 2.1, we see that the mid-point of $W(xx^*A_1 + A_1xx^*)$ is equal to $\operatorname{tr}(A_1xx^*) = (A_1x, x)$, and, the mid-point of $W(xx^*A_2 + A_2xx^*)$ is equal to $\operatorname{tr}(A_2xx^*) = (A_2x, x)$. They give the real and imaginary parts of μ_0. Since $W(xx^*A + Axx^*) = W(\phi(xx^*)\phi(A) + \phi(A)\phi(xx^*))$, thus $\operatorname{tr}(Axx^*) = \operatorname{tr}(\phi(A)\phi(xx^*))$ as asserted. ∎

Assertion 2.2.3 The mapping ϕ has the asserted form.

Proof. We first show that it is impossible to have $\phi(xx^*) = yy^*$ and $\phi(uu^*) = -vv^*$. In fact, consider the unit vector $w = (x+u)/\|x+u\|$. Then $\phi(ww^*) = \pm zz^*$ for some unit vector z. But then by Assertion 2.2.2 we cannot have $0 < \operatorname{tr}(xx^*ww^*) = \operatorname{tr}(yy^*\phi(ww^*))$ and $0 < \operatorname{tr}(uu^*ww^*) = \operatorname{tr}(-vv^*\phi(ww^*))$, which is a contradiction.

Now assume that we always have $\phi(xx^*) = yy^*$. Otherwise, replace ϕ by $-\phi$. Since $E_{ii}E_{jj} + E_{ii}E_{jj} = 0$, we have $\phi(E_{ii})\phi(E_{jj}) + \phi(E_{jj})\phi(E_{ii}) = 0$ for all $1 \le i < j \le n$. So, there is a unitary matrix $U \in M_n$ such that $\phi(E_{ii}) = UE_{ii}U^*$ for $i = 1, \ldots, n$. In fact, $U = [y_1\ y_2 \ldots\ y_n]$ where $y_i y_i^* = \phi(E_{ii})$ for $i = 1, \ldots, n$. Now, replacing $X \mapsto U\phi(X)U^*$, we may assume that $\phi(E_{ii}) = E_{ii}$ for $i = 1, \ldots, n$.

Finally, since $W(E_{ii}\phi(I_n) + \phi(I_n)E_{ii}) = W(\phi(E_{ii})\phi(I_n) + \phi(I_n)\phi(E_{ii})) = W(2E_{ii}) = [0,2]$ for $i = 1, \ldots, n$, by Assertion 2.2.2, we obtain that $(\phi(I_n)e_i, e_i) = \operatorname{tr}(\phi(I_n)e_ie_i^*) = 1$ for $i = 1, \ldots, n$. On the other hand, $W(\phi(I_n)^2) = W(I_n) = \{1\}$ implies that $\phi(I_n)$ is a contraction. We conclude that $\phi(I_n) = I_n$. Consequently, $W(\phi(A)) = W(A)$ for all $A \in M_n$. By Assertion 2.2.2, $\operatorname{tr}(xx^*A) = \operatorname{tr}(\phi(xx^*)\phi(A))$ for every $A \in M_n$. By Proposition 3, we see that ϕ is invertible and linear. By Theorem 2 in [4], the result follows. ∎

3 The infinite dimensional case

Let $x, y \in H$. Denote by xy^* the rank-one operator $(xy^*)z = (z, y)x$ for z in H.

3.1 Self-adjoint operators

In this subsection, we give a proof of Theorem 1 when $\mathbf{V} = S(H)$. Suppose that $\phi : S(H) \to S(H)$ is surjective and satisfies (1.1).

Assertion 3.1.1 $\phi(I) = \pm I$.

Proof. Since $W(\phi(I)^2) = W(I^2) = \{1\}$, we see that $\phi(I) = I_{H_1} \oplus -I_{H_2}$ on $H = H_1 \oplus H_2$. Let $f_1 \in H_1$, $f_2 \in H_2$, and $Y = f_1f_2^* + f_2f_1^*$. Then there is $X \in S(H)$ such that $\phi(X) = Y$. But then $W(X^2) = W(Y^2) = [0,1]$ and $W(2X) = W(Y\phi(I) + \phi(I)Y) = \{0\}$, which is a contradiction. ∎

Assertion 3.1.2 Let $A \in S(H)$ with $W(A) = [0,1]$. Then $A = xx^*$ for some unit vector x if and only if the following holds:

For any $Y \in S(H)$ satisfying $W(2Y) = [0,2] = W(AY + YA)$ we have

$$\{Z \in S(H) : YZ + ZY = 0\} \subseteq \{Z \in S(H) : AZ + ZA = 0\}.$$

Proof. We first prove the sufficiency part. Assume otherwise that $A \ne xx^*$ for all unit vector x. Since $W(A) = [0,1]$, A has an eigenvalue 1 with a unit eigenvector u. Then $A = [1] \oplus A_2$ on $H = \operatorname{span}\{u\} \oplus \{u\}^\perp$, where A_2 is non-zero. Then for $Y = [1] \oplus 0$, the operator $Z = [0] \oplus I_{\{u\}^\perp}$ satisfies $YZ + ZY = 0$ but $AZ + ZA = [0] \oplus 2A_2 \neq 0$. This contradiction yields that $A_2 = 0$ and therefore $A = uu^*$.

We now prove the necessity part. Suppose $A = xx^* = [1] \oplus 0$ on $H = \operatorname{span}\{x\} \oplus \{x\}^\perp$. For any $Y \in S(H)$ satisfying $W(2Y) = [0,2] = W(AY + YA)$, then $Y = [1] \oplus Y_1$ where Y_1 is positive semi-definite. If $Z \in S(H)$ satisfying $ZY + YZ = 0$, says, $Z = \begin{pmatrix} \alpha & z_1^* \\ z_1 & Z_2 \end{pmatrix}$ on $\operatorname{span}\{x\} \oplus \{x\}^\perp$. Then

$$0 = ZY + YZ = \begin{pmatrix} 2\alpha & z_1^* Y_1 + z_1^* \\ z_1 + Y_1 z_1 & Z_2 Y_1 + Y_1 Z_2 \end{pmatrix}$$

It follows that $\alpha = 0$ and $(I_{\{x\}^\perp} + Y_1) z_1 = 0$. Note that $I_{\{x\}^\perp} + Y_1$ is invertible, thus $z_1 = 0$ and $Z = [0] \oplus Z_2$. It is easy to see that $ZA + AZ = 0$ as asserted. ∎

Assertion 3.1.3 The mapping ϕ has the asserted form.

Proof. Assume that $\phi(I) = I$. Otherwise, replace ϕ by $-\phi$. Then $W(A) = W(\phi(A))$ for all A. In particular, $W(\phi(xx^*)) = W(xx^*) = [0,1]$. We want to show that $\phi(xx^*) = yy^*$ for some unit vector y. We will verify the condition in Assertion 3.1.2. If $Y \in S(H)$ satisfies $W(2Y) = [0,2] = W(\phi(xx^*)Y + Y\phi(xx^*))$, then

$$\{Z \in S(H) : YZ + ZY = 0\} \subseteq \{Z \in S(H) : \phi(xx^*)Z + Z\phi(xx^*) = 0\}.$$

If $YZ + ZY = 0$, there exist X and T in $S(H)$ such that $\phi(X) = Y$ and $\phi(T) = Z$, then $XT + TX = 0$. On the other hand, $W(2X) = W(2Y) = [0,2] = W(\phi(xx^*)Y + Y\phi(xx^*)) = W(xx^*X + Xxx^*)$, by Assertion 3.1.2, we obtain $xx^*T + Txx^* = 0$. This implies that $\phi(xx^*)Z + Z\phi(xx^*) = 0$. By Assertion 3.1.2, we have $\phi(xx^*) = yy^*$ for some unit vector y. The same arguments show that $\phi(\mathcal{R}) = \mathcal{R}$.

Finally, since $\phi(\mathcal{R}) = \mathcal{R}$, the fact that $\operatorname{tr}(AB) = \operatorname{tr}(\phi(A)\phi(B))$ for all $A \in S(H)$ and $B \in \mathcal{R}$ follows immediately from the same arguments in the proof of Section 2.1. By Proposition 3, ϕ is linear surjective. Hence the assertion follows from [4, Theorem 2]. ∎

3.2 General operators

In this last subsection, we consider the case $\mathbf{V} = B(H)$ of Theorem 1. Suppose that $\phi : B(H) \to B(H)$ is surjective and satisfies (1.1).

Assertion 3.2.1 $\phi(I) = I$.

Proof. Because $\phi(I)^2 = I$, we see that $\phi(I)$ is unitarily similar to $\begin{pmatrix} I_{H_1} & R \\ 0 & -I_{H_1} \end{pmatrix} \oplus I_{H_2} \oplus -I_{H_3}$ on $H = (H_1 \oplus H_1) \oplus H_2 \oplus H_3$, where R is positive definite, that is, $(Rx, x) > 0$ for all nonzero vector x in H_1 (e.g, see [7, Theorem 1.1]). Since $\|R\|$ is in the closure of $W(R)$, we take two orthonormal vectors $f_1, f_2 \in H_1$ so that $a \equiv (Rf_1, f_1)$ is very close to $\|R\|$, $\operatorname{span}\{f_1, Rf_1\} \subseteq \operatorname{span}\{f_1, f_2\}$ and $b \equiv (Rf_1, f_2) \geq 0$. Then R has the form

$$\begin{pmatrix} a & b & 0 \\ b & * & * \\ 0 & * & * \end{pmatrix}$$

on $H_1 = \operatorname{span}\{f_1\} \oplus \operatorname{span}\{f_2\} \oplus (H_1 \ominus \operatorname{span}\{f_1, f_2\})$. Since $a^2 + b^2 \leq \|R\|^2$, we obtain that $b \to 0$ as $a \to \|R\|$, thus we may assume that $b, b/a \in [0, 1/10)$. Let $K = \operatorname{span}\{f_1, f_2\}$, the compression of $\phi(I)$ on $K \oplus K$ equals

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & b & * \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

That is, $\phi(I)$ has the form

$$\left(\begin{array}{cccc|cc} 1 & 0 & a & b & 0 & 0 \\ 0 & 1 & b & * & 0 & * \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & * & I & * \\ 0 & 0 & 0 & 0 & 0 & -I \end{array}\right) \oplus I_{H_2} \oplus -I_{H_3}$$

on $H = ((K \oplus K) \oplus ((H_1 \ominus K) \oplus (H_1 \ominus K))) \oplus H_2 \oplus H_3$. Let

$$Y_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2/a & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and consider the operator

$$Y = \begin{pmatrix} Y_1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 0_{H_2} \oplus 0_{H_3}$$

on $H = ((K \oplus K) \oplus ((H_1 \ominus K) \oplus (H_1 \ominus K))) \oplus H_2 \oplus H_3$. Since ϕ is surjective, there exists $X \in B(H)$ such that $\phi(X) = Y$. Then $W(X^2) = W(Y^2) = [0, 1]$ and $W(2X) = W(XI + IX) = W(Y\phi(I) + \phi(I)Y) = W(2Z)$ with

$$Z = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & -b \\ 2b/a & 0 & b & 0 \\ 0 & 0 & 0 & 2b/a \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that all the eigenvalues of X lie in $W(X) = W(Z) \subseteq \{\mu \in \mathbb{C} : |\mu|^2 \leq \operatorname{tr}(Z^*Z)\}$. Thus, all eigenvalues of X have modulus less than 1, and X^2 cannot have an eigenvalue 1, which contradicts the fact that $W(X^2) = [0, 1]$. This contradiction yields that H_1 cannot appear in the above decomposition of H. Thus we may assume that $\phi(I) = I_{H_2} \oplus -I_{H_3}$ on $H = H_2 \oplus H_3$. Applying the same argument in the proof of Assertion 3.1.1, we obtain that $\phi(I) = \pm I$. ■

Assertion 3.2.2 The mapping has the asserted form.

Proof. Assume $\phi(I) = I$. So, $W(A) = W(\phi(A))$ for all A. Consequently, $\phi(S(H)) = S(H)$. Applying the result in Section 3.1, we conclude that $\phi(\mathcal{R}) = \mathcal{R}$ for $\mathcal{R} = \{xx^* : x \in H,\ (x, x) = 1\}$. Thus, ϕ is linear on $B(H)$. The result then follows from [4, Theorem 2]. ■

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