A GEOMETRIC CHARACTERIZATION OF INVERTIBLE QUANTUM MEASUREMENT MAPS

KAN HE, JIN-CHUAN HOU, AND CHI-KWONG LI

ABSTRACT. A geometric characterization is given for invertible quantum measurement maps. Denote by $\mathcal{S}(H)$ the convex set of all states (i.e., trace-1 positive operators) on Hilbert space H with $\dim H \leq \infty$, and $[\rho_1, \rho_2]$ the line segment joining two elements ρ_1, ρ_2 in $\mathcal{S}(H)$. It is shown that a bijective map $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ satisfies $\phi([\rho_1, \rho_2]) \subseteq [\phi(\rho_1), \phi(\rho_2)]$ for any $\rho_1, \rho_2 \in \mathcal{S}$ if and only if ϕ has one of the following forms

$$\rho \mapsto \frac{M\rho M^*}{\operatorname{tr}(M\rho M^*)} \quad \text{or} \quad \rho \mapsto \frac{M\rho^T M^*}{\operatorname{tr}(M\rho^T M^*)},$$

where M is an invertible bounded linear operator and ρ^T is the transpose of ρ with respect to an arbitrarily fixed orthonormal basis.

1. Introduction and the main result

In the mathematical framework of the theory of quantum information, a state is a positive operator of trace 1 acting on a complex Hilbert space H. Denote by $\mathcal{S}(H)$ the set of all states on H, that is, of all positive operators with trace 1. It is clear that $\mathcal{S}(H)$ is a closed convex subset of $\mathcal{T}(H)$, the Banach space of all trace-class operators on H endowed with the trace-norm $\|\cdot\|_{\mathrm{Tr}}$. In quantum information science and quantum computing, it is important to understand, characterize, and construct different classes of maps on states. For instance, all quantum channels and quantum operations are completely positive linear maps; in quantum error correction, one has to construct the recovery map for a given channel; to study the entanglement of states, one constructs entanglement witnesses, which are special types of positive maps; see [11]. In this connection, it is helpful to know the characterizations of maps leaving invariant some important subsets or quantum properties. Such questions have attracted the attention of many researchers; for example, see [1, 2, 4, 6, 8, 9, 10].

In this paper, we characterize invertible maps $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ that satisfies

$$\phi([\rho_1, \rho_2]) \subseteq [\phi(\rho_1), \phi(\rho_2)] \quad \text{for any } \rho_1, \rho_2 \in \mathcal{S}(H),$$

where $[\rho_1, \rho_2] = \{t\rho_1 + (1-t)\rho_2 : t \in [0, 1]\}$ denotes the closed line segment joining two states ρ_1, ρ_2. In other words, we characterize maps on states such that for any $\rho_1, \rho_2 \in \mathcal{S}(H)$ and $0 \leq t \leq 1$, there is some s with $0 \leq s \leq 1$ such that

$$\phi(t\rho_1 + (1-t)\rho_2) = s\phi(\rho_1) + (1-s)\phi(\rho_2).$$

2002 Mathematical Subject Classification. 47B49, 47L07, 47N50.

Key words and phrases. Quantum states, Quantum measurement, Segment preserving maps.

This work is partially supported by National Natural Science Foundation of China (11171249, 11271217, 11201329), a grant to International Cooperating Research from Shanxi (2011081039). Li was also supported by a USA NSF grant and a HK RCG grant.

This question is motivated by the study of affine isomorphisms on $\mathcal{S}(H)$; see [2]. Recall that an affine isomorphism on $\mathcal{S}(H)$ is a bijective map $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ satisfying

$$\phi(t\rho_1 + (1-t)\rho_2) = t\phi(\rho_1) + (1-t)\phi(\rho_2) \quad \text{for all } t \in [0,1] \text{ and } \rho_1, \rho_2 \in \mathcal{S}(H).$$

Evidently, we have the implications (c) $\Rightarrow$ (b) $\Rightarrow$ (a) for a bijective map $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ for the following conditions.

(a) $\phi([\rho_1, \rho_2]) \subseteq [\phi(\rho_1), \phi(\rho_2)]$ for any $\rho_1, \rho_2 \in \mathcal{S}(H)$.
(b) $\phi([\rho_1, \rho_2]) = [\phi(\rho_1), \phi(\rho_2)]$ for any $\rho_1, \rho_2 \in \mathcal{S}(H)$.
(c) ϕ is an affine isomorphism.

It was shown in [2] that an affine isomorphism $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ has the form

$$\rho \mapsto U\rho U^* \quad \text{or} \quad \rho \mapsto U\rho^T U^*, \tag{1.1}$$

where U is a unitary operator and ρ^T is the transpose of ρ with respect to a certain orthonormal basis for H. Note that unitary similarity transforms correspond to evolutions of quantum systems, and many maps that leave invariant subsets or quantum properties of the states have the form described in (1.1). One may be tempted to conjecture that maps on states satisfying (a) or (b) above also have the forms described in (1.1). However, this is not true as shown by our results. It turns out that the maps satisfying condition (a) and (b) are closely related to quantum measurements.

Recall that in quantum mechanics a fine-grained quantum measurement is described by a collection $\{M_m\}$ of measurement operators acting on the Hilbert space H corresponding to the system satisfying $\sum_m M_m^* M_m = I$. Let M_j be a measurement operator. If the state of the quantum system is $\rho \in \mathcal{S}(H)$ before the measurement, then the state after the measurement is $\frac{M_j \rho M_j^*}{\mathrm{tr}(M_j \rho M_j^*)}$ whenever $M_j \rho M_j^* \neq 0$. If M_j is fixed, we get a measurement map ϕ_j defined by $\phi_j(\rho) = \frac{M_j \rho M_j^*}{\mathrm{tr}(M_j \rho M_j^*)}$ from the convex subset $\mathcal{S}_M(H) = \{\rho : M_j \rho M_j^* \neq 0\}$ of the (convex) set $\mathcal{S}(H)$ of states into $\mathcal{S}(H)$. If M_j is invertible, then $\phi_j : \mathcal{S}(H) \to \mathcal{S}(H)$ is bijective and will be called an invertible measurement map. Observe that a measurement map ϕ_j satisfies (a), (b), and is not of the standard form (1.1) in general.

In this paper, we show that, up to the transpose, bijective maps on states satisfying (a) or (b) are precisely invertible measurement maps. The following is our main result.

Theorem 1. *Let $\mathcal{S}(H)$ be the convex set of all states on Hilbert space H with $2 \leq \dim H \leq \infty$. The following statements are equivalent for a bijective map $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$.*

(a) $\phi([\rho_1, \rho_2]) \subseteq [\phi(\rho_1), \phi(\rho_2)]$ *for any* $\rho_1, \rho_2 \in \mathcal{S}(H)$.
(b) $\phi([\rho_1, \rho_2]) = [\phi(\rho_1), \phi(\rho_2)]$ *for any* $\rho_1, \rho_2 \in \mathcal{S}(H)$.
(c) *There is an invertible bounded linear operator $M \in \mathcal{B}(H)$ such that ϕ has the form*

$$\rho \mapsto \frac{M\rho M^*}{\mathrm{tr}(M\rho M^*)} \quad \text{or} \quad \rho \mapsto \frac{M\rho^T M^*}{\mathrm{tr}(M\rho^T M^*)},$$

where ρ^T is the transpose of ρ with respect to an orthonormal basis.

It is interesting to note that condition (a) is much weaker than condition (b). For example, condition (a) does not even ensure that $\phi([\rho_1, \rho_2])$ is a convex (connected) subset of $[\phi(\rho_1), \phi(\rho_2)]$. It turns out that the two conditions (a) and (b) are equivalent for a bijective map, and the map must be a measurement map or the composition of the transpose map with a measurement map.

The proof of Theorem 1 is done in the next few sections. In Section 2, we will establish the equivalence of (a) and (b) using a result of Păles [12]. Then we verify the equivalence of (b) and (c). We treat the finite dimensional case in Section 3. Using the result in Section 3, we complete the proof for the infinite dimensional case in Section 4.

2. The equivalence of the first two conditions

The implication of (b) $\Rightarrow$ (a) is clear. We consider the implication (a)$\Rightarrow$(b).

Assume (a) holds. We will prove that $\phi([\rho,\sigma]) = [\phi(\rho), \phi(\sigma)]$ for any quantum states ρ, σ. If $\rho = \sigma$, it is trivial. Suppose $\rho \neq \sigma$.

Note that $\rho, \sigma \in \mathcal{S}(H)$ are linearly dependent if and only if $\rho = \sigma$. So, if ρ, σ are linearly independent, then $\phi(\rho), \phi(\sigma)$ are linearly independent as $\phi(\rho) \neq \phi(\sigma)$ by the injectivity of ϕ. Let $\mathcal{HT}(H)$ be the real linear space of all self-adjoint trace-class operators on H. As ϕ is injective, we must have $\phi(]\rho,\sigma[) \subset]\phi(\rho), \psi(\sigma)[$ for any $\rho, \sigma \in \mathcal{S}(H)$, where $]\rho, \sigma_2[= [\rho,\sigma] \setminus \{\rho,\sigma\}$ is the open line segment joining ρ, σ. So by Păles' result [12, Theorem 2], there exists a real linear map $\psi : \mathcal{HT}(H) \to \mathcal{HT}(H)$, a real linear functional $f : \mathcal{HT}(H) \to \mathbb{R}$, an operator $B \in \mathcal{H}T(H)$ and a real number c such that

$$\phi(\rho) = \frac{\psi(\rho) + B}{f(\rho) + c} \quad \text{and} \quad f(\rho) + c > 0 \tag{2.1}$$

hold for all $\rho \in \mathcal{S}(H)$. Thus, for any $\rho, \sigma \in \mathcal{S}(H)$ with $\rho \neq \sigma$ and any $t \in [0,1]$, there exists $s \in [0,1]$ such that

$$\phi(t\rho + (1-t)\sigma) = s\phi(\rho) + (1-s)\phi(\sigma) = s\frac{\psi(\rho)+B}{f(\rho)+c} + (1-s)\frac{\psi(\sigma)+B}{f(\sigma)+c}.$$

On the other hand, by the linearity of ψ and f, we have

$$\begin{aligned}\phi(t\rho + (1-t)\sigma) &= \frac{\psi(t\rho + (1-t)\sigma) + B}{f(t\rho + (1-t)\sigma) + c} \\ &= t\frac{\psi(\rho)+B}{f(t\rho + (1-t)\sigma)+c} + (1-t)\frac{\psi(\sigma)+B}{f(t\rho + (1-t)\sigma)+c}.\end{aligned}$$

Write $\lambda_{t,\rho,\sigma} = f(t\rho + (1-t)\sigma) + c$, we get

$$\left(\frac{s}{f(\rho)+c} - \frac{t}{\lambda_{t,\rho,\sigma}}\right)(\psi(\rho)+B) + \left(\frac{1-s}{f(\sigma)+c} - \frac{1-t}{\lambda_{t,\rho,\sigma}}\right)(\psi(\sigma)+B) = 0.$$

As $\rho \neq \sigma$, $\phi(\rho)$ and $\phi(\sigma)$ are linearly independent. This implies that $\psi(\rho) + B$ and $\psi(\sigma) + B$ are linearly independent, too. It follows that

$$\frac{t}{f(t\rho + (1-t)\sigma)+c} = \frac{s}{f(\rho)+c} \text{ and } \frac{1-t}{f(t\rho + (1-t)\sigma)} = \frac{1-s}{f(\sigma)+c}.$$

Clearly, s is continuously dependent of t such that $\lim_{t\to 0} s = 0$ and $\lim_{t\to 1} s = 1$. Hence we must have $\phi([\rho,\sigma]) = [\phi(\rho), \phi(\sigma)]$. Thus, condition (b) holds. $\square$

Denote by $\mathcal{P}ur(H) = \{x \otimes x : x \in H, \|x\| = 1\}$ the set of pure states in $\mathcal{S}(H)$. The following lemma is useful for our future discussion.

Lemma 2.1. If condition (b) of Theorem 1 holds, then ϕ preserves pure states in both directions, that is, $\phi(\mathcal{P}ur(H)) = \mathcal{P}ur(H)$.

Proof It is clear that $\mathcal{S}(H)$ is a convex set and its extreme point set is the set $\mathcal{P}ur(H)$ of all pure states (rank-1 projections). For any $P \in \mathcal{P}ur(H)$, if $\phi^{-1}(P) \notin \mathcal{P}ur(H)$, then there are two states $Q, R \in \mathcal{S}(H)$ such that $Q \neq R$ and $\phi^{-1}(P) = tQ + (1-t)R$. As $\phi([\rho,\sigma]) \subseteq [\phi(\rho), \phi(\sigma)]$ for any ρ, σ, there is some $s \in [0,1]$ such that $P = \phi(\phi^{-1}(P)) = \Phi(tQ + (1-t)R) = s\phi(Q) + (1-s)\phi(R)$. Since $\phi(Q) \neq \phi(R)$, this contradicts the fact that P is extreme point. So ϕ^{-1} sends pure states to pure states. Similarly, since $\phi([\rho,\sigma]) \supseteq [\phi(\rho), \phi(\sigma)]$ for any states ρ, σ, one can show that ϕ maps pure states into pure states. $\square$

3. Proof of Theorem 1: finite dimensional case

In this section we assume that $\dim H = n < \infty$. In such a case, we may regard $\mathcal{HT}(H)$ the same as $\mathbf{H}_n$, the real linear space of $n \times n$ Hermitian matrices. Since the implication (c) $\Rightarrow$ (b) obviously holds, we needs only prove the implication (b) $\Rightarrow$ (c). We divide the proof of this implication into several assertions. Assume (b) holds.

Assertion 3.1. $\phi(\frac{I}{n})$ is invertible.

Let $\phi(\frac{I}{n}) = T$. In order to prove T is invertible, we show that ϕ maps invertible states to invertible states. Note that ϕ has the form of Eq.(2.1), that is, for any $\rho \in \mathcal{S}(H)$, $\phi(\rho) = \frac{\psi(\rho)+B}{f(\rho)+c}$. Since $\mathbf{H}_n$ is finite dimensional, the linear map ψ and the linear functional f are bounded. So ϕ is continuous. ϕ^{-1} is also continuous as ϕ preserves line segment and hence has the form of Eq.(2.1). Thus ϕ maps open sets to open sets. Denote by $G(\mathcal{S}(H))$ the subset of all invertible states. $G(\mathcal{S}(H))$ is an open subset of $\mathcal{S}(H)$. In fact, $G(\mathcal{S}(H))$ is the maximal open set of all interior points of $\mathcal{S}(H)$. To see this, assume that a state ρ is not invertible; then there are mutually orthogonal rank-one projections P_i $(i = 1, 2, \ldots n)$, an integer $1 \leq k < n$ and scalars $t_i > 0$ with $\sum_{i=1}^k t_i = 1$ such that $\rho = \Sigma_{i=1}^k t_i P_i$. For any $\varepsilon > 0$ small enough so that $\frac{\varepsilon}{2k} < \min\{t_1, t_2 \ldots, t_k\}$, let

$$\rho_\varepsilon = \Sigma_{i=1}^k (t_i - \frac{\varepsilon}{2k}) P_i + \Sigma_{j=k+1}^n (\frac{\varepsilon}{2(n-k)}) P_j.$$

Then ρ_ε is an invertible state and

$$\|\rho - \rho_\varepsilon\|_{\rm tr} \leq \Sigma_{i=1}^k \frac{\varepsilon}{2k} + \Sigma_{j=k+1}^n \frac{\varepsilon}{2(n-k)} = \varepsilon.$$

It follows that for any state ρ and any $\varepsilon > 0$, there is an invertible state σ such that $\rho \in \{\tau \in \mathcal{S}(H) : \|\tau - \sigma\|_{\rm Tr} < \varepsilon\}$. So the trace norm closure of $G(\mathcal{S}(H))$ equals $\mathcal{S}(H)$. Thus $G(\mathcal{S}(H))$ is the set of all interior points of $\mathcal{S}(H)$. Since ϕ preserves the open sets, we have $\phi(G(\mathcal{S}(H))) \subseteq G(\mathcal{S}(H))$. So ϕ preserves the invertible states. In particular, $\phi(\frac{I}{n})$ is invertible. $\square$

By Assertion 1, there is an invertible operator $R \in \mathcal{B}(H)$ such that $\phi(\frac{I}{n}) = RR^*$. Let $S = R^{-1}$; then the map $\tilde{\phi} : \mathcal{S}(H) \to \mathcal{S}(H)$ defined by

$$\rho \mapsto \frac{S\phi(\rho)S^*}{\mathrm{tr}(S\phi(\rho)S^*)}$$

is bijective, sends line segments to line segments in both directions, i.e., $\tilde{\phi}([\rho, \sigma]) = [\tilde{\phi}(\rho), \tilde{\phi}(\sigma)]$, and satisfies $\tilde{\phi}(\frac{I}{n}) = \frac{I}{n}$.

Assertion 3.2. $\tilde{\phi}$ maps orthogonal rank one projections to orthogonal rank one projections.

If $\{P_1, \ldots, P_n\}$ is an orthogonal set of rank one projections satisfying $P_1 + \cdots + P_n = I$, then there are $t_i \in [0, 1]$ $(i = 1, ..., n)$ with $\Sigma_{i=1}^n t_i = 1$ such that

$$\frac{I}{n} = \tilde{\phi}(\frac{I}{n}) = \tilde{\phi}(\frac{(P_1 + \cdots + P_n)}{n}) = t_1\tilde{\phi}(P_1) + \cdots + t_n\tilde{\phi}(P_n) \geq t_i\tilde{\phi}(P_i)$$

for each $i = 1, \ldots, n$. Because $\tilde{\phi}(P_i)$ is a rank one orthogonal projection and $I/n - t_i\tilde{\phi}(P_i)$ is positive semidefinite, we see that $1/n \geq t_i$ for $i = 1, \ldots, n$. Taking trace, we have

$$1 = \mathrm{tr}(I/n) = \sum_{i=1}^n t_i.$$

Thus, $t_1 = \cdots = t_n = 1/n$. So, $I = \sum_{i=1}^n \tilde{\phi}(P_i)$. This implies that $\{\tilde{\phi}(P_1), \ldots, \tilde{\phi}(P_n)\}$ is an orthogonal set of rank one projections. Hence, $\tilde{\phi}$ sends orthogonal rank one projections to orthogonal rank one projections. □

By [12, Theorem 2] again, $\tilde{\phi}$ has the form of Eq.(2.1), that is,

$$\tilde{\phi}(\rho) = \frac{\psi(\rho) + B}{f(\rho) + c} \tag{3.1}$$

holds for any $\rho \in \mathcal{S}(H)$, where $\psi : \mathbf{H}_n(\mathbb{C}) \to \mathbf{H}_n(\mathbb{C})$ is a real linear map, $\mathbf{H}_n(\mathbb{C})$ is the real linear space of all $n \times n$ hermitian matrices, $B \in \mathbf{H}_n(\mathbb{C})$, $f : \mathbf{H}_n(\mathbb{C}) \to \mathbb{R}$ is a real linear functional and c is a real constant with $f(\rho) + c > 0$ for all $\rho \in \mathcal{S}(H)$.

Next we consider the two cases of $\dim H > 2$ and $\dim H = 2$ respectively.

Assertion 3.3. Assume $\dim H > 2$. The functional f in Eq.(3.1) is a constant on $\mathcal{S}(H)$, that is, there is a real number a such that $f(\rho) = a$ for all $\rho \in \mathcal{S}(H)$.

For any normalized orthogonal basis $\{e_i\}_{i=1}^n$, let $P_i = e_i \otimes e_i$. We first claim that $f(e_i \otimes e_i) = f(e_j \otimes e_j)$ for any i and j. Since $\tilde{\phi}$ preserves the rank one projections in both directions, there is a rank one projection $Q_i = x_i \otimes x_i$ such that

$$x_i \otimes x_i = Q_i = \tilde{\phi}(P_i) = \frac{\psi(e_i \otimes e_i) + B}{f(e_i \otimes e_i) + c}.$$

So

$$\psi(e_i \otimes e_i) + B = (f(e_i \otimes e_i) + c)(x_i \otimes x_i).$$

As $\tilde{\phi}(\frac{I}{n}) = \frac{I}{n}$ and $\frac{I}{n} = \frac{1}{n}\sum_{i=1}^n e_i \otimes e_i$, we have

$$\frac{I}{n} = \tilde{\phi}(\frac{1}{n}\sum_{i=1}^n e_i \otimes e_i) = \frac{\psi(\sum_{i=1}^n \frac{1}{n} e_i \otimes e_i) + B}{f(\sum_{i=1}^n \frac{1}{n} e_i \otimes e_i) + c} = \frac{\sum_{i=1}^n \frac{1}{n}\psi(e_i \otimes e_i) + n\frac{1}{n}B}{\sum_{i=1}^n \frac{1}{n} f(e_i \otimes e_i) + n\frac{1}{n}c}.$$

Then

$$\frac{I}{n} = \frac{\frac{1}{n}(\sum_{i=1}^n \psi(e_i \otimes e_i) + B)}{\frac{1}{n}(\sum_{i=1}^n f(e_i \otimes e_i) + c)} = \frac{\sum_{i=1}^n(\psi(e_i \otimes e_i) + B)}{\sum_{i=1}^n(f(e_i \otimes e_i) + c)}. \tag{3.2}$$

On the other hand, by Assertion 3.2, we have

$$\frac{I}{n} = \tilde{\phi}(\frac{I}{n}) = \frac{1}{n}\sum_{i=1}^n \tilde{\phi}(e_i \otimes e_i) = \frac{1}{n}\sum_{i=1}^n \frac{\psi(e_i \otimes e_i) + B}{f(e_i \otimes e_i) + c}.$$

Thus we get

$$I = \sum_{i=1}^n \frac{\psi(e_i \otimes e_i) + B}{f(e_i \otimes e_i) + c}. \tag{3.3}$$

Let $A_i = \psi(e_i \otimes e_i) + B$ and $a_i = f(e_i \otimes e_i) + c$. Then Eq.(3.2) and Eq.(3.3) imply that

$$I = n(\frac{A_1 + A_2 + \ldots + A_n}{a_1 + a_2 + \ldots + a_n}) = \frac{A_1}{a_1} + \frac{A_2}{a_2} + \ldots + \frac{A_n}{a_n}.$$

Note that $A_i = a_i Q_i$, where $Q_i = \tilde{\phi}(e_i \otimes e_i) = x_i \otimes x_i$. Therefore, we get that

$$I = n(\frac{a_1Q_1 + a_2Q_2 + \ldots + a_nQ_n}{a_1 + a_2 + \ldots + a_n}) = \frac{a_1Q_1}{a_1} + \frac{a_2Q_2}{a_2} + \ldots + \frac{a_nQ_n}{a_n}.$$

It follows that

$$n(\frac{a_1Q_1 + a_2Q_2 + \ldots + a_nQ_n}{a_1 + a_2 + \ldots + a_n}) = Q_1 + Q_2 + \ldots + Q_n.$$

Since $\{Q_i\}_{i=1}^n$ is an orthogonal set of rank one projections, we see that

$$\frac{a_1 + a_2 + \ldots + a_n}{n} = a_1 = a_2 = \ldots = a_n.$$

This implies that there is some scalar a such that $f(e_i \otimes e_i) = a$ holds for all i. Now for arbitrary unit vectors $x, y \in H$, as $\dim H > 2$, there is a unit vector $z \in H$ such that $z \in [x, y]^\perp$. It follows from the above argument that $f(x \otimes x) = f(z \otimes z) = f(y \otimes y)$. So $f(x \otimes x) = a$ for all unit vectors $x \in H$. Since each state is a convex combination of pure states, by the linearity of f, we get that $f(\rho) = a$ holds for every state ρ. □

Assertion 3.4. Assume $\dim H > 2$. ϕ has the form stated in Theorem 1 (c).

Every state is a convex combination of some pure states, i.e. convex combination of some rank one projections. Therefore, by Assertion 3.3, we have

$$\tilde{\phi}(\rho) = \frac{\psi(\rho) + B}{\alpha + c}$$

holds for all ρ. Then by the linearity of ψ, it is clear that $\tilde{\phi}$ is an affine isomorphism, i.e., for any states ρ, σ and scalar λ with $0 \le \lambda \le 1$, $\tilde{\phi}(\lambda\rho + (1-\lambda)\sigma) = \lambda\tilde{\phi}(\rho) + (1-\lambda)\tilde{\phi}(\sigma)$.

By a result due to Kadison (Ref. [2, Theorem 8.1]), $\tilde{\phi}$ has the standard form, that is, there exists a unitary operator $U \in \mathcal{B}(H)$ such that $\tilde{\phi}$ has the form

$$\tilde{\phi}(\rho) = U\rho U^* \text{ for all } \rho \quad \text{or} \quad \rho \mapsto U\rho^T U^* \text{ for all } \rho.$$

Now recalled that $\tilde{\phi}$ is defined by $\tilde{\phi}(\rho) = S\phi(\rho)S^*/\mathrm{tr}(S\phi(\rho)S^*)$. If $\tilde{\phi}$ takes the first form, then we have

$$\phi(\rho) = \mathrm{tr}(S\phi(\rho)S^*)S^{-1}\tilde{\phi}(\rho)(S^*)^{-1} = \mathrm{tr}(S\phi(\rho)S^*)S^{-1}U\rho U^*(S^*)^{-1}.$$

As $1 = \mathrm{tr}(\phi(\rho)) = \mathrm{tr}(S\phi(\rho)S^*)\mathrm{tr}(S^{-1}U\rho U^*(S^*)^{-1})$, so

$$\mathrm{tr}(S\phi(\rho)S^*) = \frac{1}{\mathrm{tr}(S^{-1}U\rho U^*(S^*)^{-1})}.$$

Letting $M = S^{-1}U$, we get $\phi(\rho) = \frac{M\rho M^*}{\mathrm{tr}(M\rho M^*)}$ for all ρ, that is, ϕ has the first form stated in (c) of Theorem 1.

Similarly, if $\tilde{\phi}$ takes the second form, then ϕ takes the second form stated in (c) of Theorem 1. □

Assertion 3.5. Condition (c) of Theorem 1 holds for the case of $\dim H = 2$.

Assume that $\dim H = 2$. Denote by $\mathcal{S}_2 = \mathcal{S}(H)$ the convex set of 2×2 positive matrices with the trace 1. Then the map $\tilde{\phi} : \mathcal{S}_2 \to \mathcal{S}_2$ is a bijective map preserving segment in both directions satisfying $\tilde{\phi}(\frac{1}{2}I_2) = \frac{1}{2}I_2$. Let us identify $\mathcal{S}_2$ with the unit ball $(\mathbb{R}^3)_1 = \{(x, y, z)^T \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le 1\}$ of $\mathbb{R}^3$ by the following way. Let $\pi : (\mathbb{R}^3)_1 \to \mathcal{S}_2$ be the map defined by

$$(x, y, z)^T \mapsto \frac{1}{2}I_2 + \frac{1}{2}\begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}.$$

π is a bijective affine isomorphism. Note that $v = (x, y, z)^T$ satisfies $x^2 + y^2 + z^2 = 1$ if and only if the corresponding matrix $\pi(v)$ is a rank one projection, and $0 = (0, 0, 0)^T$ if and only if the corresponding matrix is $\pi(0) = \frac{1}{2}I$. The map $\tilde{\phi} : \mathcal{S}_2 \to \mathcal{S}_2$ induces a map $\hat{\phi} : (\mathbb{R}^3)_1 \to (\mathbb{R}^3)_1$ by the following equation

$$\tilde{\phi}(\rho) = \frac{1}{2}I + \pi(\hat{\phi}(\pi^{-1}(\rho))).$$

Since $\tilde{\phi}$ is a segment preserving bijective map and π is an affine isomorphism, the map $\hat{\phi}$ is a bijective map preserving segment in both directions, that is, $\hat{\phi}([u, v]) = [\hat{\phi}(u), \hat{\phi}(v)]$ for $u, v \in (\mathbb{R}^3)_1$. So $\hat{\phi}$ maps the surface of $(\mathbb{R}^3)_1$ onto the surface of $(\mathbb{R}^3)_1$. Since $\tilde{\phi}(\frac{1}{2}I) = \frac{1}{2}I$, we have that $\hat{\phi}((0, 0, 0)^T) = (0, 0, 0)^T$.

Applying the Pǎles' result [12, Theorem 2] to $\hat{\phi}$, there exists a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$, a linear functional $f : \mathbb{R}^3 \to \mathbb{R}$, a vector $u_0 \in \mathbb{R}^3$ and a scalar $r \in \mathbb{R}$ such that $f((x, y, z)^T) + r > 0$ and

$$\hat{\phi}((x, y, z)^T) = \frac{L((x, y, z)^T) + u_0}{f((x, y, z)^T) + r}$$

for each $(x,y,z)^T \in (\mathbb{R}^3)_1$. Since $\hat{\phi}((0,0,0)^T) = (0,0,0)^T$, we have $u_0 = 0$ and $r > 0$. Furthermore, the linearity of f implies that there are real scalars r_1, r_2, r_3 such that $f((x,y,z)^T) = r_1x + r_2y + r_3z$. We claim that $r_1 = r_2 = r_3 = 0$ and hence $f = 0$. If not, then there is a vector $(x_0, y_0, z_0)^T$ satisfying $x_0^2 + y_0^2 + z_0^2 = 1$ such that $f((x_0, y_0, z_0)^T) = r_1x_0 + r_2y_0 + r_3z_0 \neq 0$. It follows that

$$1 = \|\hat{\phi}((x_0, y_0, z_0)^T)\| = \|\frac{L((x_0, y_0, z_0)^T)}{r_1x_0 + r_2y_0 + r_3z_0 + r}\|,$$

and thus

$$\|L((x_0, y_0, z_0)^T)\| = r_1x_0 + r_2y_0 + r_3z_0 + r.$$

Similarly

$$\|L((-x_0, -y_0, -z_0)^T)\| = -r_1x_0 - r_2y_0 - r_3z_0 + r.$$

By the linearity of L we have $r_1x_0 + r_2y_0 + r_3z_0 + r = -r_1x_0 - r_2y_0 - r_3z_0 + r$. Hence $r_1x_0 + r_2y_0 + r_3z_0 = 0$, a contradiction. So, we have $f = 0$, and thus $\hat{\phi} = \frac{L}{r}$ is linear. Now it is clear that $\tilde{\phi}$ is an affine isomorphism as π is an affine isomorphism. Applying a similar argument to the proof of Assertion 3.4 and the Kadison's result, one sees that $\tilde{\phi}$ has the standard form. Thus, Theorem 1 (c) holds. □

By Assertions 3.4 and 3.5, we get the proof of Theorem 1 for finite-dimensional case.

4. Proof: infinite dimensional case

In this section we give a proof of our main result for infinite dimensional case. Similar to the previous section, we need only establish the implication (b) ⇒ (c). We begin with two lemmas.

Let V_1, V_2 be linear spaces on a field $\mathbb{F}$, $\upsilon : \mathbb{F} \to \mathbb{F}$ a nonzero ring automorphism. An additive map $A : V_1 \to V_2$ is called a υ-linear transformation if $A(\lambda x) = \upsilon(\lambda)Ax$ for all $x \in V_1$. The following lemma is similar to [7, Lemma 2.3.1].

Lemma 4.1 *Let V_1, V_2 be linear spaces on a field $\mathbb{F}$, $\tau, \upsilon : \mathbb{F} \to \mathbb{F}$ nonzero ring auto-isomorphisms. Suppose $A : V_1 \to V_2$ is a τ-linear transformation, $B : V_1 \to V_2$ is a υ-linear transformation, and* $\dim \operatorname{span}(\operatorname{ran}(B)) \geq 2$. *If* $\ker B \subseteq \ker A$ *and Ax and Bx are linearly dependent for all $x \in V$, then $\tau = \upsilon$ and $A = \lambda B$ for some scalar λ.*

Proof As $\ker B \subseteq \ker A$, for every $x \in V_1$, there is some scalar λ_x such that $Ax = \lambda_x Bx$. If $Bx \neq 0$, then there exists $y \in V_1$ such that Bx, By are linearly independent. Then $\lambda_{x+y}(Bx + By) = A(x + y) = \lambda_x Bx + \lambda_y By$. This implies that $\lambda_x = \lambda_{x+y} = \lambda_y$. Moreover, for any $\alpha \in \mathbb{F}$, we have $\lambda_{\alpha x} = \lambda_x$. If $Bx = 0$, then $Ax = 0$. Thus it follows that there exists a scalar λ such that $Ax = \lambda Bx$ holds for all $x \in V_1$. So, $A = \lambda B$ and $\tau = \upsilon$. □

Lemma 4.2 *Let $\mathcal{S}(H)$ be the set of all states on Hilbert space H with* $\dim H = \infty$, *and $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ a bijective map. If ϕ satisfies that, for any $t \in [0,1]$ and $\rho, \sigma \in \mathcal{S}(H)$, there is $s \in [0,1]$ such that*

$$\phi(t\rho + (1-t)\sigma) = s\phi(\rho) + (1-s)\phi(\sigma),$$

then, ϕ is continuous and there is an invertible bounded linear or conjugate linear operator T such that

$$\phi(x \otimes x) = \frac{Tx \otimes Tx}{\|Tx\|^2} \text{ for all unit vectors } x \in H.$$

Proof We complete the proof by checking several assertions. First we restate Lemma 2.1 as:

Assertion 4.1. ϕ preserves pure states (rank one projections) in both directions.

Assertion 4.2. For any $x_i \otimes x_i \in \mathcal{P}ur(H)$ with $\{x_1, x_2 \ldots, x_n\}$ linearly independent, let

$$F(x_1, \ldots, x_n) = C(x_1, \ldots, x_n) \cup F_0(x_1, \ldots, x_n),$$

where $C(x_1, \ldots, x_n) = \text{cov}\{x_i \otimes x_i : i = 1, 2 \ldots, n\}$ is the convex hull of $\{x_i \otimes x_i\}_{i=1}^n$,

$$\begin{aligned} F_0(x_1, \ldots, x_n) = \ & \{Z \in \mathcal{S}(H) \setminus C(x_1, \ldots, x_n) : \text{there exists some} \\ & W \in \mathcal{S}(H) \setminus C(x_1, \ldots, x_n) \text{ such that } [Z, W] \cap C(x_1, \ldots, x_n) \neq \emptyset\}. \end{aligned}$$

Let $H_0 = \text{span}\{x_1, \ldots, x_n\}$. Then we have

$$F(x_1, \ldots, x_n) = \mathcal{S}(H_0) \oplus \{0\}. \tag{4.1}$$

Obviously, $C(x_1, \ldots, x_n) \subset \mathcal{S}(H_0) \oplus \{0\}$. If $Z \in F_0(x_1, \ldots, x_n)$, then there exists some $W \in \mathcal{S}(H) \setminus C(x_1, \ldots, x_n)$, $t_i > 0$ with $\sum_{i=1}^n t_i = 1$ and $t \in (0, 1)$ such that

$$\sum_{i=1}^n t_i x_i \otimes x_i = tZ + (1-t)W.$$

Let $P_0 \in \mathcal{B}(H)$ be the projection from H onto H_0. As $\sum_{i=1}^n t_i x_i \otimes x_i - tZ = (1-t)W \geq 0$ and $(I - P_0) \sum_{i=1}^n t_i x_i \otimes x_i = \sum_{i=1}^n t_i x_i \otimes x_i (I - P_0) = 0$, we see that $(I - P_0)Z = Z(I - P_0) = 0$, which implies that $P_0 Z P_0 = Z$ and hence $Z \in \mathcal{S}(H_0) \oplus \{0\}$.

Conversely, assume that $Z \in \mathcal{S}(H_0) \oplus \{0\}$. Since $C(x_1, \ldots, x_n) \subset \mathcal{S}(H_0) \oplus \{0\}$, we may assume that Z is not a convex combination of $\{x_i \otimes x_i\}_{i=1}^n$. Because $\{x_i\}_{i=1}^n$ is a linearly independent set, there exists an operator $S \in \mathcal{B}(H_0)$ such that $\{e_i = Sx_i\}_{i=1}^n$ is an orthonormal basis of H_0. Then, consider

$$S(\sum_{i=1}^n a_i x_i \otimes x_i - Z)S^* = \sum_{i=1}^n a_i Sx_i \otimes Sx_i - SZS^* = \sum_{i=1}^n a_i e_i \otimes e_i - SZS^*.$$

It is clear that for sufficient large $a_i > 0$, $\sum_{i=1}^n a_i e_i \otimes e_i - SZS^* \geq 0$, and hence, $W = \sum_{i=1}^n a_i x_i \otimes x_i - Z \geq 0$. This entails that

$$\frac{\sum_{i=1}^n a_i x_i \otimes x_i}{\sum_{i=1}^n a_i} = \frac{1}{\sum_{i=1}^n a_i} Z + \frac{\text{tr}(W)}{\sum_{i=1}^n a_i} (\frac{W}{\text{tr}(W)}),$$

that is, $Z \in F_0(x_1, \ldots, x_n) \subset F(x_1, \ldots, x_n)$. This finishes the proof of Eq.(4.1). □

Assertion 4.3. For any finite-dimensional subspace $H_0 \subset H$, there exists a subspace H_1 with $\dim H_1 = \dim H_0$ such that

$$\phi(\mathcal{S}(H_0) \oplus \{0\}) = \mathcal{S}(H_1) \oplus \{0\}.$$

Assume that $\dim H_0 = n$. Choose an orthonormal basis $\{x_i\}_{i=1}^n$ of H_0. Then by Assertion 4.1, there are unit vectors $u_i \in H$ such that $\phi(x_i \otimes x_i) = u_i \otimes u_i$. It is clear that $\{u_i\}_{i=1}^n$ is a linearly independent set. Let $H_1 = \mathrm{span}\{u_i\}_{i=1}^n$. Then $\dim H_1 = n$, and by Eq.(4.1) in Assertion 4.2, we have $F(x_1, \ldots, x_n) = \mathcal{S}(H_0) \oplus \{0\}$, $F(u_1, \ldots, u_n) = \mathcal{S}(H_1) \oplus \{0\}$. Since the bijection ϕ preserves segments and pure states in both directions, it is easily checked that $\phi(F(x_1, \ldots, x_n)) = F(u_1, \ldots, u_n)$, and the conclusion of Assertion 4.3 follows. $\square$

Assertion 4.4. For any finite dimensional subspace $\Lambda \subset H$, there exists a subspace $H_\Lambda \subset H$ with $\dim H_\Lambda = \dim \Lambda$ and an invertible linear or conjugate linear operator $M_\Lambda : \Lambda \to H_\Lambda$ such that

$$\phi(P_\Lambda \rho P_\Lambda) = \frac{Q_\Lambda M_\Lambda \rho M_\Lambda^* Q_\Lambda}{\mathrm{tr}(M_\Lambda \rho M_\Lambda^*)}$$

for all $\rho \in \mathcal{S}(\Lambda)$, where P_Λ and Q_Λ are respectively the projections onto Λ and H_Λ. Moreover, the M_Λ can be chosen so that $M_{\Lambda_1} = M_{\Lambda_2}|_{\Lambda_1}$ whenever $\Lambda_1 \subseteq \Lambda_2$.

Let H_0 be a finite dimensional subspace of H and let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of H_0. By Assertion 4.1 there exist unit vectors $\{u_1, u_2, \ldots, u_n\}$ such that $\phi(e_i \otimes e_i) = u_i \otimes u_i$. Let $H_1 = \mathrm{span}\{u_1, u_2, \ldots, u_n\}$. By Assertion 4.1 again, $\dim H_1 = n = \dim H_0$. It follows from Assertion 4.3 that, for any $\rho \in \mathcal{S}(H)$, $P_0 \rho P_0 = \rho$ implies that $P_1 \phi(\rho) P_1 = \phi(\rho)$. Thus ϕ induces a bijective map $\phi_0 : \mathcal{S}(H_0) \to \mathcal{S}(H_1)$ by $\phi_0(\rho) = \phi(P_0 \rho P_0)|_{H_1}$. Applying Theorem 1 for finite dimensional case just proved in Section 2, we obtain that there is an invertible bounded linear operator $M : H_0 \to H_1$ such that ϕ_0 has the form

$$\rho \mapsto \frac{M \rho M^*}{\mathrm{tr}(M^* M \rho)} \quad \text{or} \quad \rho \mapsto \frac{M \rho^T M^*}{\mathrm{tr}(M^* M \rho^T)},$$

where ρ^T is the transpose of ρ with respect to the orthonormal basis $\{e_1, e_2, \ldots, e_n\}$. In the last case, we let $J : H_0 \to H_0$ be the conjugate linear operator defined by $J(\sum_{i=1}^n \xi_i e_i) = \sum_{i=1}^n \bar{\xi}_i e_i$, and let $M' = MJ$. Then, $M' : H_0 \to H_1$ is invertible conjugate linear and $\phi_0(\rho) = \frac{M' \rho M'^*}{\mathrm{tr}(M'^* M' \rho)}$ for all $\rho \in \mathcal{S}(H_0)$. Therefore, the first part of the Assertion 4.4 is true.

Let Λ_i, $i = 1, 2$, are finite dimensional subspaces of H and M_is are associated operators as that obtained above way. If $\Lambda_1 \subseteq \Lambda_2$, then, for any unit vector $x \in \Lambda_1$, we have $\frac{M_1 x \otimes M_1 x}{\|M_1 x\|^2} = \phi(x \otimes x) = \frac{M_2 x \otimes M_2 x}{\|M_2 x\|^2}$. It follows that $M_1 x$ and $M_2 x$ are linearly dependent. By Lemma 4.2 we see that $M_2|_{\Lambda_1} = \lambda M_1$ for some scalar λ. As $\frac{(\lambda M) \rho (\lambda M)^*}{\mathrm{tr}((\lambda M)^* (\lambda M) \rho)} = \frac{M \rho M^*}{\mathrm{tr}(M^* M \rho)}$, we may choose M_2 so that $M_2|_{\Lambda_1} = M_1$. $\square$

Assertion 4.5. There exists a linear or conjugate linear bijective transformation $T : H \to H$ such that

$$\phi(x \otimes x) = \frac{Tx \otimes Tx}{\|Tx\|^2}$$

for every unit vector $x \in H$ and $T|_\Lambda = M_\Lambda$ for every finite dimensional subspace Λ of H.

For any $x \in H$, there is finite dimensional subspace Λ such that $x \in \Lambda$. Let $Tx = M_\Lambda x$. Then, by Assertion 4.4, $T : H \to H$ is well defined, linear or conjugate linear. And by Assertion 4.1, T is bijective.

Note that Păles' result (Theorem 2 in [12]) holds true for the infinite dimensional case. Since ϕ preserves segment, by [12, Theorems 1-2], there exists a linear operator $\Gamma : \mathcal{HT}(H) \to \mathcal{HT}(H)$, a linear functional $g : \mathcal{HT}(H) \to \mathbb{R}$, a scalar $b \in \mathbb{R}$ and some operator $B \in \mathcal{HT}(B)$ such that

$$\phi(\rho) = \frac{\Gamma\rho + B}{g(\rho) + b} \tag{4.2}$$

for all $\rho \in \mathcal{S}(H)$, where $\mathcal{HT}(H)$ denotes the set of all self-adjoint Trace-class operators in $\mathcal{B}(H)$ and $g(\rho) + b > 0$ for all $\rho \in \mathcal{S}(H)$. $\square$

Assertion 4.6. The functions g, Γ in Eq.(4.2) are bounded and hence ϕ is continuous.

Note that, for any $\rho_1, \rho_2 \in \mathcal{S}(H)$ and any $t \in (0,1)$, there exists some $s(t) \in (0,1)$ such that

$$\phi(t\rho_1 + (1-t)\rho_2) = s(t)\phi(\rho_1) + (1 - s(t))\phi(\rho_2).$$

Combining this with Eq.(4.2), one gets

$$\frac{t\Gamma\rho_1 + (1-t)\Gamma\rho_2 + B}{tg(\rho_1) + (1-t)g(\rho_2) + b} = s(t)\frac{\Gamma\rho_1 + B}{g(\rho_1) + b} + (1 - s(t))\frac{\Gamma\rho_2 + B}{g(\rho_2) + b}. \tag{4.3}$$

Note that different states are linearly independent. Comparing the coefficients of $\Gamma\rho_1$ in Eq.(4.3), one sees that

$$s(t) = \frac{t(g(\rho_1) + b)}{tg(\rho_1) + (1-t)g(\rho_2) + b}. \tag{4.4}$$

It follows that $s(t) \to 1$ when $t \to 1$. If $\rho = \sum_{i=1}^n t_i\rho_i \in \mathcal{S}(H)$ with $\rho_i \in \mathcal{S}(H)$, one can get some p_i so that $\phi(\rho) = \phi(\sum_{i=1}^n t_i\rho_i) = \sum_{i=1}^n p_i\phi(\rho_i)$, where $\sum_{i=1}^n t_i = \sum_{i=1}^n p_i = 1$. Similarly we can check that

$$p_i = \frac{t_i(g(\rho_i) + b)}{\sum_{i=1}^n t_i g(\rho_i) + b}. \tag{4.5}$$

Suppose that $\rho, \rho_i \in \mathcal{S}(H)$ with $\rho = \sum_{i=1}^\infty t_i\rho_i$, where $t_i > 0$ and $\sum_{i=1}^\infty t_i = 1$. Then

$$\begin{aligned} \phi(\rho) = {} & \phi(\textstyle\sum_{i=1}^\infty t_i\rho_i) \\ = {} & \phi((\textstyle\sum_{j=1}^k t_j)\sum_{i=1}^k(\frac{t_i}{\sum_{j=1}^k t_j})\rho_i + (1 - \sum_{j=1}^k t_j)\sum_{i=k+1}^\infty(\frac{t_i}{1-\sum_{j=1}^k t_j})\rho_i) \\ = {} & s_k\phi(\textstyle\sum_{i=1}^k(\frac{t_i}{\sum_{j=1}^k t_j})\rho_i) + (1 - s_k)\phi(\sum_{i=k+1}^\infty(\frac{t_i}{1-\sum_{j=1}^k t_j})\rho_i). \end{aligned} \tag{4.6}$$

Thus there exist scalars $q_i^{(k)} > 0$ with $\sum_{i=1}^k q_i^{(k)} = 1$ such that

$$s_k\phi(\sum_{i=1}^k(\frac{t_i}{\sum_{j=1}^k t_j})\rho_i) = \sum_{i=1}^k s_k q_i^{(k)}\phi(\rho_i).$$

According to Eq.(4.4), Eq.(4.5), and keeping in mind that g is a linear functional, a simple calculation reveals that

$$\begin{aligned} s_k = & \frac{(\sum_{j=1}^k t_j)(g(\sum_{i=1}^k(\frac{t_i}{\sum_{j=1}^k t_j})\rho_i)+b)}{(\sum_{j=1}^k t_j)(g(\sum_{i=1}^k(\frac{t_i}{\sum_{j=1}^k t_j})\rho_i))+(1-\sum_{j=1}^k t_j)g(\sum_{i=k+1}^\infty(\frac{t_i}{1-\sum_{j=1}^k t_j})\rho_i))+b} \\ = & \frac{(\sum_{j=1}^k t_j)(g(\sum_{i=1}^k(\frac{t_i}{\sum_{j=1}^k t_j})\rho_i)+b)}{g(\rho)+b}, \end{aligned} \tag{4.7}$$

$$q_i^{(k)} = \frac{(\frac{t_i}{\sum_{j=1}^k t_j})(g(\rho_i)+b)}{\sum_{i=1}^k(\frac{t_i}{\sum_{j=1}^k t_j})g(\rho_i)+b} = \frac{(\frac{t_i}{\sum_{j=1}^k t_j})(g(\rho_i)+b)}{g(\sum_{i=1}^k(\frac{t_i}{\sum_{j=1}^k t_j})\rho_i)+b}, \tag{4.8}$$

and

$$s_k q_i^{(k)} = \frac{t_i(g(\rho_i)+b)}{g(\rho)+b}. \tag{4.9}$$

Observe that $s_k q_i^{(k)}$ is independent of k. Since $\sum_{i=1}^k t_i \to 1$ as $k \to \infty$, we must have $s_k \to 1$ as $k \to \infty$. Eqs.(4.6)-(4.9) imply that

$$\sum_{i=1}^\infty \frac{t_i(g(\rho_i)+b)}{g(\rho)+b} = 1$$

and

$$\phi(\sum_{i=1}^\infty t_i\rho_i) = \sum_{i=1}^\infty(\frac{t_i(g(\rho_i)+b)}{g(\rho)+b})\phi(\rho_i). \tag{4.10}$$

In particular, we have

$$g(\sum_{i=1}^\infty t_i\rho_i) = \sum_{i=1}^\infty t_i g(\rho_i). \tag{4.11}$$

We assert that $\sup\{g(\rho) : \rho \in \mathcal{S}(H)\} < \infty$. Assume that $\sup\{g(\rho) : \rho \in \mathcal{S}(H)\} = \infty$. Then, for any positive integer i, there exists $\rho_i \in \mathcal{S}(H)$ satisfying that $g(\rho_i) > 2^i$. Let $\rho_0 = \sum_{i=1}^\infty \frac{1}{2^i}\rho_i$, $\sigma_k = \sum_{i=1}^k \frac{1}{2^i}\rho_i$, then $\sigma_k \to \rho_0$, and

$$g(\sigma_k) = \sum_{i=1}^k \frac{1}{2^i} g(\rho_i) \geq \sum_{i=1}^k 1 = k.$$

Since $g(\rho_i) \geq 0$, by Eq.(4.11), we have $g(\rho_0) \geq g(\sigma_k) \geq k$ for every k, contradicting to the fact that $g(\rho_0) < \infty$. Now the fact $g(\rho)+b > 0$ for all ρ entails that there exists a positive number c such that $\sup\{|g(\rho)| : \rho \in \mathcal{S}(H)\} = c$. Thus g is continuous on $\mathcal{HT}(H)$ and

$$\|g\| = c < \infty. \tag{4.12}$$

Since

$$\|\Gamma\rho\| \leq \|\Gamma\rho + B\| + \|B\| \leq \|\Gamma\rho + B\|_{\rm Tr} + \|B\| = g(\rho) + b + \|B\| \leq c + |b| + \|B\|$$

holds for all $\rho \in \mathcal{S}(H)$, it follows that Γ is $\|\cdot\|_{\rm tr}$-$\|\cdot\|$ continuous from $\mathcal{HT}(H)$ into itself. Hence, if $\rho_n, \rho \in \mathcal{S}(H)$ and $\|\cdot\|_{\rm tr}\text{-}\lim_{n\to\infty}\rho_n = \rho$, then $\|\cdot\|\text{-}\lim_{n\to\infty}\phi(\rho_n) = \phi(\rho)$. However, convergence under trace-norm topology and convergence under uniform-norm topology are the same for states [15]. Hence we have $\|\cdot\|_{\rm tr}$-$\lim_{n\to\infty}\phi(\rho_n)=\phi(\rho)$, i.e., ϕ is continuous under the trace-norm topology. $\square$

Assertion 4.7. The operator T in Assertion 4.5 is bounded.

For any finite dimensional subspace $\Lambda\subset H$, let M_Λ be the invertible linear or conjugate linear operator stated in Assertion 4.4. Then for any $\rho\in\mathcal{S}(H)$ with range in Λ, we have $\frac{\Gamma\rho+B}{g(\rho)+b}=\frac{Q_\Lambda M_\Lambda\rho M_\Lambda^* Q_\Lambda}{{\rm tr}(M_\Lambda\rho M_\Lambda^*)}$. Thus

$$\Gamma\rho+B=\lambda_\rho Q_\Lambda M_\Lambda\rho M_\Lambda^* Q_\Lambda,$$

where

$$\lambda_\rho=\frac{g(\rho)+b}{{\rm tr}(M_\Lambda\rho M_\Lambda^*)}.$$

For any $\sigma\in\mathcal{S}(H)$ with range in Λ and $\sigma\neq\rho$, and for any $0<t<1$, by considering $t\rho+(1-t)\sigma$ one gets

$$\lambda_\rho=\lambda_{t\rho+(1-t)\sigma}=\lambda_\sigma.$$

This implies that there exists a scalar $d>0$ such that $\lambda_\rho=d$ for all ρ with range in Λ. Use Assertion 4.4 again, it is clear that d is not dependent of Λ. Thus, the equation

$${\rm tr}(M_\Lambda\rho M_\Lambda^*)=d^{-1}(g(\rho)+b)$$

holds for all finite rank $\rho\in\mathcal{S}(H)$. In particular, for any unit vector $x\in\Lambda$, by Assertion 4.6, $\|g\|<\infty$ and we have

$$\|M_\Lambda x\|^2=d^{-1}(g(x\otimes x)+b)\le d^{-1}(\|g\|+|b|)<\infty,$$

which implies that $\|M_\Lambda\|\le\sqrt{d^{-1}(\|g\|+|b|)}$. It follows that, for any unit vector $x\in H$, we have $\|Tx\|\le\sqrt{d^{-1}(\|g\|+|b|)}$ and hence $\|T\|\le\sqrt{d^{-1}(\|g\|+|b|)}$.

The proof is finished. $\square$

Now we are in a position to give a proof of the main theorem for infinite dimensional case.

Proof of Theorem 1: infinite dimensional case. Similar to the finite dimensional case, we need only to show (b) $\Rightarrow$ (c).

Assume (b). By Lemma 4.2, there is a bounded invertible linear or conjugate linear operator T such that $\phi(x\otimes x)=\frac{Tx\otimes Tx}{\|Tx\|^2}=\frac{Tx\otimes xT^*}{\|Tx\|^2}$ for all unit vectors $x\in H$. Let ρ be any finite rank state. Then there exists a finite dimensional subspace Λ of H such that the range of ρ is contained in Λ. By Assertion 4.4 in the proof of Lemma 4.3, we have $\phi(\rho)=\frac{(Q_\Lambda M_\Lambda)\rho(Q_\Lambda M_\Lambda)^*}{{\rm tr}((Q_\Lambda M_\Lambda)\rho(Q_\Lambda M_\Lambda)^*)}=\frac{T\rho T^*}{{\rm tr}(T\rho T^*)}$. Since the set of finite-rank states is dense in $\mathcal{S}(H)$ and, by Lemma 4.3, ϕ is continuous, we get that $\phi(\rho)=\frac{T\rho T^*}{{\rm tr}(T^*T\rho)}$ for all states ρ as desired, completing the proof. $\square$

Acknowldegement

We thank Wen-ling Huang for helpful discussions on the segment preserving bijections $\hat{\phi}$ in Assertion 3.5.

References

[1] E. Alfsen, F. Shultz, Unique decompositions, faces, and automorphisms of separable states, Journal of Mathematical Physics, 51(2010), 052201.

[2] I. Bengtsson, K. Zyczkowski, Geometry of quantum states: an introduction to quantum entanglement, Cambridge University Press, Cambridge, 2006.

[3] C.-A. Faure, An elementary proof of the fundamental theorem of projective geometry, Geom. Dedicata, 90(2002), 145-151.

[4] S. Friedland, C.-K. Li Y.-T. Poon and N.-S. Sze, The automorphism group of separable states in quantum information theory, Journal of Mathematical Physics, 52(2011), 042203.

[5] S. Gudder, A structure for quantum measurements, Reports on Mathematical Physics, 55(2005) 2, 249-267.

[6] J. C. Hou, A characterization of positive linear maps and criteria of entanglement for quantum states, J. Phys. A: Math. Theor., 43(2010) 385201.

[7] J. C. Hou, J. L. Cui, Introduction to linear maps on operator algebras, Scince Press in China, Beijing, 2004.

[8] L. Molnár, Characterizations of the automorphisms of Hilbert space effect algebras, Commun. Math. Phys., 223(2001), 437-450.

[9] L. Molnár, On some automorphisms of the set of effects on Hilbert space. Lett. Math. Phys., 51(2000), 37-45

[10] L. Molnár, W. Timmermann, Mixture preserving maps on von Neumann algebra effects, Lett. Math. Phys., 79(2007), 295-302

[11] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, 2000.

[12] Z. Păles, Characterization of segment and convexity preserving maps, preprint.

[13] U. Uhlhorn, Representation of symmetry transformations in quantum mechanics, Ark. Fysik, 23(1963), 307-340.

[14] E. P. Wigner, Group theory: And its application to the quantum mechanics of atomic spectra, Academic Press, 1959.

[15] S. Zhu, Z.-H. Ma, Topologies on quantum states, Phys. Lett. A, 374(2010), 1336-1341.

(Kan He) Faculty of Mathematics, Institute of Mathematics, Taiyuan University of Technology, Taiyuan, 030024, P. R. China

E-mail address, K. He: kanhemath@yahoo.com.cn

(Jin-Chuan Hou) Faculty of Mathematics, Institute of Mathematics, Taiyuan University of Technology, Taiyuan, 030024, P. R. China

E-mail address, J. Hou: jinchuanhou@yahoo.com.cn

(Chi-Kwong Li) Department of Mathematics, College of William Mary, Williamsburg, VA 23187-8795, USA; Faculty of Mathematics, Institute of Mathematics, Taiyuan University of Technology, Taiyuan, 030024, P. R. China

E-mail address, C.K. Li: ckli@math.wm.edu