MULTIPLICATIVE MAPS ON INVERTIBLE MATRICES THAT PRESERVE MATRICIAL PROPERTIES

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Abstract. Descriptions are given of multiplicative maps on complex and real matrices that leave invariant a certain function, property, or set of matrices: norms, spectrum, spectral radius, elementary symmetric functions of eigenvalues, certain functions of singular values, (p, q) numerical ranges and radii, sets of unitary, normal, or Hermitian matrices, as well as sets of Hermitian matrices with fixed inertia. The treatment of all these cases is unified, and is based on general group theoretic results concerning multiplicative maps of general and special linear groups, which in turn are based on classical results by Borel - Tits. Multiplicative maps that leave invariant elementary symmetric functions of eigenvalues and spectra are described also for matrices over a general commutative field.

1. Introduction

There has been considerable interest in studying linear or multiplicative maps on matrices that leave invariant some special functions, sets, and relations, see [4, 24, 10]. In this paper, we study multiplicative maps on invertible matrices with such preserver properties. Although our main interest is in complex and real matrices, we have found it advantageous to present first descriptions of multiplicative maps of the general linear group and the special linear groups over any (commutative) field. These descriptions are based on the Borel–Tits results, and are presented in the next section. In this connection we note that multiplicative maps on full matrix algebras are well understood [14].

Our main results for the complex field are presented in Section 3. Here, we describe multiplicative maps that preserve a certain function, property, or set of matrices: norms, spectrum, spectral radius, elementary symmetric functions of eigenvalues, certain functions of singular values, (p, q) numerical ranges and radii, sets of unitary,

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normal, or Hermitian matrices, as well as sets of Hermitian matrices with fixed inertia. The corresponding results for the real field are presented in Section 4. There, we also describe multiplicative preservers of elementary symmetric functions of eigenvalues and of spectra for matrices over a general field.

2. Group Theory Results

The following notation and conventions will be used in this section (some of the notation will be used in subsequent sections as well).

$|\mathbf{X}|$ cardinality of a set $\mathbf{X}$.

All fields $\mathbb{F}$ are commutative.

$\overline{\mathbb{F}}$ the algebraic closure of a field $\mathbb{F}$.

$\mathbb{F}^*$ the group of nonzero elements of a field $\mathbb{F}$.

$GL(n,\mathbb{F})$ the group of $n \times n$ invertible matrices with entries in a field $\mathbb{F}$.

$SL(n,\mathbb{F}) = \{A \in GL(n,\mathbb{F}) \,:\, \det(A) = 1\}$.

$PSL(n,\mathbb{F}) = SL(n,\mathbb{F})/\{xI \,:\, x \in \mathbb{F}^*,\ x^n = 1\}$.

A^t the transpose of a matrix A.

$M_n(\mathbb{F})$ the algebra of $n \times n$ matrices over the field $\mathbb{F}$. *To avoid trivialities, we assume $n \geq 2$ everywhere.*

$\mathbb{F}_q$ the finite field of q elements.

S_m the group of permutations of m elements.

$G_1 \leq G_2$ means that G_1 is a subgroup of G_2.

A map $\alpha : \mathbf{X} \to \mathbf{Y}$ is called *nontrivial* if its image $\alpha(\mathbf{X})$ consists of more than one element.

2.1. Multiplicative maps on $SL(n,\mathbb{F})$.

The results in this and the next subsection are based on a special case of the Borel–Tits results [2]. We need to do some work in order to apply them. Let $\mathbb{E},\mathbb{F}$ be fields. We first dispose of the case where $\mathbb{E}$ and $\mathbb{F}$ have different characteristics. The following is known:

Lemma 2.1. *Let $\mathbb{E}$ and $\mathbb{F}$ be fields of distinct characteristics $p_{\mathbb{E}}$ and $p_{\mathbb{F}}$. Assume that $n > 1$. Let $\phi : SL(n,\mathbb{F}) \to M_m(\mathbb{E})$ be a nontrivial multiplicative map. Then $\mathbb{F}$ is finite, say of order q, and one of the following holds:*

(1) *$n = 2$, and $m \geq (q-1)/2$; or*
(2) *$n = 2$, $q = 3$ and $m = 1$; or*
(3) *$q = 2$ and $n \leq 4$; or*
(4) *$n > 2$ and $m \geq (q^n - 1)/(q-1) - 2$.*

Proof. There is no harm in assuming that $\mathbb{E}$ is algebraically closed and also that ϕ maps into $GL(m,\mathbb{E})$ (passing to a smaller m if necessary). If $\mathbb{F}$ is infinite, then the only proper normal subgroups of $SL(n,\mathbb{F})$ are contained in the scalars. Consider the subgroup consisting of $I + tE_{12}, t \in \mathbb{F}$, where E_{12} is the matrix with 1 in the $(1,2)$ position and zeros elsewhere. We note that any two nontrivial elements in this group are conjugate in $SL(n,\mathbb{F})$. The image of this group has the same property but also is an abelian group of semisimple elements. In particular, this group can be diagonalized – but all elements are conjugate and so in particular have the same set of eigenvalues. The image is thus finite and so this subgroup intersects the kernel nontrivially, whence ϕ is trivial.

In the finite case, this result is known – see [9] (or [17] for a somewhat weaker result). □

We assume from now on that $\mathbb{F}$ and $\mathbb{E}$ have the same characteristic.

Fix a positive integer $n > 1$. We make the assumption that if $n = 2$, then $|\mathbb{F}| > 3$. This is equivalent to assuming that $SL(n,\mathbb{F})$ is perfect (cf. [3]). Recall that a group G is called *perfect* if it coincides with its commutator $[G,G]$:

$$G = [G,G] := \{\text{the subgroup generated by } xyx^{-1}y^{-1} \,:\, x,y \in G\}.$$

We first record the well known fact (see, e.g., [6]):

Lemma 2.2. *Let $f_i, i = 1,2$ be two representations of a semigroup $\mathbf{A}$ into $\mathrm{End}(V)$ with V a finite dimensional vector space over $\mathbb{E}$. If f_1 and f_2 are equivalent over any extension field of $\widetilde{\mathbb{E}}$ of $\mathbb{E}$, i.e., there exists $Q \in GL(n,\widetilde{\mathbb{E}})$ such that $f_1(x) = Q^{-1}f_2(x)Q$ for every $x \in \mathbf{A}$, then f_1 and f_2 are equivalent over $\mathbb{E}$.*

Lemma 2.3. *There are no nontrivial homomorphisms from $SL(n,\mathbb{F})$ to $GL(m,\mathbb{E})$ for $m < n$.*

Proof. There is no harm in assuming that $\mathbb{E}$ is algebraically closed. We induct on n. If $n = 2$ and $|\mathbb{F}| > 3$, then the result is clear (since $SL(n,\mathbb{F})$ is perfect and so any 1-dimensional representation is trivial, because $GL(1,\mathbb{E})$ is abelian). Still assuming $n = 2$, consider the remaining cases $SL(2,\mathbb{F}_3)$ and $SL(2,\mathbb{F}_2)$. We have $|SL(2,\mathbb{F}_3)| = 24$ and $|[SL(2,\mathbb{F}_3), SL(2,\mathbb{F}_3)]| = 8$. So if there were a nontrivial homomorphism $SL(2,\mathbb{F}_3) \to GL(1,\mathbb{E})$, then $GL(1,\mathbb{E})$ would have an element of order 3, which is impossible because $\mathbb{E}$ has characteristic 3. For $SL(2,\mathbb{F}_2)$ we have that $|SL(2,\mathbb{F}_2)| = 6$ and $|[SL(2,\mathbb{F}_2), SL(2,\mathbb{F}_2)]| = 3$ (because $SL(2,\mathbb{F}_2)$ is isomorphic to S_3), and a similar argument applies.

So assume that $n > 2$. Since $PSL(n, \mathbb{F})$ is simple (see [3], for example), any nontrivial representation of $SL(n, \mathbb{F})$ has kernel contained in the center of $SL(n, \mathbb{F})$. Consider the embedding

$$SL(n-1, \mathbb{F}) \quad \longrightarrow \quad SL(n, \mathbb{F})$$

given by

$$A \in SL(n-1, \mathbb{F}) \quad \mapsto \quad \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \in SL(n, \mathbb{F}).$$

Thus, the restriction to $SL(n-1, \mathbb{F})$ (via the above embedding) of a nontrivial representation of $SL(n, \mathbb{F})$ is again a nontrivial representation of $SL(n-1, \mathbb{F})$, and this nontrivial representation of $SL(n-1, \mathbb{F})$ is one-to-one. By induction, $m \geq n-1$. If $m = n-1$, then $SL(n-1, \mathbb{F})$ acts irreducibly (otherwise, the matrices of the representation of $SL(n-1, \mathbb{F})$ are nontrivially block triangular with respect to a certain basis, and by restricting the representation to one of the diagonal blocks, we obtain a contradiction with the induction hypothesis). The centralizer of $SL(n-1, \mathbb{F})$ in $SL(n, \mathbb{F})$ is the set

$$\left\{ \begin{bmatrix} xI_{n-1} & 0 \\ 0 & y \end{bmatrix} \, : \, x, y \in \mathbb{F} \text{ and } x^{n-1}y = 1 \right\}$$

(recall that the centralizer of a group G_1 in an overgroup G_2 is the set $\{y \in G_2 \, : \, yx = xy \text{ for every } x \in G_1\}$). Assume first that the centralizer is nontrivial (which is always the case if $\mathbb{F}$ is infinite). The centralizer maps (under the irreducible representation of $SL(n-1, \mathbb{F})$ which is the restriction of a nontrivial representation of $SL(n, \mathbb{F})$) into the center of $GL(n-1, \mathbb{E})$. The inverse image of the center is a normal subgroup of $SL(n, \mathbb{F})$ and so is either contained in the center of $SL(n, \mathbb{F})$, or coincides with $SL(n, \mathbb{F})$, a contradiction.

If $\mathbb{F} = \mathbb{F}_q$, $q = p^\alpha$, we note that $SL(n-1, \mathbb{F})$ is contained in the normalizer of an elementary abelian subgroup $\mathbf{A}$ of order q^{n-1}. (An abelian group is called *elementary* if the order of any non-identity element is a prime.) Namely,

$$\mathbf{A} = \left\{ \begin{bmatrix} I_{n-1} & y \\ 0 & 1 \end{bmatrix} \, : \, y \in \mathbb{F}^{n-1 \times 1} \right\},$$

and $fgf^{-1} \in \mathbf{A}$ for every $f \in SL(n-1, \mathbb{F})$, $g \in \mathbf{A}$. It is well known that any representation of a p-group in characteristic p has a nonzero vector subspace of vectors that are fixed by the representation, which must be invariant under its normalizer. Thus, there is a nonzero vector subspace which consists of vectors fixed by $\mathbf{A}$ and which is invariant under $SL(n-1,\mathbb{F})$. Since $SL(n-1,\mathbb{F})$ acts irreducibly, this subspace must be the whole $n-1$-dimensional space. Thus, the kernel of the original nontrivial representation of $SL(n,\mathbb{F})$ contains $\mathbf{A}$, a contradiction with the previously established fact that the kernel of this representation must be contained in the center of $SL(n,\mathbb{F})$. □

A large part of Lemma 2.3 is contained in [5, Theorem 2.3] which deals with homomorphisms of special linear groups into general linear groups over division rings. Lemma 2.3 also follows from well known results about the representation theory of $SL(n,\mathbb{F}_p)$ and $SL(n,\mathbb{Q})$; $\mathbb{Q}$ is the field of rational numbers (see [25] if the characteristic is positive and [26] if $\mathbb{F}=\mathbb{Q}$).

Note that we do need to exclude the cases $n=2$ and $|\mathbb{F}|\le 3$ in the next result.

Lemma 2.4. *Let ϕ be a nontrivial multiplicative map from $SL(n,\mathbb{F})$ into $M_n(\mathbb{E})$. If $n=2$, assume that $|\mathbb{F}|>3$. Then ϕ is injective and $\phi(SL(n,\mathbb{F}))\le SL(n,\mathbb{E})$.*

Proof. The only nontrivial normal subgroups of $SL(n,\mathbb{F})$ are contained in the group of scalars in $SL(n,\mathbb{F})$. Since $|\phi(SL(n,\mathbb{F})|>1$, the element $\phi(1)$ is an idempotent and is the identity of the group $\phi(SL(n,\mathbb{F}))$. If $\phi(1)$ is not the identity matrix, then we have a nontrivial group homomorphism from $SL(n,\mathbb{F})$ to $GL(m,\mathbb{F})$ where m is the rank of $\phi(1)$. The previous lemma implies that $m=n$. Since $SL(n,\mathbb{F})$ is perfect, so is its image, whence the image is contained in $SL(n,\mathbb{E})$.

Next, we show that ϕ is injective. First suppose that $n=2$. Then the only nontrivial normal subgroup consists of $\pm I$. Since the image of ϕ is contained in $SL(2,\mathbb{E})$ and the only elements of order 2 in $SL(2,\mathbb{E})$ are scalars, it follows that ϕ is injective. So assume that $n>2$. Recall that any nontrivial proper normal subgroup of $SL(n,\mathbb{F})$ consist of the scalars matrices of order dividing n. If ϕ is not injective, then there is a prime r with $r|n$ and $\mathbb{F}$ containing the rth roots of unity, with $aI\in\operatorname{Ker}\phi$, where $a\in\mathbb{F}$ is an rth root of unity. Let M be the subgroup of monomial matrices of determinant 1 all of whose nonzero entries are rth roots of unity (recall that a monomial matrix is one which has exactly one nonzero entry in each row and column). The representation theory of this group is well known and it is easy to see that the smallest representation that has kernel properly contained in D, the subgroup of diagonal matrices in $SL(n,\mathbb{F})$, has dimension n and any such representation of degree n is faithful on D (and in particular, $aI\in D$ is not in the kernel of ϕ). The main idea is the following. Let Z be the subgroup of scalar matrices in D. Then M permutes the nontrivial characters of D/Z a group of order r^{n-2} and that and M/Z-module must be a direct sum of characters for D that are permuted by M. If $n > 4$, then the smallest such orbit has size $n(n-1)/2 > n$. If $n = r = 3$ or $n = 4, r = 2$, , then any such representation has kernel not contained in D, a contradiction. □

Note that if $\phi : SL(n, \mathbb{F}) \to M_m(\mathbb{E})$, then the space V of column vectors is a module for $SL(n, \mathbb{F})$ in the obvious way (any representation gives a module). We denote this module by V^ϕ.

Define the map τ by $\tau(A) = (A^t)^{-1}$, $A \in GL(n, \mathbb{F})$. If $\sigma : \mathbb{F} \to \mathbb{E}$ is a field embedding, then we can extend this to a homomorphism $SL(n, \mathbb{F}) \to SL(n, \mathbb{E})$ in the obvious manner (entrywise). Indeed, we can extend this to an embedding from $GL(n, \mathbb{F}) \to GL(n, \mathbb{E})$. As above, we let V^σ or $V^{*\sigma}$ denote the corresponding $SL(n, \mathbb{F})$-module over $\mathbb{E}$.

Theorem 2.5. *Let $\phi : SL(n, \mathbb{F}) \to M_n(\mathbb{E})$ be a nontrivial multiplicative map. Assume that $|\mathbb{F}| > 3$ if $n = 2$. Then ϕ is a group homomorphism into $SL(n, \mathbb{E})$ and there exists a field embedding $\sigma : \mathbb{F} \to \mathbb{E}$ and $S \in GL(n, \mathbb{E})$ such that ϕ has the form*

$$A \mapsto S\sigma(A)S^{-1} \quad \text{or} \quad A \mapsto S\sigma(\tau(A))S^{-1}. \tag{1}$$

Proof. If $\mathbb{F}$ is finite and $SL(n, \mathbb{F})$ is perfect, then the result of Theorem 2.5 follows easily from basic results in the representation theory of the finite Chevalley groups, see [25, 13].

So assume that $\mathbb{F}$ is infinite. First consider the case that $\mathbb{E}$ is algebraically closed. Then the closure $\mathbf{H}$ of $\phi(SL(n, \mathbb{F}))$ (in the Zariski topology of $GL(n, \mathbb{E})$) is an algebraic group. Since $\phi(SL(n, \mathbb{F}))$ is infinite (because $SL(n, \mathbb{F})$ is infinite and ϕ is injective by the previous lemma) and simple modulo the center (because $SL(n, \mathbb{F})$ has the this property and ϕ is injective), it follows that $\phi(SL(n, \mathbb{F}))$ is connected (here we use the fact that an infinite simple group cannot have subgroups of finite index). Since the closure of a connected group is itself connected, we conclude that $\mathbf{H}$ is connected. Since $\phi(SL(n, \mathbb{F}))$ is perfect, the same is true for $\mathbf{H}$ (here we use the properties that homomorphic images of a perfect group are perfect and the closure of a perfect group in Zariski topology is perfect), and so $\mathbf{H} \le SL(n, \mathbb{E})$. It follows that $\mathbf{H}$ is semisimple and again by the previous lemma, $\mathbf{H}$ is a simple algebraic group (otherwise, we have a nontrivial map from $SL(n, \mathbb{F})$ into a group with a smaller representation).

Now apply the main result in [2] to conclude that ϕ is as above. The hypotheses of [2] are: ϕ is a homomorphism from a subgroup of $GL(n, \mathbb{F})$ which is generated by unipotent elements (i.e., matrices all of which eigenvalues are equal to 1) and the closure of the image of ϕ is a simple algebraic group over an algebraically closed field $\mathbb{E}$; it is also hypothesized that $\mathbb{E}$ and $\mathbb{F}$ have the same characteristic.

We now need to descend to $\mathbb{E}$ from the algebraic closure of $\mathbb{E}$. What we have shown so far amounts to saying that the only irreducible representations of $SL(n,\mathbb{F})$ over $\overline{\mathbb{E}}$ of dimension n are V^σ or $V^{*\sigma}$ where σ is a field embedding of $\mathbb{F}$ into $\overline{\mathbb{E}}$. Note that the character of the representation of V^σ is just $\sigma \circ \text{trace}$ (and similarly for V^*). Since we have a representation into $GL(n,\mathbb{E})$, the character takes on values in $\mathbb{E}$, whence $\sigma(\mathbb{F}) \subseteq \mathbb{E}$. Thus, the representation corresponding to ϕ is equivalent to V^σ or $V^{*\sigma}$ over $\overline{\mathbb{E}}$, whence also over $\mathbb{E}$. Thus, the conjugating element S can always be taken to be in $GL(n,\mathbb{E})$ rather than just in $GL(n,\overline{\mathbb{E}})$. $\square$

A special case of Theorem 2.5 (for $n = 2$) is contained in [5].

2.2. Multiplicative Maps on $GL(n,\mathbb{F})$. It is quite easy to determine the multiplicative maps on $GL(n,\mathbb{F})$ using Theorem 2.5. We first prove:

Lemma 2.6. *Let $\mathbf{X},\mathbf{Y}$ be groups and $\alpha,\beta : \mathbf{X} \to \mathbf{Y}$ group homomorphisms. Let $\mathbf{Z}$ denote the center of $\mathbf{Y}$. Let $\mathbf{W}$ be a normal subgroup of $\mathbf{X}$. Assume the following:*

(a) *α is one-to-one on $\mathbf{X}$;*
(b) *$\alpha(w) = \beta(w)$ for all $w \in \mathbf{W}$;*
(c) *$\mathbf{X}/\mathbf{W}$ is abelian; and*
(d) *the centralizer of $\alpha(\mathbf{W})$ in $\mathbf{Y}$ coincides with $\mathbf{Z}$;*

Then $\beta(x) = \alpha(x)\gamma(x\mathbf{W})$ where $\gamma : \mathbf{X}/\mathbf{W} \to \mathbf{Z}$ is a homomorphism.

Proof. Since α is one-to-one, there is no harm in identifying $\mathbf{X}$ with $\alpha(\mathbf{X})$ and assuming that $\mathbf{X}$ is a subgroup of $\mathbf{Y}$ (and α is the identity). Define $\gamma : \mathbf{X} \to \mathbf{X}$ by $\gamma(x) = x^{-1}\beta(x)$.

Note that if $w \in \mathbf{W}, x \in \mathbf{X}$, then $wxw^{-1}x^{-1} \in \mathbf{W}$ and so

$$\beta(wxw^{-1}x^{-1}) = wxw^{-1}x^{-1} = w\beta(x)w^{-1}\beta(x)^{-1}.$$

Thus, $\beta(x)^{-1}x$ commutes with w^{-1} for all $w \in \mathbf{W}$. Hence $\beta(x)^{-1}x \in \mathbf{Z}$. This implies that $\beta(x)$ commutes with x and $\gamma(x) \in \mathbf{Z}$.

It follows that

$$\gamma(x)\gamma(y) = x^{-1}\beta(x)y^{-1}\beta(y) = y^{-1}x^{-1}\beta(x)\beta(y) = (xy)^{-1}\beta(xy) = \gamma(xy), \quad x,y \in \mathbf{X}.$$

So $\gamma : \mathbf{X} \to \mathbf{Z}$ is a homomorphism. Clearly, $\gamma(\mathbf{W}) = 1$ and we can view γ as a homomorphism from $\mathbf{X}/\mathbf{W} \to \mathbf{Z}$. $\square$

Fix a field $\mathbb{F}$. We assume that $n > 1$ and if $n = 2$, then $|\mathbb{F}| > 3$. Let $\phi : GL(n, \mathbb{F}) \to M_n(\mathbb{F})$ be a multiplicative map. We consider two cases.

Case 1. $|\phi(SL(n, \mathbb{F}))| = 1$. Then $\phi(SL(n, \mathbb{F}))$ is just an idempotent $\mathbb{E}$ of rank $m \leq n$. It follows that ϕ is a group homomorphism from $GL(n, \mathbb{F})/SL(n, \mathbb{F})$ into $GL(m, \mathbb{F})$ (where we view $GL(m, \mathbb{F})$ inside $GL(n, \mathbb{F})$ in the obvious way). There is not much to say here.

Case 2. $|\phi(SL(n, \mathbb{F}))| > 1$. By Theorem 2.5 we know the possible forms for ϕ restricted to $SL(n, \mathbb{F})$. Since $\phi(I) = I$, it follows that ϕ maps $GL(n, \mathbb{F})$ into $GL(n, \mathbb{F})$. Formula (1) gives a homomorphism $\psi : GL(n, \mathbb{F}) \to GL(n, \mathbb{F})$.

In the previous lemma take $\mathbf{X} = GL(n, \mathbb{F})$, $\mathbf{Y} = GL(n, \mathbb{F})$, $\mathbf{W} = SL(n, \mathbb{F})$, $\alpha = \psi$ and $\beta = \phi$. Note that $\phi(SL(n, \mathbb{F})) = \psi(SL(n, \mathbb{F}))$ acts irreducibly (since there are no representations of dimension less than n) and so by Schur's Lemma its centralizer in $\mathbf{Y}$ is just the group of scalars. The lemma implies that $\phi(A) = \psi(A)f(\det(A))$.

In the excluded cases $GL(2, \mathbb{F}_2) \cong S_3$ and $GL(2, \mathbb{F}_3)$, there are more normal subgroups and one can get other homomorphisms which are easy to describe. We leave this as an exercise to the reader.

This leads to the following result.

Theorem 2.7. *Let $\mathbb{E}/\mathbb{F}$ be an extension of fields. Let $\phi : GL(n, \mathbb{F}) \to M_n(\mathbb{E})$ be a multiplicative map. Assume that $|\mathbb{F}| > 3$ if $n = 2$. Then one of the following holds true:*

(a) $|\phi(SL(n, \mathbb{F}))| = 1$*; or*

(b) *there exists a field embedding $\sigma : \mathbb{F} \to \mathbb{E}$, a homomorphism $f : F^* \to E^*$, and $S \in GL(n, \mathbb{E})$ such that ϕ has the form*

$$A \mapsto f(\det(A))S\sigma(A)S^{-1} \quad \textit{or} \quad A \mapsto f(\det(A))S\sigma(\tau(A))S^{-1}. \tag{2}$$

3. Multiplicative Preservers: The complex field

In this section, we use the group theory results of Section 2 to study multiplicative preservers problems on matrices. In particular, we show that many results on classical linear preserver problems have nice multiplicative analogs. We also derive general results to connect the group theory results and the multiplicative preserver applications. Sometimes, we will reduce the multiplicative preserver problem to well studied linear preserver problems, and use the known results on linear preservers. Since there are many interesting preserver problems, presenting all the multiplicative preserver results we can obtain will be too lengthy. We will select a list of well known examples from linear preserver problems and show how our techniques can be used to derive results on multiplicative preservers. In our discussion, we will focus on the complex field $\mathbb{F} = \mathbb{C}$, and the group $\mathbf{H}$ to be either $\mathbf{H} = SL(n, \mathbb{C})$ or $\mathbf{H} = GL(n, \mathbb{C})$. Analogous results on real matrices and other fields will be discussed in the next section. The following notations will be used.

$\mathbb{C}_k = \{\cos(\frac{2j\pi}{k}) + i\sin(\frac{2j\pi}{k}) : j = 0, \ldots, k-1\}$ the group of k-th roots of unity.
$\mathbf{H}$: either $\mathbf{H} = SL(n, \mathbb{C})$ or $\mathbf{H} = GL(n, \mathbb{C})$.
$\mathrm{Spec}\,(A)$ the spectrum of $A \in M_n(\mathbb{C})$.
$r(A)$ the spectral radius of $A \in M_n(\mathbb{C})$.
$\mathbb{T}$ the unit circle in $\mathbb{C}$.
$\{e_1, \ldots, e_n\}$ the standard orthonormal basis for $\mathbb{C}^n$.
$M_n(\mathbb{C})$ the algebra of complex $n \times n$ matrices.
$\{E_{11}, E_{12}, \ldots, E_{nn}\}$ the standard basis for $M_n(\mathbb{C})$.
$\mathrm{diag}\,(a_1, \ldots, a_n)$ diagonal matrix with diagonal entries $a_1, \ldots, a_n$ (in that order).
$\tau(A) = (A^{-1})^t$, for an invertible matrix A.
$\sigma : \mathbb{C} \longrightarrow \mathbb{C}$ a complex field embedding.
$\|x\|$ the Euclidean length of a vector $x \in \mathbb{C}^n$.

3.1. Preliminary results. The following characterizations of continuous complex field embeddings will be useful.

Lemma 3.1. *The following statements for a complex field embedding σ are equivalent:*

(a) *σ is continuous.*
(b) *either $\sigma(z) = z$ for every $z \in \mathbb{C}$ or $\sigma(z) = \overline{z}$ for every $z \in \mathbb{C}$.*
(c) *$\sigma(\mathbb{R}) \subseteq \mathbb{R}$.*
(d) *$\sigma(z) > 0$ for every positive z.*
(e) *$|\sigma(z)| = |z|$ for every $z \in \mathbb{C}$.*
(f) *there exist rational numbers s and $r \neq 0$ such that*

$$z \in \mathbb{C} \text{ and } |s + rz| = 1 \quad \Longrightarrow \quad |s + r\sigma(z)| = 1.$$

Proof. For the equivalence of (a) – (c), see [29]. Evidently, (b) $\Longrightarrow$ (d), (e), and (f). It is also clear that (d) $\Longrightarrow$ (c) and (e) $\Longrightarrow$ (a). We prove the part (f) $\Longrightarrow$ (a). First, using the transformation $w = s + rz$ and replacing z by w we see that (f) holds with $s = 0$, $r = 1$. Note that $|z| < 1$, $z \in \mathbb{C}$, if and only if

$$|z - w| = |z + w| = 1 \quad \text{for some nonzero } w \in \mathbb{C}. \tag{3}$$

Thus, assuming (f) holds we have for every $z \in \mathbb{C}^*$, $|z| < 1$:

$$|\sigma(z) - \sigma(w)| = |\sigma(z) + \sigma(w)| = 1,$$

where w is taken from (3). So

$$0 < |z| < 1 \quad \Longrightarrow \quad |\sigma(z)| < 1.$$

By scaling z, we obtain

$$0 < |z| < \frac{1}{m} \quad \Longrightarrow \quad |\sigma(z)| < \frac{1}{m},$$

for $m = 1, 2, \ldots$, and the continuity of σ follows easily. $\square$

The following observation will also be used frequently.

Lemma 3.2. *Let $f : \mathbb{C}^* \to \mathbb{C}^*$ be a multiplicative map.*

(a) Suppose k is a positive integer such that $f(\mu)^k = 1$ for all $\mu \in \mathbb{C}^*$. Then $f(\mu) = 1$ for all $\mu \in \mathbb{C}^*$.

(b) Suppose m, n are positive integers such that $(f(\mu))^n = \mu^m$ for all $\mu \in \mathbb{C}^*$. Then n divides m.

Proof. Part (a). The assumption implies that f maps into $\mathbb{C}_k$. Since the multiplicative group $\mathbb{C}^*$ is infinitely divisible (for every $x \in \mathbb{C}^*$ and every positive integer m there is $y \in \mathbb{C}^*$ such that $y^m = x$), the only multiplicative map $f : \mathbb{C}^* \longrightarrow \mathbb{C}_k$ is the trivial one.

Part (b). Considering the multiplicative map

$$g(\mu) = f(\mu)^{n/(\gcd(m,n))} \mu^{-m/(\gcd(m,n))}, \quad \mu \in \mathbb{C}^*,$$

and using the part (a) of the lemma, we may assume that n and m are relatively prime. Let q be a primitive m-th root of 1, and let $\alpha = f(q)$. Then

$$\alpha^n = f(q)^n = q^m = 1 \quad \text{and} \quad 1 = f(1) = f(q^m) = (f(q))^m = \alpha^m,$$

hence $\alpha = 1$. Let r be a primitive mn-th root of 1 such that $q = r^n$. Then

$$1 = f(q) = f(r^n) = (f(r))^n = r^m,$$

a contradiction, unless $n = 1$. $\square$

Next, we have the following general result.

Theorem 3.3. *Suppose $S \in SL(n, \mathbb{C})$ and a multiplicative map $f : \mathbb{C}^* \to \mathbb{C}^*$ are given, and $\phi : \mathbf{H} \to M_n(\mathbb{C})$ is defined by*

(a) $\phi(A) = f(\det(A)) S\sigma(A)S^{-1}$, *or*

(b) $\phi(A) = f(\det(A))S\sigma(\tau(A))S^{-1}$.

Assume that there exists a function $\gamma : M_n(\mathbb{C}) \to [0, \infty)$ satisfying the following properties:

(i) *There exists $k_1 > 0$ such that*

$$\gamma\left(S^{-1}\phi(\mathrm{diag}\,(a_1, \ldots, a_n))S\right) \le k_1 \max\{|a_1|, \ldots, |a_n|\}$$

for all $a_1, \ldots, a_n \in \mathbb{C}$ such that $\prod_{1 \le j \le n} a_j = 1$;

(ii) *There exists $k_2 > 0$ such that*

$$\gamma(\mathrm{diag}\,(a_1, \ldots, a_n)) \ge k_2 \max\{|a_1|, \ldots, |a_n|\}$$

for all $a_1, \ldots, a_n \in \mathbb{C}$ such that $\prod_{1 \le j \le n} a_j = 1$.

Then (a) holds true with σ such that $\sigma(A) = A$ for all $A \in \mathbf{H}$, or $\sigma(A) = \overline{A}$ for all $A \in \mathbf{H}$.

Proof. We give the proof for the case $\mathbf{H} = GL(n, \mathbb{C})$; the case $\mathbf{H} = SL(n, \mathbb{C})$ is completely analogous.

First, we show that (b) cannot occur. To this end, assume $n > 2$ (if $n = 2$, forms (a) and (b) are the same). Consider the matrix $A = \mathrm{diag}\,(m, \ldots, m, m^{-n+1})$, where m is a positive integer. Note that $\det(A) = 1$. If ϕ is given by (b), then $\phi(A) = S\mathrm{diag}\,(m^{-1}, \ldots, m^{-1}, m^{n-1})S^{-1}$, and therefore

$$\begin{aligned} m^{n-1} &\le \frac{1}{k_2}\gamma(\mathrm{diag}\,(m^{-1}, \ldots, m^{-1}, m^{n-1})) \\ &= \frac{1}{k_2}\gamma(S^{-1}\phi(A)S) \;\le\; \frac{1}{k_2}k_1 \max\{m, m^{-n+1}\} \;=\; \frac{k_1}{k_2}m, \end{aligned}$$

which is a contradiction for large m.

Next, we show that σ is either identity or complex conjugation. It is enough to prove that $\sigma(x) > 0$ for every positive x. Assume not, then there exists $x > 0$ such that $\sigma(x) = re^{i\theta}$, where $r > 0$, $0 < \theta < 2\pi$. Obviously, x is irrational. By a theorem in number theory ([23, Theorem 6.9]), for $m = 2, 3, \ldots$ there exist integers a_m, $b_m \neq 0$ such that $|a_m + b_m x - 1| < 1/m$. Clearly, the sets $\{a_m : m = 2, 3, \ldots, \}$ and $\{b_m : m = 2, 3, \ldots, \}$ are unbounded, and without loss of generality (passing to a subsequence if necessary) we may assume that

$$\lim_{m\to\infty} |a_m| = \lim_{m\to\infty} |b_m| = \infty.$$

Clearly, a_m and b_m are of opposite signs, at least for m large enough, say $a_m < 0$, $b_m > 0$. Let

$$A_m = \text{diag}\,\left(a_m + b_m x, (a_m + b_m x)^{-1}, 1, \ldots, 1\right), \quad m = 2, 3, \ldots.$$

Then

$$\max\{|a_m + b_m x|, |(a_m + b_m x)^{-1}|\} \leq 2,$$

and

$$\begin{aligned} & \max\{|a_m + b_m\sigma(x)|, |(a_m + b_m\sigma(x))^{-1}|\} \\ \leq\ & \frac{1}{k_2}\gamma(\sigma(A_m)) \ = \ \frac{1}{k_2}\gamma(S^{-1}\phi(A_m)S) \\ \leq\ & \frac{k_1}{k_2}\max\{|a_m + b_m x|, |(a_m + b_m x)^{-1}|\} \\ \leq\ & \frac{2k_1}{k_2}, \end{aligned}$$

and therefore the set $\{a_m + b_m\sigma(x) : m = 2, 3, \ldots\}$ must be bounded. But in fact $|a_m + b_m\sigma(x)| > |a_m|$ if $\theta = \pi$, and $|a_m + b_m\sigma(x)| \geq |b_m||\sin\theta|$ if $\theta \neq \pi$, a contradiction in both cases. □

Although the above theorem seems to be artificial, it can be used to deduce results on multiplicative preservers very effectively, as shown in the following theorem and results in the next few subsections.

Theorem 3.4. *Let $\|\cdot\|$ be a norm on $M_n(\mathbb{C})$. Then a multiplicative map $\phi : \mathbf{H} \to M_n(\mathbb{C})$ satisfies*

$$\|\phi(A)\| = \|A\| \quad \text{for all } A \in \mathbf{H}$$

if and only if there exist $S \in SL(n, \mathbb{C})$ and a multiplicative map $f : \mathbb{C}^* \to \mathbb{T}$, which collapses to the constant function $f(\mu) = 1$ when $\mathbf{H} = SL(n, \mathbb{C})$, such that ϕ has the form

$$A \mapsto f(\det(A))SAS^{-1} \quad \text{or} \quad A \mapsto f(\det(A))S\overline{A}S^{-1},$$

where S is such that $\|SAS^{-1}\| = \|A\|$ for every $A \in \mathbf{H}$, or $\|S\overline{A}S^{-1}\| = \|A\|$ for every $A \in \mathbf{H}$, as the case may be.

Proof. The "if" part is clear. For the "only if" part apply Theorems 2.5 and 2.7 to conclude that ϕ has the form (a) or (b) in Theorem 3.3, where f is the constant function if $\mathbf{H} = SL(n, \mathbb{C})$, and $S \in GL(n, \mathbb{C})$. We may assume that $S \in SL(n, \mathbb{C})$; otherwise, replace S by $S/\det(S)^{1/n}$. Now, the result follows from Theorem 3.3 with

$\gamma(A) = \|A\|$. Note that f maps $\mathbb{C}^*$ into the unit circle in the case $\mathbf{H} = GL(n, \mathbb{C})$ because for any $\mu \in \mathbb{C}^*$ we have

$$\|\mu I\| = \|\phi(\mu I)\| = \|f(\mu^n)\sigma(\mu)I\| = |f(\mu)^n||\mu|\|I\|,$$

as $\sigma(\mu) = \mu$ or $\bar{\mu}$. $\square$

3.2. Functions of Eigenvalues. Many researchers have studied linear preservers of the functions of eigenvalues; see [24]. We have the following multiplicative analogs. We begin with the preservers of spectral radius $r(A)$.

Theorem 3.5. *A multiplicative map $\phi : \mathbf{H} \to M_n(\mathbb{C})$ satisfies*

$$r(\phi(A)) = r(A) \quad \textit{for all } A \in \mathbf{H}$$

if and only if there exist $S \in SL(n, \mathbb{C})$ and a multiplicative map $f : \mathbb{C}^* \to \mathbb{T}$, which collapses to the constant function $f(\mu) = 1$ when $\mathbf{H} = SL(n, \mathbb{C})$, such that ϕ has the form

$$A \mapsto f(\det(A))SAS^{-1} \quad \textit{or} \quad A \mapsto f(\det(A))S\overline{A}S^{-1}.$$

Proof. Use arguments similar to those in the proof of Theorem 3.4, and use Theorem 3.3 with $\gamma(A) = r(A)$. $\square$

The following result describes multiplicative preservers of $\mathrm{Spec}(A)$; see [10, Theorem 2].

Corollary 3.6. *A multiplicative map $\phi : \mathbf{H} \to M_n(\mathbb{C})$ satisfies*

$$\mathrm{Spec}(\phi(A)) = \mathrm{Spec}(A) \quad \textit{for all } A \in \mathbf{H}$$

if and only if there exist $S \in SL(n, \mathbb{C})$ such that ϕ has the form

$$A \mapsto SAS^{-1}.$$

As we will see in the next section, the result of Corollary 3.6 extends to matrices over any field (with few exceptions).

Multiplicative preservers behave much better than linear preservers. For example, let $E_k(A)$ be the kth elementary symmetric function of the eigenvalues of $A \in M_n(\mathbb{C})$. Thus, $E_1(A) = \mathrm{trace}\,(A)$ and $E_n(A) = \det(A)$. The linear preservers for $E_1(A)$ do not have much structure, and the description of the linear preservers for $E_2(A)$ is very involved; see [15, 24, 1, 22]. In contrast, for multiplicative preservers, we have the following transparent description.

Theorem 3.7. *Fix $1 \le k < n$. A multiplicative map $\phi : \mathbf{H} \to M_n(\mathbb{C})$ satisfies $E_k(\phi(A)) = E_k(A)$ for all $A \in \mathbf{H}$ if and only if there is an $S \in SL(n, \mathbb{C})$ such that*

(a) *ϕ has the form $X \mapsto SXS^{-1}$, or*

(b) *$n = 2k$, $\mathbf{H} = SL(n, \mathbb{C})$, and ϕ has the form $X \mapsto S\tau(X)S^{-1}$.*

Proof. The "if" part is clear if (a) holds true. If (b) holds true, then for $X \in \mathbf{H}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, we have

$$E_k(\phi(X)) = \det(X)E_k(\tau(X)) = \left(\prod_{j=1}^{n} \lambda_j\right) E_k(1/\lambda_1, \ldots, 1/\lambda_n) = E_k(\lambda_1, \ldots, \lambda_n).$$

Consider the "only if" part. Clearly, ϕ cannot be trivial. Suppose $\mathbf{H} = SL(n, \mathbb{C})$. So, ϕ has one of the two standard forms of Theorem 2.5. If $\phi(X) = S\tau(\sigma(X))S^{-1}$, then for any integer m and $X = \operatorname{diag}(m, \ldots, m, 1/m^{n-1})$,

$$\begin{aligned} &E_k(\operatorname{diag}(m, \ldots, m, 1/m^{n-1})) = E_k(X) \\ &= E_k(S\tau(X)S^{-1}) = E_k(\operatorname{diag}(1/m, \ldots, 1/m, m^{n-1})); \end{aligned}$$

thus

$$\binom{n-1}{k} m^{k+n} + \binom{n-1}{k-1} m^k = \binom{n-1}{k} m^{n-k} + \binom{n-1}{k-1} m^{2n-k}$$

for all positive integers m, which is impossible unless $n = 2k$. So,

(i) $\phi(X) = S\sigma(X)S^{-1}$, or

(ii) $n = 2k$ and $\phi(X) = S\sigma(\tau(X))S^{-1}$.

In both cases of (i) and (ii), consider

$$X = I_n + aE_{11} + aE_{12} + E_{21} \in \mathbf{H}, \qquad \text{for } \ a \in \mathbb{C}.$$

If $k = 1$, then

$$n + a = E_1(X) = E_1(\phi(X)) = n + \sigma(a)$$

implies that $\sigma(a) = a$. If $k \ge 2$, then

$$E_k(X) = (a+2)\binom{n-2}{k-1} + \binom{n-2}{k-2} + \binom{n-2}{k}$$

and

$$E_k(\phi(X)) = (\sigma(a)+2)\binom{n-2}{k-1} + \binom{n-2}{k-2} + \binom{n-2}{k}.$$

So, $E_k(X) = E_k(\phi(X))$ implies that $\sigma(a) = a$. Hence, our assertion is proved if $\mathbf{H} = SL(n, \mathbb{C})$.

Let $\mathbf{H} = GL(n,\mathbb{C})$. If (i) holds true on $SL(n,\mathbb{C})$, then $\phi(X) = f(\det(X))SXS^{-1}$, where $f : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ is multiplicative. Applying the equality $E_k(\mu I) = E_k(\phi(\mu I))$, we have $f(\mu)^{nk} = 1$ for every $\mu \in \mathbb{C}^*$. By Lemma 3.2, we see that $f(\mu) = 1$ for all $\mu \in \mathbb{C}^*$. If (ii) holds true on $SL(n,\mathbb{C})$, then $\phi(X) = f(\det(X))S\tau(X)S^{-1}$. Since $E_k(X) = E_k(\phi(X))$ for all $X \in \mathbf{H}$, we see, by considering $X = \mu I$ and using Lemma 3.2, that $f(\det(X))^n = \det(X)^2$ for all $X \in \mathbf{H}$. However, when $n > 2$, by Lemma 3.2(b) there is no such multiplicative map, and when $n = 2$, the form (ii) is the same as (i). □

3.3. Functions related to singular values, norms, numerical ranges. Recall that a norm $\|\cdot\|$ on $M_n(\mathbb{C})$ is unitary similarity invariant if $\|A\| = \|U^*AU\|$ for every $A \in M_n(\mathbb{C})$ and every unitary U. We have the following.

Theorem 3.8. *Let $\|\cdot\|$ be a unitary similarity invariant norm on $M_n(\mathbb{C})$. Then a multiplicative map $\phi : \mathbf{H} \to M_n(\mathbb{C})$ satisfies*

$$\|\phi(A)\| = \|A\| \quad \text{for all } A \in \mathbf{H}$$

if and only if there exist a unitary $S \in SL(n,\mathbb{C})$ and a multiplicative map $f : \mathbb{C}^* \to \mathbb{T}$ such that ϕ has the form

$$A \mapsto f(\det(A))SAS^{-1} \quad \text{or} \quad A \mapsto f(\det(A))S\overline{A}S^{-1},$$

where the latter form holds true if and only if $\|A\| = \|\overline{A}\|$ for all $A \in M_n(\mathbb{C})$.

Proof. "Only if" part. By Theorem 3.4, there exists $S \in SL(n,\mathbb{C})$ such that either $\|SAS^{-1}\| = \|A\|$ for every $A \in \mathbf{H}$, or $\|S\overline{A}S^{-1}\| = \|A\|$ for every $A \in \mathbf{H}$. Suppose the former case holds true. By continuity and homogeneity of the norm function, $\|SAS^{-1}\| = \|A\|$ for every $A \in M_n(\mathbb{C})$.

To prove that S is unitary, let $S = UDV$ for some unitary matrices U and V, and $D = \operatorname{diag}(d_1,\ldots,d_n)$ with $d_1 \geq \cdots \geq d_n > 0$. Now, $A_1 = V^*E_{1n}V$ and $A_2 = V^*E_{n1}V$ are unitarily similar, and thus $\|A_1\| = \|A_2\|$. However, $SA_1S^{-1} = (d_1/d_n)UE_{1n}U^*$ and $SA_2S^{-1} = (d_n/d_1)UE_{n1}U^*$ so that $\|SA_1S^{-1}\| = (d_n/d_1)^2\|SA_2S^{-1}\|$. Thus, $d_1 = d_n$. Analogously one proves $d_1 = d_j$ for $j = 2,\ldots,n-1$, and hence it follows that S is unitary. Similarly, one can show that S is unitary if ϕ has the form $A \mapsto S\overline{A}S^{-1}$. □

Theorem 3.8 covers all the norms on square matrices depending only on the singular values $s_1(A) \geq \cdots \geq s_n(A)$ of a matrix $A \in M_n(\mathbb{C})$. Well–known examples include:

(i) the spectral norm: the largest singular value;

(ii) the Ky Fan k-norm: the sum of the k largest singular values for $1 \leq k \leq n$;

(iii) the Schatten p-norm: the ℓ_p norm of the vector of singular values for $p \geq 1$.

Of course, the Schatten 2-norm is just the Frobenius norm, which admits all unitary operators on $M_n(\mathbb{C})$ as linear preservers (isometries), but the multiplicative Frobenius norm preservers have rather specific form.

One can use a similar technique to study other functions on matrices induced by singular values. For example, in the linear preserver context, researchers have studied $F(A) = \sum_{j=1}^n s_j(A)^p$ with $p \neq 0$ or $F(A) = E_k(s_1(A), \ldots, s_n(A))$ – the kth elementary symmetric function of the singular values, see [24, Chapter 5] and its references. We will prove a general result for multiplicative preservers of functions of matrices depending only on singular values. To achieve that, we need the following lemma.

Lemma 3.9. *Let $S \in SL(n, \mathbb{C})$. If $SE_{ij}S^{-1}$ has singular values $1, 0, \ldots, 0$ for all $(i,j) \in \{(r,s) : 1 \leq r, s \leq n, r \neq s\}$, then S is unitary.*

Proof. Suppose S has columns $u_1, \cdots, u_n$ and S^{-1} has rows $v_1^*, \ldots, v_n^*$ with

$$u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{C}^n.$$

Since $s_1(SE_{ij}S^{-1}) = \|u_i\|\|v_j\|$, the given condition ensures that $\|u_i\|\|v_j\| = 1$ for any $i \neq j$. Suppose $\|u_i\| = r_i$ for $i = 1, \ldots, n$. Then $\|v_j\| = 1/r_i$ for any $j \neq i$. If $n \geq 3$, then $r_1 = \cdots = r_n$. If $n = 2$, then $\|u_1\| = r_1 = 1/\|v_2\|$ and $\|u_2\| = r_2 = 1/\|v_1\|$. Since $1 = v_1^* u_1 \leq \|v_1\|\|u_1\| = r_1/r_2$, and $1 = v_2^* u_2 \leq \|v_2\|\|u_2\| = r_2/r_1$, we see that $r_1 = r_2 = 1$.

Now, by Hadamard inequality (see, e.g., [12, Corollary 7.8.2]), $1 = |\det(S)| \leq r_1^n$ and $1 = |\det(S^{-1})| \leq r_1^{-n}$. It follows that $r_1 = 1$ and $1 = |\det(S)| = \prod_{j=1}^n \|u_j\|$. Using the conditions for equality in Hadamard inequality (see the same [12, Corollary 7.8.2]), it follows that the columns of S are orthogonal. We conclude that S is unitary. $\square$

Theorem 3.10. *Suppose $F : M_n(\mathbb{C}) \to \mathbb{R}$ is a function depending only on singular values of matrices such that $F(D) \neq F(\tilde{D})$ whenever*

(i) *$D = aI$ and $\tilde{D} = bI$ with $0 < a < b$, or*

(ii) *$D \in SL(n, \mathbb{C})$ is a diagonal matrix with positive diagonal entries and $\tilde{D}$ is obtained from $D = \mathrm{diag}\,(d_1, \ldots, d_n)$ by replacing two of the diagonal entries of D, say $d_r \geq d_s$, by td_r and d_s/t, respectively, where $t > 1$.*

If $\phi : \mathbf{H} \to M_n(\mathbb{C})$ is a multiplicative map such that $F(A) = F(\phi(A))$ for all $A \in \mathbf{H}$, then there is a unitary $U \in M_n(\mathbb{C})$ such that one of the following holds true.

(a) *There is a multiplicative map $f : \mathbb{C}^* \to \mathbb{T}$ such that ϕ has the form*

$$A \mapsto f(\det(A))UAU^* \quad \text{or} \quad A \mapsto f(\det(A))U\overline{A}U^*.$$

(b) *$F(D) = F(D^{-1})$ for every diagonal matrix $D \in SL(n, \mathbb{C})$, and there is a multiplicative map $f : \mathbb{C}^* \to \mathbb{C}^*$ satisfying $|f(z)| = |z^{2/n}|$ for every $z \in \mathbb{C}^*$ such that ϕ has the form*

$$A \mapsto f(\det(A))U\tau(A)U^* \quad \text{or} \quad A \mapsto f(\det(A))U\tau(\overline{A})U^*.$$

Proof. The map ϕ has the form as in Theorem 2.5 (if $\mathbf{H} = SL(n, \mathbb{C})$), or as in Theorem 2.7(b) (if $\mathbf{H} = GL(n, \mathbb{C})$).

Suppose $\mathbf{H} = SL(n, \mathbb{C})$. First, we show that S is unitary. If it is not true, then by Lemma 3.9 there exist $i \neq j$ such that the rank one matrix $SE_{ij}S^{-1}$ has singular values $r, 0, \ldots, 0$ with $r \neq 1$. Thus, $A = I + E_{ij}$ has singular values $\gamma_1, 1, \ldots, 1, 1/\gamma_1$, where $\gamma_1 = \left\{1 + \sqrt{5}\right\}/2$, and $\phi(A) = \phi(I + E_{ij}) = I + SE_{ij}S^{-1}$ (or $\phi(A) = I - SE_{ji}S^{-1}$ if ϕ is given by the second formula in (1)) has singular values $\gamma_2, 1, \ldots, 1, 1/\gamma_2$, where $\gamma_2 = \left\{r + \sqrt{r^2 + 4}\right\}/2$. (To verify the last assertion, write $SE_{ij}S^{-1} = uv^*$ for some $u, v \in \mathbb{C}^n$, and note that $\operatorname{tr}(SE_{ij}S^{-1}) = 0$, hence $v^*u = 0$, and therefore there exists a unitary V such $V(uv^*)V^*$ is a scalar multiple of E_{ij}.) Let $D = \operatorname{diag}(\gamma_1, 1, \ldots, 1, 1/\gamma_1)$ and $\tilde{D} = \operatorname{diag}(\gamma_2, 1, \ldots, 1, 1/\gamma_2)$. By condition (ii) and the fact that $r \neq 1$, we have $F(D) \neq F(\tilde{D})$. Since F depends only on singular values, we have $F(A) = F(D) \neq F(\tilde{D}) = F(\phi(A))$, which is a contradiction. So, S is unitary.

Next, we show that $|\sigma(z)| = |z|$ for all $z \in \mathbb{C}$. To this end, let $B = I + zE_{12}$. Then B has singular values $\mu_1, 1, \ldots, 1, 1/\mu_1$ and $\phi(B)$ has singular values $\mu_2, 1, \ldots, 1, 1/\mu_2$, where

$$\mu_1 = \left\{|z| + \sqrt{|z|^2 + 4}\right\}/2 \quad \text{and} \quad \mu_2 = \left\{|\sigma(z)| + \sqrt{|\sigma(z)|^2 + 4}\right\}/2.$$

Since $F(B) = F(\phi(B))$, we see that $\mu_1 = \mu_2$, and hence $|\sigma(z)| = |z|$. As a result, σ has the form $z \mapsto z$ or $z \mapsto \bar{z}$ by Lemma 3.1.

Suppose $F(D) \neq F(D^{-1})$ for some $D \in SL(n, \mathbb{C})$. Then (b) cannot hold, and condition (a) follows. If $F(D) = F(D^{-1})$ for every $D \in SL(n, \mathbb{C})$, then either (a) or (b) holds true.

Now, suppose $\mathbf{H} = GL(n, \mathbb{C})$ and condition (a) holds true for $A \in SL(n, \mathbb{C})$. Then $\phi(A) = f(\det(A))S\sigma(A)S^{-1}$ for every $A \in GL(n, \mathbb{C})$, where S is unitary and $\sigma(A) = A$ or $\overline{A}$. Now, $F(A) = F(\phi(A))$ implies that $|f(\det(A))| = 1$ for all $A \in GL(n, \mathbb{C})$. Thus, $f(z) = 1$ for all $z \in \mathbb{C}^*$. If condition (ii) holds true for $A \in SL(n, \mathbb{C})$, then $\phi(A) = f(\det(A))S\tau(\sigma(A))S^{-1}$ for every $A \in GL(n, \mathbb{C})$, where S is unitary and $\sigma(A) = A$ or $\overline{A}$. Then for $A = zI$ we have

$$F(zI) = F(\phi(zI)) = F(f(z^n)z^{-1}I).$$

Thus, $|f(z^n)z^{-1}| = |z|$ for all $z \in \mathbb{C}^*$. It follows that $|f(z)| = |z^{2/n}|$. □

By the above result, one easily checks that ϕ has the form (a) if $p \neq 0$ and $F(A) = \sum_{j=1}^n s_j(A)^p$; for $1 < k < n$ and $F(A) = E_k(s_1(A), \ldots, s_n(A))$, ϕ has the form (a), or ϕ has the form (b) provided $n = 2k$.

Next, we turn to functions F on $M_n(\mathbb{C})$ that are invariant under unitary similarity, i.e., $F(A) = F(U^*AU)$ for any $A \in M_n(\mathbb{C})$ and unitary $U \in M_n(\mathbb{C})$, but do not necessarily depend only on the singular values of matrices. Examples of such functions include the numerical radius $w(A)$ of $A \in M_n(\mathbb{C})$, the numerical range $W(A)$, and their generalizations. For example, for $1 \le p \le q \le n$ integers such that $(p, q) \neq (n, n)$, the (p, q)-numerical range and (p, q)-numerical radius are defined by

$$W_{p,q}(A) = \{E_p(X^*AX) : X \in M_{n\times q}(\mathbb{C}),\ X^*X = I_q\},$$

where $E_p(Y)$ denotes the pth elementary symmetric function of the eigenvalues of Y, and

$$w_{p,q}(A) = \max\{|z| : z \in W_{p,q}(A)\}.$$

Note that $W_{1,1}(A) = W(A)$ and $w_{1,1}(A) = w(A)$. The proofs for the linear preservers of these functions are very intricate [7, 8, 18, 19, 21], whereas the proofs for multiplicative preservers are easier using the group theory results.

The multiplicative preservers of the (p, q)-numerical range and (p, q)-numerical radius are described in the following theorems. We exclude the case $(p, q) = (n, n)$, as in this case $W_{n,n}(A) = \det A$, and there exist multiplicative maps ϕ such that $\det(\phi(A)) = \det A$ for all $A \in \mathbf{H}$ of various forms, e.g., $\phi(A) = \mathrm{diag}(\det A, 1, \ldots, 1)$.

Theorem 3.11. *Suppose p, q are integers, $1 \le p \le q \le n$, and $(p, q) \neq (n, n)$. A multiplicative map $\phi : \mathbf{H} \to M_n(\mathbb{C})$ satisfies $w_{p,q}(\phi(A)) = w_{p,q}(A)$ for all $A \in \mathbf{H}$ if and only if there is $S \in SL(n, \mathbb{C})$ such that one of the following conditions holds true.*

(a) *There is a multiplicative map* $f : \mathbb{C}^* \to \mathbb{T}$ *such that* ϕ *has the form*

$$A \mapsto f(\det A) SAS^{-1} \quad \text{or} \quad A \mapsto f(\det A) S\overline{A}S^{-1},$$

where S *is unitary if* $n > q$.

(b) $2p = 2q = n > 2$, *and there is a multiplicative map* $f : \mathbb{C}^* \to \mathbb{T}$ *such that* ϕ *has the form*

$$A \mapsto f(\det A) U\tau(A)U^{-1} \quad \text{or} \quad A \mapsto f(\det A) U\tau(\overline{A})U^{-1},$$

where U *is unitary.*

(c) $2p = q = n > 2$, *and there is a multiplicative map* $f : \mathbb{C}^* \to \mathbb{C}^*$ *such that* $|f(\mu)|^n = |\mu|^2$ *for all* $\mu \in \mathbb{C}^*$ *and* ϕ *has the form*

$$A \mapsto f(\det A) S\tau(A)S^{-1} \quad \text{or} \quad A \mapsto f(\det A) S\tau(\overline{A})S^{-1}.$$

Proof. Note that $w_{p,q}(UAU^*) = w_{p,q}(A) = w_{p,q}(\overline{A})$ for any $A \in \mathbf{H}$ and unitary U. So, if (a) holds, then ϕ is multiplicative and preserves $w_{p,q}$.

Denote the $k \times k$ principal submatrix in the top left corner of $A \in \mathbf{H}$ by $A[k]$ and its complementary $(n-k) \times (n-k)$ principal submatrix by $A(k)$. It is well known that $\det(A[k]) = \det(A)\det(A^{-1}(k))$. Thus, for any unitary U of the form $[U_1|U_2]$ where U_1 is $n \times k$, we have

$$\det(U_1^* A^t U_1) = \det(A)\det(U_2^*(A^{-1})^t U_2) = \det(A)\det(U_2^* \tau(A) U_2).$$

Consequently, if $A \in SL(n, \mathbb{C})$ we have

$$W_{p,n-p}(A) = W_{p,n-p}(A^t) = W_{n-p,p}(\tau(A)). \tag{4}$$

So, if (b) holds, then $w_{n/2,n/2}(A) = w_{n/2,n/2}(\phi(A))$ for all $A \in \mathbf{H}$.

Suppose (c) holds, say, ϕ has the first asserted form. Assume $A \in \mathbf{H}$ has eigenvalues $a_1, \ldots, a_n$. Then

$$\begin{aligned}
w_{p,q}(\phi(A)) &= |E_{n/2}(\phi(A))| \\
&= |E_{n/2}(f(\det(A)) S\tau(A)S^{-1})| \\
&= |f(\det(A))|^{n/2} |E_{n/2}(1/a_1, \ldots, 1/a_n)| \\
&= |\det(A)|\, |E_{n/2}(1/a_1, \ldots, a_n)| \\
&= |E_{n/2}(a_1, \ldots, a_n)| \\
&= w_{p,q}(A).
\end{aligned}$$

Next, we consider the converse. Suppose $q < n$. If $p = 1$, then $w_{1,q}(A)$ is a unitary similarity invariant norm, and the result follows from Theorem 3.8.

Suppose $2 \le p \le q < n$. Assume $\mathbf{H} = SL(n, \mathbb{C})$. Then ϕ is clearly non-trivial, and has the standard form

$$A \mapsto S\sigma(A)S^{-1} \quad \text{or} \quad A \mapsto S\tau(\sigma(A))S^{-1}$$

for some $S \in SL(n, \mathbb{C})$. We first show that S is unitary. Note that if $A = I + rZ$ such that Z is a rank one matrix with trace zero and singular values $1, 0, \dots, 0$, then

$$w_{p,q}(A) = E_p(1 + r/2, 1, \dots, 1).$$

To see this, consider any $n \times q$ matrix X with $X^*X = I_q$; then the matrix X^*ZX has rank at most one. Thus, X^*ZX is unitarily similar to $\lambda E_{11} + \mu E_{12} \in M_q$. In particular, X^*ZX has eigenvalues $\lambda, 0, \dots, 0$ with $\lambda = v^*Zv$ for some unit vector $v \in \mathbb{C}^n$, i.e., $\lambda \in W(Z)$. Since Z is unitarily similar to $E_{12} \in M_n(\mathbb{C})$, we have

$$W(Z) = W(E_{12}) = \{z \in \mathbb{C} : |z| \le 1/2\};$$

see [11, Section 22]. Thus, $|\lambda| \le 1/2$, and

$$|E_p(X^*(I + rZ)X)| \le E_p(1 + r|\lambda|, 1, \dots, 1) \le E_p(1 + r/2, 1, \dots, 1).$$

Now, suppose U is unitary with columns $u_1, \dots, u_n$ such that $U^*ZU = E_{12}$. Let X have columns $(u_1 + u_2)/\sqrt{2}, u_3, \dots, u_{q+1}$. Then $X^*ZX = E_{11}/2 \in M_q$, and

$$E_p(X^*(I + rZ)X) = E_p(1 + r/2, 1, \dots, 1).$$

Suppose S is not unitary. By Lemma 3.9, there is E_{ij} with $i \ne j$ so that $\tilde{Z} = SE_{ij}S^{-1}$ has trace zero and singular values $r, 0, \dots, 0$ such that $r \ne 1$. If ϕ has the form $A \mapsto S\sigma(A)S^{-1}$, then

$$w_{p,q}(I + E_{ij}) \ne w_{p,q}(I + \tilde{Z}) = w_{p,q}(\phi(I + E_{ij})).$$

If ϕ has the form $A \mapsto S\tau(\sigma(A))S^{-1}$, then

$$w_{p,q}(I - E_{ji}) = w_{p,q}(I + E_{ji}) \ne w_{p,q}(I + \tilde{Z}) = w_{p,q}(\tau(I - E_{ji})) = w_{p,q}(\phi(I - E_{ji})).$$

In both cases, we have a contradiction.

Next, we show that σ is the identity map, or the complex conjugation $z \mapsto \bar{z}$. For any positive number a, let m be a positive integer such that $|m + \sigma(a)| > 1$. For $D_m = \mathrm{diag}(m + a, 1, \dots, 1, (m + a)^{-1})$, we have

$$E_p(m + a, \underbrace{1, \dots, 1}_{q-1}) = w_{p,q}(D_m) = w_{p,q}(\phi(D_m)) = E_p(|m + \sigma(a)|, \underbrace{1, \dots, 1}_{q-1}).$$

Thus,

$$m + a = |m + \sigma(a)| = |m + a + (\sigma(a) - a)|. \tag{5}$$

Since there is more than one positive integer m such that $|m + \sigma(a)| > 1$, using (5) for two different such values of m leads to equality $\sigma(a) = a$. By Lemma 3.1, σ has the form $z \mapsto z$ or $z \mapsto \bar{z}$. Hence, ϕ has the form

(i) $A \mapsto UAU^*$, (ii) $A \mapsto U\overline{A}U^*$, (iii) $A \mapsto U\tau(A)U^*$, or (iv) $A \mapsto U\tau(\overline{A})U^*$,

for some unitary U.

Next, we show that the (iii) and (iv) cannot hold if it is not the case that $2p = 2q = n \geq 4$. To this end, consider two cases. If $n \leq 2p$, let $A_m = \mathrm{diag}(1/m, \ldots, 1/m, m^{n-1})$. Since it is not the case that $2p = 2q = n \geq 4$, we have $n - p < p$ or $\binom{q-1}{p-1} < \binom{q}{p}$. Then for a sufficiently large positive integer m we have

$$\begin{aligned} w_{p,q}(A_m) &= E_p(\underbrace{1/m, \ldots, 1/m,}_{q-1} m^{n-1}) \\ &= \binom{q-1}{p-1}\frac{m^{n-1}}{m^{p-1}} + \binom{q-1}{p}\frac{1}{m^p} \\ &< \binom{q}{p} m^p \\ &\leq w_{p,q}(S\tau(A_m)S^{-1}) \end{aligned}$$

as $W_{p,q}(S\tau(A_m)S^{-1})$ contains $E_p(\lambda_1, \ldots, \lambda_q)$ for any q eigenvalues of $\tau(A_m)$. This contradicts the fact that $w_{p,q}(A) = w_{p,q}(\phi(A))$. If $n > 2p$, let

$$B_m = \mathrm{diag}(m, \ldots, m, 1/m^{n-1}).$$

Then for a sufficiently large positive integer m we have

$$w_{p,q}(B_m) = \binom{q}{p} m^p < \binom{q-1}{p-1}\frac{m^{n-1}}{m^{p-1}} + \binom{q-1}{p}\frac{1}{m^p} \leq w_{p,q}(S\tau(B_m)S^{-1}),$$

which is a contradiction.

Now continue to assume that $2 \leq p \leq q < n$, and consider $\mathbf{H} = GL(n, \mathbb{C})$. Since the restriction of ϕ on $SL(n, \mathbb{C})$ has one of the forms (i) – (iv), we have by Theorem 2.7:

$$\binom{q}{p}|f(z^n)z|^p = w_{p,q}(\phi(zI)) = w_{p,q}(zI) = \binom{q}{p}|z|^p$$

for any $z \in \mathbb{C}$, and the conclusion on f follows.

Now suppose $n = q > p$. One can modify the proof of Theorem 3.7 to get the conclusion as follows. When $\mathbf{H} = SL(n, \mathbb{C})$, the map ϕ has the standard form

$$A \mapsto S\sigma(A)S^{-1} \quad \text{or} \quad A \mapsto S\tau(\sigma(A))S^{-1}.$$

If the latter form holds true, one can consider $A = \operatorname{diag}(m, \ldots, m, 1/m^{n-1})$, and conclude that

$$\left|\binom{n-1}{p} m^{p+n} + \binom{n-1}{p-1} m^p\right| = \left|\binom{n-1}{p} m^{n-p} + \binom{n-1}{p-1} m^{2n-p}\right|$$

for all positive integers m, which is impossible unless $n = 2p$. So,

(v) $\phi(X) = S\sigma(X)S^{-1}$, or (vi) $n = 2p$ and $\phi(X) = S\sigma(\tau(X))S^{-1}$.

In both cases of (v) and (vi), consider

$$X = I_n + aE_{11} + aE_{12} + E_{21} \in \mathbf{H}, \qquad \text{for } a \in \mathbb{C}.$$

If $p = 1$, then

$$|n + a| = |E_1(X)| = |E_1(\phi(X))| = |n + \sigma(a)|$$

implies (in view of Lemma 3.1) that either $\sigma(a) = a$ or $\sigma(a) = \overline{a}$. If $p \geq 2$, then

$$|E_p(X)| = \left|(a+2)\binom{n-2}{p-1} + \binom{n-2}{p-2} + \binom{n-2}{p}\right|$$

and

$$|E_p(\phi(X))| = \left|(\sigma(a)+2)\binom{n-2}{p-1} + \binom{n-2}{p-2} + \binom{n-2}{p}\right|.$$

So, $|E_p(X)| = |E_p(\phi(X))|$ implies that either $\sigma(a) = a$ or $\sigma(a) = \overline{a}$. Hence, our assertion is proved if $\mathbf{H} = SL(n, \mathbb{C})$.

Let $\mathbf{H} = GL(n, \mathbb{C})$. If (v) holds true on $SL(n, \mathbb{C})$, then

$$\phi(X) = f(\det(X))SXS^{-1} \quad \text{or} \quad \phi(X) = f(\det(X))S\overline{X}S^{-1},$$

where $f : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ is a multiplicative map. Applying the equality $|E_p(\mu I)| = |E_p(\phi(\mu I))|$, $\mu \in \mathbb{C}^*$, one shows that $|f(\mu)| = 1$ for every $\mu \in \mathbb{C}^*$. If (vi) holds true on $SL(n, \mathbb{C})$, then

$$\phi(X) = f(\det(X))S\tau(X)S^{-1} \quad \text{or} \quad \phi(X) = f(\det(X))S\tau(\overline{X})S^{-1}.$$

Since $|E_p(X)| = |E_p(\phi(X))|$ for all $X \in \mathbf{H}$, we see that $|f(\det(X))|^n = |\det(X)|^2$ for all $X \in \mathbf{H}$. $\square$

Theorem 3.12. *Suppose p, q are integers, $1 \le p \le q \le n$, and $(p,q) \neq (n,n)$. A multiplicative map $\phi : \mathbf{H} \to M_n(\mathbb{C})$ satisfies $W_{p,q}(\phi(A)) = W_{p,q}(A)$ for all $A \in \mathbf{H}$ if and only if there is an $S \in SL(n,\mathbb{C})$ such that one of the following conditions holds true.*

(a) *ϕ has the form*

$$A \mapsto SAS^{-1},$$

where S is unitary if $n > q$.

(b) *$2p = 2q = n > 2$, and ϕ has the form $A \mapsto S\tau(A)S^{-1}$, where S is unitary.*

(c) *$n = q = 2p > 2$, $\mathbf{H} = SL(n,\mathbb{C})$, and ϕ has the form*

$$A \mapsto S\tau(A)S^{-1}.$$

Proof. The "if part" is clear (use (4)), and we prove the "only if" part.

Let $q < n$. Any mapping preserving $W_{p,q}(A)$ also preserves $w_{p,q}(A)$; thus, we need to look for multiplicative $W_{p,q}(A)$ preservers among the set of multiplicative $w_{p,q}(A)$ preservers. Consider

$$A = \frac{1}{(1+i)^{1/n}} \operatorname{diag}(1+i, 1, \ldots, 1).$$

Then we have $W_{p,q}(A) = (1+i)^{-p/n} W_{p,q}(I + iE_{11})$. For any $n \times q$ matrix X with $X^*X = I_q$, the matrix $X^*(I + iE_{11})X \in M_q$ has eigenvalues $1 + ti, 1, \ldots, 1$ with $t \in [0,1]$, and all such eigenvalues can be constructed in this way. Thus,

$$\begin{aligned} W_{p,q}(I + iE_{11}) &= \{E_p(1 + ti, \underbrace{1, \ldots, 1}_{q-1}) : t \in [0,1]\} \\ &= \left\{ \binom{q-1}{p} + \binom{q-1}{p-1}(1+ti) : t \in [0,1] \right\}. \end{aligned}$$

Similarly, $W_{p,q}(\overline{A}) = (1-i)^{-p/n} W_{p,q}(I - iE_{11})$ with

$$W_{p,q}(I - iE_{11}) = \left\{ \binom{q-1}{p} + \binom{q-1}{p-1}(1-ti) : t \in [0,1] \right\}.$$

Clearly, $W_{p,q}(A) \neq W_{p,q}(\overline{A})$. Also, if $2p = 2q = n > 2$, then

$$W_{p,q}(A) \neq W_{p,q}(\tau(\overline{A})),$$

in view of (4). Thus, ϕ on $SL(n,\mathbb{C})$ has the form $A \mapsto UAU^*$, or in case $2p = 2q = n > 2$ the map ϕ may also have the form $A \mapsto U\tau(A)U^*$, where U is unitary.

Consequently, ϕ on $GL(n,\mathbb{C})$ has the form

$$\phi(A) = f(\det(A))UAU^*, \tag{6}$$

or possibly

$$\phi(A) = f(\det(A))U\tau(A)U^* \qquad (\text{if } 2p = 2q = n > 2). \tag{7}$$

Now,

$$W_{p,q}(zI) = \left\{ \binom{q}{p} z^p \right\}.$$

Thus, assuming ϕ has the form (6), we have

$$\binom{q}{p} z^p = W_{p,q}(zI) = W_{p,q}(\phi(zI)) = W_{p,q}\left((f(z))^n zI\right) = \binom{q}{p} (f(z))^{np} z^p.$$

Thus, $f(z)^{np} = 1$ for any $z \in \mathbb{C}^*$. By Lemma 3.2, $f(z) = 1$ for all $z \in \mathbb{C}^*$. If ϕ has the form (7) we analogously obtain

$$z^p = (f(z))^{np} z^{-p}, \quad z \in \mathbb{C}^*.$$

Thus, $(f(z))^{np} = z^{2p}$, which is impossible by Lemma 3.2(b).

If $n = q > p$, then $W_{p,n}(A) = \{E_p(A)\}$, and the multiplicative preservers of $W_{p,n}$ are characterized in Theorem 3.7. $\square$

3.4. Multiplicative Set Preservers. Theorems 2.5 and 2.7 allow one to obtain results on multiplicative maps of $GL(n,\mathbb{C})$ and $SL(n,\mathbb{C})$ that preserve various sets of matrices. We demonstrate this approach in the case of multiplicative preservers of the sets of unitary matrices, normal matrices, invertible Hermitian matrices, or invertible Hermitian matrices with prescribed inertia. The interest in these classes is motivated by extensive results concerning *linear* preservers (see [24, Chapter 3] and references there, [16, 20, 28]). Denote

$$\mathcal{U}_n = \{U \in M_n(\mathbb{C}) : U \text{ is unitary}\}, \quad \mathcal{N}_n = \{A \in M_n(\mathbb{C}) : A \text{ is normal}\}, \tag{8}$$

$$\mathcal{H}_n = \{A \in M_n(\mathbb{C}) : A \text{ is invertible and Hermitian }\},$$

$$\mathcal{H}_n(k, n-k) = \{A \in \mathcal{H}_n : A \text{ has } k \text{ positive and } n-k \text{ negative eigenvalues}\}.$$

Here k is a fixed integer, $0 \le k \le n$. Clearly, trivial maps such as $A \mapsto \det(A)I$ are multiplicative preservers of some of these sets. We exclude multiplicative maps that map into the set of scalar multiples of I from our consideration.

Theorem 3.13. *Let Ξ be one of the sets $\mathcal{U}_n$, $\mathcal{N}_n$, $\mathcal{H}_n$, or $\mathcal{H}_n(k, n-k)$. Assume that $n-k$ is even if $\mathbf{H} = SL(n, \mathbb{C})$ (otherwise $\mathcal{H}_n(k, n-k)$ does not intersect $SL(n, \mathbb{C})$). Then $\phi : \mathbf{H} \to M_n(\mathbb{C})$ is a multiplicative map such that*

$$\phi(\mathbf{H} \cap \Xi) \subseteq \Xi \tag{9}$$

and $\phi(SL(n, \mathbb{C}))$ is not a singleton if and only if there is a unitary $U \in SL(n, \mathbb{C})$ and a multiplicative map $f : \mathbb{C}^* \to \mathbb{C}^*$ such that ϕ has one of the following four forms:

$$A \mapsto f(\det(A))UAU^*, \qquad A \mapsto f(\det(A))U\tau(A)U^*,$$

$$A \mapsto f(\det(A))U\overline{A}U^*, \qquad A \mapsto f(\det(A))U\tau(\overline{A})U^*,$$

where:

(a) $|f(z)| = 1$ *for all* $z \in \mathbb{T}$ *if* $\Xi = \mathcal{U}_n$;
(b) $f(\mathbb{R}^*) \subseteq \mathbb{R}^*$ *if* $\Xi = \mathcal{H}_n$ *or if* $\Xi = \mathcal{H}_n(k, k)$ *(and* $n = 2k$*);*
(c) $f(\mathbb{R}^*) \subseteq \mathbb{R}^*$ *if* $\Xi = \mathcal{H}_n(k, n-k)$ *with* $k \neq n-k$ *and* $n-k$ *even;*
(c') $f(\mathbb{R}^*) \subseteq (0, \infty)$ *if* $\Xi = \mathcal{H}_n(k, n-k)$ *with* $k \neq n-k$ *and* $n-k$ *odd.*

The condition that $\phi(SL(n, \mathbb{C}))$ is not a singleton is automatic if $\Xi = \mathcal{H}_n(k, n-k)$ with $k < n$ and $n-k$ even.

Proof. The "if" part is clear; note that if σ is a complex field automorphism such that $\sigma(\mathbb{R}) = \mathbb{R}$ then $\sigma(\Xi) = \Xi$ for $\Xi = \mathcal{U}_n$, $\Xi = \mathcal{H}_n$, and $\Xi = \mathcal{N}_n$.

We prove the "only if" part. Suppose $\mathbf{H} = SL(n, \mathbb{C})$. Then ϕ has the standard form $A \mapsto S\sigma(A)S^{-1}$ or $A \mapsto S\tau(\sigma(A))S^{-1}$, for some $S \in SL(n, \mathbb{C})$. First, we show that S is unitary. Assume that S has columns $u_1, \dots, u_n$ and S^{-1} has rows $v_1^*, \dots, v_n^*$, where $u_1, \dots, u_n, v_1, \dots, v_n \in \mathbb{C}^n$.

Suppose Ξ is one of $\mathcal{N}_n$, $\mathcal{H}_n$, or $\mathcal{H}_n(k, n-k)$. Let

$$A_m = mE_{11} + \sum_{j=2}^{k} E_{jj} - \sum_{j=k+1}^{n-1} E_{jj} \pm E_{nn}/m,$$

where m is a positive integer, and the sign $\pm$ is chosen so that $\det(A_m) = 1$. Then

$$\phi(A_m) = mu_1v_1^* + \sum_{j=2}^{k} u_jv_j^* - \sum_{j=k+1}^{n-1} u_jv_j^* \pm \frac{u_nv_n^*}{m}$$

or

$$\phi(A_m) = \frac{u_1v_1^*}{m} + \sum_{j=2}^{k} u_jv_j^* - \sum_{j=k+1}^{n-1} u_jv_j^* \pm mu_nv_n^*.$$

Clearly,

$$\lim_{m\to\infty}\frac{\phi(A_m)}{m}=\begin{cases} u_1v_1^* & \text{or}\\ \pm u_nv_n^* \end{cases}$$

is normal, and hence v_1 and u_1 (or v_n and u_n) are multiples of each other. Applying the above argument to any pair of indices $i\neq j$ instead of 1 and n, we conclude that u_r is a multiple of v_r for every $r\in\{1,\ldots,n\}$. Since $v_j^*u_j=1$ for $j=1,\ldots,n$, it follows that $S^{-1}=DS^*$ for some diagonal matrix D with positive diagonal entries. Then $S=U\sqrt{D}^{-1}$, where U is unitary, and where $\sqrt{D}$ is the positive definite diagonal square root of D. We show next that $\sqrt{D}=I$. Consider the matrix

$$A=\begin{bmatrix}0&1\\1&0\end{bmatrix}\oplus\delta_3\oplus\delta_4\oplus\cdots\oplus\delta_n,$$

where $\delta_j=\pm1$, and the signs are adjusted so that $A\in\Xi$. (This is possible unless $k=0$ or $k=n$.) Then $\sqrt{D}A(\sqrt{D})^{-1}$ is normal, which implies that the first two diagonal entries of $\sqrt{D}$ are equal. It follows analogously that all diagonal entries of $\sqrt{D}$ are the same, hence $\sqrt{D}=rI$ for some $r>0$. Now

$$1=\det S=(\det U)(\det\sqrt{D})^{-1}=r^{-n}(\det U),$$

and since $|\det U|=1$, $r=1$. Thus, S is unitary. In the remaining cases $k=n$ or $k=0$, use

$$A=\pm\left(\begin{bmatrix}5/4&3/4\\3/4&5/4\end{bmatrix}\oplus I_{n-2}\right).$$

Then $\sqrt{D}A(\sqrt{D})^{-1}$ is normal, which implies again that the first two diagonal entries of $\sqrt{D}$ are equal, and as before we conclude that S is unitary.

Suppose $\Xi=\mathcal{U}_n$. Let A be a generalized permutation matrix, i.e., $A=DP$, where D is diagonal with ±1's on the diagonal and P is a permutation matrix, such that $\det A=1$. Then $\sigma(A)=A$, and $\phi(A)$ is unitary means: $S^*SA=AS^*S$. Thus, S^*S commutes with every matrix in the algebra $\mathcal{A}$ generated by the generalized permutation matrices with determinant 1. It is easy to see that $\mathcal{A}=M_n(\mathbb{C})$ if $n>2$. (Use the property that $\mathcal{A}$ is invariant under multiplication by diagonal matrices with ±1's on the diagonal having determinant 1.) It follows that S^*S is a scalar matrix. Since $\det S=1$, we obtain that S is unitary. If $n=2$, a similar argument shows that

S^*S commutes with the matrices:

$$I_2, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \in \mathcal{U}_2 \cap SL(2, \mathbb{C}).$$

We can also conclude that S^*S is a scalar matrix and S is unitary.

Next, consider σ. Since S is already proven to be unitary, we obviously have $\sigma(\Xi) \subseteq \Xi$. Letting $A \in \Xi \cap SL(n, \mathbb{C})$ be an appropriately chosen diagonal matrix, and using Lemma 3.1 if necessary, we conclude that σ has the desired form in the cases when $\Xi = \mathcal{U}_n$, $\Xi = \mathcal{H}_n$, $\Xi = \mathcal{H}_n(k, n-k)$. In the remaining case $\Xi = \mathcal{N}_n$ consider

$$A(q) = qE_{11} + E_{12} - E_{21} + E_{33} + \cdots + E_{nn} \in SL(n, \mathbb{C}), \quad q \in \mathbb{C}^*,$$

and observe that $A(q)$ is normal if and only if $q = ir$ for some $r \in \mathbb{R}$. Clearly $\sigma(A(q))$ is normal for every normal $A(q)$ if and only if $\sigma(\mathbb{R}) \subseteq \mathbb{R}$, and use Lemma 3.1 again.

It remains to prove the "only if" part for the case $\mathbf{H} = GL(n, \mathbb{C})$. For now we leave aside the case when $\Xi = \mathcal{H}_n(k, n-k)$ with odd $n-k$. Restricting the map ϕ to $SL(n, \mathbb{C})$, we obtain the required properties for U and σ. As for the additional properties of f specified in (a), (b), and (c), they are evident (and can be easily proved arguing by contradiction, for example) in cases $\Xi = \mathcal{N}_n$ (no additional properties), $\Xi = \mathcal{U}_n$, $\Xi = \mathcal{H}_n$, and $\Xi = \mathcal{H}_n(k, k)$. Consider $\Xi = \mathcal{H}_{k,n-k}$ with $k \neq n-k$. Then we clearly have (c) as well (recall that $n-k$ is assumed to be even).

Finally, consider the case $\mathbf{H} = GL(n, \mathbb{C})$, $\Xi = \mathcal{H}_n(k, n-k)$ with odd $n-k$. Then ϕ has one of the two forms

$$A \mapsto f(\det(A)) S\sigma(A)S^{-1} \qquad \text{or} \qquad A \mapsto f(\det(A)) S\tau(\sigma(A))S^{-1},$$

where $f : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ is a multiplicative map, σ is a complex field embedding, and $S \in SL(n, \mathbb{C})$. Using the matrices

$$A_m = mE_{11} + \sum_{j=2}^{k} E_{jj} - \sum_{j=k+1}^{n-1} E_{jj} - E_{nn}/m,$$

m an integer, we prove that S is in fact unitary, similarly to the proof given in the case $\mathbf{H} = SL(n, \mathbb{C})$. For the consideration of σ, notice that $f(\det A)\sigma(A) \in \mathcal{H}_n(k, n-k)$ for every $A \in \mathcal{H}_n(k, n-k)$. Letting here A to be a suitable real diagonal matrix with determinant -1, we see that $\sigma(\mathbb{R}) \subseteq \mathbb{R}$, hence σ is either trivial or complex conjugation. Properties (b) or (c') of f (whichever case is applicable) now follow easily. $\square$

If the hypothesis that $\phi(SL(n,\mathbb{C}))$ is not a singleton is omitted in Theorem 3.13, then various forms of ϕ are possible. For example, let A_μ be a group of invertible $n\times n$ complex matrices, indexed by $\mu\in\mathbb{T}$, so that $A_{\mu\nu}=A_\mu A_\nu$ for $\mu\nu\in\mathbb{T}$, such that $A_{-1}\in\mathcal{H}_n(k,n-k)$. For example, $A_\mu=\mathrm{diag}\,(\mu,\dots,\mu,1,1,\dots,1)$, where μ appears $n-k$ times. Then define

$$\phi(X)=A_{(\det X)/|(\det X)|},\qquad X\in GL(n,\mathbb{C}).$$

Clearly, if $n-k$ is odd, we have $\phi(\mathcal{H}_n(k,n-k))\subseteq\mathcal{H}_n(k,n-k)$. Analogous degenerate maps can be constructed for $\mathcal{U}_n$, $\mathcal{N}_n$, and $\mathcal{H}_n$.

Note also that if ϕ is a multiplicative map on $\mathbf{H}$, with the property (9), Ξ is one of $\mathcal{U}_n$, $\mathcal{H}_n$, or $\mathcal{H}_n(k,n-k)$ (with $n-k$ even if $\mathbf{H}=SL(n,\mathbb{C})$), and $\phi(SL(n,\mathbb{C}))$ is a singleton, then necessarily $\phi(SL(n,\mathbb{C}))=\{I\}$.

4. Multiplicative Preservers on Matrices over Other Fields

One may use the techniques of the previous section to obtain results on multiplicative preservers on matrices over fields other than $\mathbb{C}$, such as the real field $\mathbb{R}$. For the real field, the situation simplifies somewhat because there is only the trivial field embedding of $\mathbb{R}$ (see, e.g., [29]). For example, the real analog of Theorem 3.5 reads as follows (the spectral radius of a real matrix is defined as in the complex case, i.e., non-real eigenvalues, if any, are taken into account):

Theorem 4.1. *Let $\mathbf{H}=SL(n,\mathbb{R})$ or $\mathbf{H}=GL(n,\mathbb{R})$. A multiplicative map $\phi:\mathbf{H}\to M_n(\mathbb{R})$ satisfies*

$$r(\phi(A))=r(A)\quad \text{for all } A\in\mathbf{H}$$

if and only if there exist an $S\in GL(n,\mathbb{R})$ and a multiplicative map $f:\mathbb{R}^*\to\{1,-1\}$, which collapses to the constant function $f(\mu)=1$ when $\mathbf{H}=SL(n,\mathbb{R})$, such that ϕ has the form

$$\phi(A)=f(\det(A))SAS^{-1},\qquad A\in\mathbf{H}.$$

Real analogs of Theorems 3.3, 3.8, and 3.4 can be formulated analogously, with essentially the same proofs (note that the proof of Theorem 3.3 simplifies considerably in the real case); as in Theorem 4.1, here we require only that S be real orthogonal, not necessarily having determinant 1.

Next, we present a real analog of Theorem 3.13. We denote $\mathcal{U}_n(\mathbb{R})=\mathcal{U}_n\cap M_n(\mathbb{R})$, and analogously $\mathcal{N}_n(\mathbb{R})$, $\mathcal{H}_n(\mathbb{R})$, $\mathcal{H}_n(k,n-k)(\mathbb{R})$. Multiplicative maps ϕ that map into the set of scalar multiples of I are excluded in the next theorem.

Theorem 4.2. *Let* $\mathbf{H} = SL(n, \mathbb{R})$ *or* $\mathbf{H} = GL(n, \mathbb{R})$*, and let* Ξ *be one of the sets* $\mathcal{U}_n(\mathbb{R})$*,* $\mathcal{N}_n(\mathbb{R})$*,* $\mathcal{H}_n(\mathbb{R})$*, or* $\mathcal{H}_n(k, n-k)(\mathbb{R})$*. Assume that* $n-k$ *is even if* $\mathbf{H} = SL(n, \mathbb{R})$*. Then* $\phi : \mathbf{H} \to M_n(\mathbb{R})$ *is a multiplicative map such that* $\phi(\mathbf{H} \cap \Xi) \subseteq \Xi$ *and* $\phi(SL(n, \mathbb{R}))$ *is not a singleton if and only if there is a real orthogonal* $U \in M_n(\mathbb{R})$*, and a multiplicative map* $f : \mathbb{R}^* \to \mathbb{R}^*$ *such that* ϕ *has one of the following two forms:*

$$A \mapsto f(\det(A))UAU^*, \qquad A \mapsto f(\det(A))U\tau(A)U^*,$$

where $f(\mathbb{R}^*) \subseteq (0, \infty)$ *if* $\Xi = \mathcal{H}_n(k, n-k)$ *with* $k \neq n-k$ *and* $n-k$ *odd.*

The proof is completely analogous to that of Theorem 3.13. The only detail that perhaps requires additional explanation is the proof that S^*S is a scalar matrix in the case $\Xi = \mathcal{U}_n$ and $n = 2$. In this case the algebra $\mathcal{A}$ generated by the generalized permutation 2×2 matrices having determinant 1 coincides with

$$\left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

It is easy to see that every real 2×2 matrix that commutes with every element of $\mathcal{A}$ actually belongs to $\mathcal{A}$. So $S^*S \in \mathcal{A}$. Since in addition S^*S is real and positive definite we must have that S^*S is a scalar matrix.

Note that the kth elementary symmetric function $E_k(A)$ of eigenvalues of $A \in M_n(\mathbb{F})$ can be viewed as the sum of determinants of the $k \times k$ principal submatrices of A, or as $\pm$ the coefficient of λ^{n-k} in the characteristic polynomial $\det(A - \lambda I)$; thus, $E_k(A) \in \mathbb{F}$. So for multiplicative preservers of elementary symmetric functions of eigenvalues it makes sense to consider matrices over any field $\mathbb{F}$. We have an analog of Theorem 3.7. If $n = 2$, there is not much to say. So we assume that $n > 2$.

Theorem 4.3. *Let* $\mathbb{F}$ *be a field, and let* $\mathbf{H} = SL(n, \mathbb{F})$ *or* $\mathbf{H} = GL(n, \mathbb{F})$*, with* $n > 2$*. Fix* $1 \leq k < n$*. A multiplicative map* $\phi : \mathbf{H} \to M_n(\mathbb{F})$ *satisfies* $E_k(\phi(A)) = E_k(A)$ *for all* $A \in \mathbf{H}$ *if and only if there is an* $S \in GL(n, \mathbb{F})$ *and a multiplicative map* $f : \mathbb{F}^* \to \mathbb{F}^*$ *such that*

(a) ϕ *has the form* $X \mapsto f(\det X)SXS^{-1}$ *and* f *is such that* $f(\mu)^k = 1$ *for every* $\mu \in \mathbb{F}^*$*, or*

(b) $n = 2k$*, and* ϕ *has the form* $X \mapsto f(\det X)S\tau(X)S^{-1}$*, where* f *is such that* $f(\mu)^k = \mu$ *for every* $\mu \in \mathbb{F}^*$*.*

Proof. Note that in particular there is no restriction on the characteristic of $\mathbb{F}$ in Theorem 4.3. Clearly, ϕ is nontrivial on $SL(n, \mathbb{F})$. So we assume that ϕ has the usual form (2), as given in Theorem 2.7. There is no loss of generality in taking $S = I$

(since conjugation by S preserves E_k). Now take A to be the companion matrix of a polynomial $f(x) \in \mathbb{F}[x]$ with $\det(A) = 1$. Applying σ preserves the corresponding coefficient of $f(x)$ (since that is $\pm E_k(A)$). Since that coefficient is arbitrary, it follows that σ is the identity map.

First assume that $n \neq 2k$. We show that τ cannot be involved. Consider monic polynomials $f(x)$ of degree n with constant term $(-1)^n$ and let A be the companion matrix of $f(x)$. Then applying τ gives a matrix similar to the companion matrix of $g(x) := x^n f(1/x)$. Aside from the case $n = 2k$, there will always be such a polynomial so that the kth coefficient of $f(x)$ is not the kth coefficient of $g(x)$. Since $\det(A) = 1$, this shows that τ cannot be involved when $n \neq 2k$.

So assume (even for the case $n = 2k$) that τ is not involved. Noting that $E_k(\mu A) = \mu^k E_k(A)$ shows that $f(\det A)^k = 1$ for any A with $E_k(A) \neq 0$. We can always find A with $\det(A)$ arbitrary and $E_k(A) \neq 0$ (take a companion matrix). Thus, f^k is identically 1.

Finally, consider the case that $n = 2k$ and τ is involved. Then $E_k(\mu\tau(A)) = \mu^k E_k(A)/\det(A)$. In particular, if $E_k(A)$ is nonzero, this forces $f(\det(A))^k = \det(A)$, whence the result. $\square$

The result of Corollary 3.6 can be generalized for arbitrary fields (with few exceptions), as stated in the next theorem. We say that a multiplicative map $\phi : \mathbf{H} \to M_n(\mathbb{F})$, where $\mathbf{H} = SL(n, \mathbb{F})$ or $\mathbf{H} = GL(n, \mathbb{F})$, *preserves spectra* if for every $A \in \mathbf{H}$ we have

$$\{\lambda \in \mathbb{F} \,:\, \det(A - \lambda I) = 0\} \supseteq \{\lambda \in \mathbb{F} \,:\, \det(\phi(A) - \lambda I) = 0\}. \tag{10}$$

The case when one or both sides of (10) are empty sets is not excluded.

Theorem 4.4. *Let $\mathbb{F}$ be a field, and let $\mathbf{H} = SL(n, \mathbb{F})$ or $\mathbf{H} = GL(n, \mathbb{F})$. Assume that $|\mathbb{F}| > 4$. Then a multiplicative map $\phi : \mathbf{H} \to M_n(\mathbb{F})$ preserves spectra if and only if ϕ has the form SXS^{-1} for some $S \in GL(n, \mathbb{F})$.*

Proof. Clearly such maps have the preservation of spectra property. Assume now that ϕ is a multiplicative map on $\mathbf{H}$ that preserves spectra. Then ϕ cannot be trivial on $SL(n, \mathbb{F})$ (indeed, the only way ϕ can be trivial is when $\phi(\mathbf{H}) = I_n$, but if $|\mathbb{F}| > 2$ then a diagonal matrix $A \in SL(n, \mathbb{F})$ can be found with some eigenvalues in $\mathbb{F}$ different from 1, a contradiction with the preservation of spectra property). So assume that $\phi(A) = f(\det(A))S\sigma(A)S^{-1}$ or $\phi(A) = f(\det(A))S\sigma(\tau(A))S^{-1}$ as in Theorem 2.7.

We first show that $\sigma = 1$. By composing f with conjugation by S, we may take $S = I$ (since conjugation certainly keeps the preservation of spectra property). Now ϕ maps diagonal matrices to diagonal matrices. We claim that σ is the identity. Assume not, and let $a \in \mathbb{F} \setminus \{0\}$ be such that $\sigma(a) \neq a$. Taking A to have exactly the distinct eigenvalues $a, a^{-1}, 1$ with $\det(A) = 1$, we see that $\sigma(a) = a^{-1}$. Thus, σ either fixes or inverts every element of $\mathbb{F}$ – this cannot happen unless $\mathbb{F}$ has characteristic 2. Indeed, clearly $\sigma^2 = 1$ on $\mathbb{F}$, and therefore, assuming the characteristic of $\mathbb{F}$ is not 2, $\mathbb{F}$ (as a vector space over the prime field) can be decomposed into a direct sum of the subspace fixed by σ, which is actually a field, and the eigenspace of σ corresponding to the eigenvalue -1, and furthermore, if $\sigma(x) = -x = x^{-1}$, $x \in \mathbb{F}^*$, then $x^2 = -1$, so the -1 eigenspace has cardinality at most 3. We note also that $x \neq \pm 1$ inverted by σ implies that $1 + x$ is also inverted by σ, whence

$$1 = (1 + x)(1 + x^{-1}) = 2 + x + x^{-1}, \tag{11}$$

and so $x^2 + x + 1 = 0$. Since this equation holds for any xy with $y \neq 0$ fixed by σ, it follows that the fixed field of σ has cardinality at most 3. These considerations rule out any characteristic of $\mathbb{F}$ but 2, and moreover the fixed field of σ must have cardinality 2, whereas the equation (11) shows that there are at most two elements x such that $\sigma(x) \neq x$. We are left with the case $|\mathbb{F}| = 4$, which is excluded by the hypotheses of the theorem.

So $\sigma = 1$. If $n > 2$, we see that τ cannot be involved, since there are matrices of determinant 1 whose spectrum in $\mathbb{F}$ is not closed under inverses (here the hypothesis that $|\mathbb{F}| > 3$ is essential). If $n = 2$, we can modify f and assume again that τ is not involved.

It remains to consider f. Suppose that $f(a) \neq 1$, $a \in \mathbb{F}^*$. Now take A to be diagonal with all but one diagonal entry 1 (and the remaining diagonal entry $a = \det(A)$). The spectrum of $\phi(A)$ consists of $f(a)$ and $af(a)$. Now $\{1, a\} = \{f(a), af(a)\}$ implies that $f(a) = a = a^{-1}$. Thus $a = -1$ and the kernel of f is trivial, whence $|\mathbb{F}| = 3$, an excluded case. $\square$

Note that Theorem 4.4 fails if $|\mathbb{F}| \leq 4$: If $|\mathbb{F}| = 4$, then the map $\phi(A) = \sigma(\tau(A))$ on $SL(n, \mathbb{F})$, where σ is the squaring map on $\mathbb{F}$, preserves spectra; if $|\mathbb{F}| \leq 3$, then the map $\phi(A) = \tau(A)$ on $SL(n, \mathbb{F})$ preserves spectra.

If we adopt a more stringent definition of the spectrum preservation property, namely that (10) is replaced by

$$\{\lambda \in \overline{\mathbb{F}} : \det(A - \lambda I) = 0\} \supseteq \{\lambda \in \overline{\mathbb{F}} : \det(\phi(A) - \lambda I) = 0\}, \tag{12}$$

where $\overline{\mathbb{F}}$ is the algebraic closure of $\mathbb{F}$, then the exceptions of Theorem 4.4 disappear:

Theorem 4.5. *Let $\mathbb{F}$ be a field, and let $\mathbf{H} = SL(n, \mathbb{F})$ or $\mathbf{H} = GL(n, \mathbb{F})$. Then a multiplicative map $\phi : \mathbf{H} \to M_n(\mathbb{F})$ satisfies* (12) *for every $A \in \mathbf{H}$ if and only if ϕ has the form SXS^{-1} for some $S \in GL(n, \mathbb{F})$.*

Proof. In view of Theorem 4.4 we may assume $|\mathbb{F}| \leq 4$. Assume now that ϕ is a multiplicative map on $\mathbf{H}$ that preserves spectra. Then ϕ cannot be trivial on $SL(n, \mathbb{F})$ (there are matrices of determinant 1 that have a nontrivial spectrum in $\overline{\mathbb{F}}$).

Assume first that $|\mathbb{F}| > 3$ if $n = 2$. Then ϕ has the form $\phi(A) = f(\det(A))S\sigma(A)S^{-1}$ or $\phi(A) = f(\det(A))S\sigma(\tau(A))S^{-1}$ as in Theorem 2.7. We show that $\sigma = 1$. Arguing as in the proof of Theorem 4.4, we need to consider only the case $|\mathbb{F}| = 4$. In that case, we choose matrix A whose eigenvalues are $a, a^4, 1, \ldots, 1$ where a is a root of the polynomial x^2+bx+1, $b \in \mathbb{F}_4 \backslash \{0, 1\}$, in a quadratic extension $\mathbb{F}_{16}$ of $\mathbb{F}_4$. Then $a^5 = 1$, and if $\phi \neq 1$, then $\phi(A)$ has eigenvalues $a^2, a^3, 1 \ldots, 1$, a contradiction. So $\sigma = 1$. If $n > 2$, we see that τ cannot be involved, since there are matrices of determinant 1 whose spectrum in $\overline{\mathbb{F}}$ is not closed under inverses (if $|\mathbb{F}| = 4$, this is clear; if $|\mathbb{F}| = 2$, take A to have three nontrivial eigenvalues of order 7 with product 1; and if $|\mathbb{F}| = 3$, take such an A with 7 replaced by 13). If $n = 2$, we can modify f and assume again that τ is not involved. It remains to consider f (in the case $\mathbf{H} = GL(n, \mathbb{F})$). The proof of Theorem 4.4 shows that we need only to consider the field of order 3. In this case, we take a matrix A whose eigenvalues are $a, a^3, 1 \ldots, 1$ where a is a primitive 8-th root of 1. So $\det(A) = -1$ and, assuming f is non-trivial, the spectrum of $\phi(A)$ is $\{-a, -a^3, -1, \ldots, -1\}$ which is different from that of A.

Finally, we treat the remaining case $n = 2$ and $|\mathbb{F}| \leq 3$. Note that $SL(2, \mathbb{F})$ has a unique minimal normal subgroup that has order prime to the characteristic of $\mathbb{F}$. This subgroup cannot be contained in the kernel of the multiplicative map (since spectra are preserved) and so the map is bijective. Since $S_3 = GL(2, \mathbb{F}_2)$ is its own automorphism group and since the automorphism group of $SL(2, \mathbb{F}_3)$ is $S_4 = PGL(3, \mathbb{F}_3)$, it follows that any automorphism of $GL(2, \mathbb{F}_2)$ or of $SL(2, \mathbb{F}_3)$ is a conjugation by an element of $GL(2, \mathbb{F})$. The same argument shows the result for $GL(2, \mathbb{F}_3)$. $\square$

In a different direction, multiplicative preservers of certain algebraic groups may be characterized.

Theorem 4.6. *Let $\mathbb{F}$ be a field and $\mathbf{H} = SL(n, \mathbb{F})$ or $GL(n, \mathbb{F})$. Assume that $|\mathbb{F}| > 3$ if $n = 2$. Let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$. Let G be an algebraic subgroup of $GL(n, \overline{\mathbb{F}})$ defined over the prime field. Then a nontrivial multiplicative map $\phi : \mathbf{H} \to M_n(\mathbb{F})$ preserves G in the sense that $\phi(G) \subseteq G$ if and only if there exists an S in $GL(n, \mathbb{F})$ and a field embedding $\sigma : \mathbb{F} \to \mathbb{F}$ such that one of the following holds:*

(1) $\phi(A) = S\sigma(A)S^{-1}$ and $SGS^{-1} = G$; or

(2) $\phi(A) = S\sigma(\tau(A))S^{-1}$ with $S\tau(G)S^{-1} = G$.

Proof. Let ϕ be a multiplicative map preserving G with the notation as in Theorem 2.7. Let σ be the associated field embedding. We can extend ϕ to $\overline{\mathbb{F}}$. Since G is defined over the prime field, $\sigma(G) \subseteq G$ and so the maps above certainly preserve G. Let P denote the prime field. Then $\sigma(\overline{P}) = \overline{P}$ and so σ is an automorphism of the $\overline{P}$ points of G. Thus, $\sigma(G)$ is Zariski dense in G. because σ sends $\overline{P}$ onto itself and so the image of σ (after extending to the algebraic closure), σ sends $G(\overline{P})$ to itself. Now it is well known that if $A < B$ are algebraically closed fields and V is a variety defined over A, then $V(A)$ is Zariski dense in $V(B)$. Thus, S or $\tau(S)$ must normalize G depending which case we are in. $\square$

There is a version of Theorem 4.6 for the excluded cases $n = 2$ and $|F| \leq 3$, but it is a bit complicated to state and has very little content.

Theorem 4.6 in particular applies to the symplectic group or orthogonal group or any split simple algebraic group (since they are defined over the prime field), as well as the groups of triangular or block triangular matrices.

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