# Graphs associated with matrices over finite fields and their endomorphisms

**In memory of Professor Michael Neumann and Professor Uri Rothblum**

Li-Ping Huang[a], Zejun Huang[b] , Chi-Kwong Li[c,d] , Nung-Sing Sze[b]

[a] School of Mathematics and Computing Science, Changsha University of Science and Technology, Changsha, 410004, P. R. China

[b] Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Hong Kong

[c] Department of Mathematics, The College of William & Mary, Williamsburg, VA 23187-8795, USA

[d] Department of Mathematics, The University of Hong Kong, Hong Kong

**Abstract**. Let $\mathbb{F}^{m\times n}$ be the set of $m \times n$ matrices over a field $\mathbb{F}$. Consider a graph $G = (\mathbb{F}^{m\times n}, \sim)$ with $\mathbb{F}^{m\times n}$ as the vertex set such that two vertices $A, B \in \mathbb{F}^{m\times n}$ are adjacent if $\operatorname{rank}(A - B) = 1$. We study graph properties of $G$ when $\mathbb{F}$ is a finite field. In particular, $G$ is a regular connected graph with diameter equal to $\min\{m, n\}$; it is always Hamiltonian. Furthermore, we determine the independence number, chromatic number and clique number of $G$. These results are used to characterize the graph endomorphisms of $G$, which extends Hua's fundamental theorem of geometry on $\mathbb{F}^{m\times n}$.

**Keywords:** graph, matrix, endomorphism, finite field, independence number, chromatic number
**2010 Mathematics Subject Classification**: 05C50, 15B33, 05C69, 05C60, 05C45.

## 1 Introduction

Let $\mathbb{F}$ be a field and $\mathbb{F}^{m\times n}$ the set of $m \times n$ matrices over $\mathbb{F}$. Define a metric $d$ on $\mathbb{F}^{m\times n}$ by

$$d(A, B) = \operatorname{rank}(A - B).$$

Two matrices $A, B \in \mathbb{F}^{m\times n}$ are *adjacent*, denoted by $A \sim B$, if $d(A, B) = \operatorname{rank}(A - B) = 1$. This metric and adjacency relation give rise to an interesting geometrical structure on $\mathbb{F}^{m\times n}$.

In mid 1940's, Hua initiated the study of the fundamental theorem of the geometry of matrices that concerns the characterization of maps $\phi : \mathbb{F}^{m\times n} \to \mathbb{F}^{m\times n}$ leaving invariant the adjacency relation, i.e., $\operatorname{rank}(\phi(A)-\phi(B)) = 1$ whenever $\operatorname{rank}(A-B) = 1$. Hua also considered the problem on matrices over a division ring, and his study generated considerable interest and led to many interesting results; for example, see [8, 9, 11, 13, 18, 20].

Suppose $\mathbb{F}$ is the finite field $\mathbb{F}_q$ with $q$ elements. Then the adjacency relation $A \sim B$ in ${\mathbb{F}_q}^{m\times n}$ defined above, i.e., $A \sim B$ if $\operatorname{rank}(A - B) = 1$, gives rise to a graph $G = (V, \sim)$ with $V = {\mathbb{F}_q}^{m\times n}$ as the vertex set and there is an edge joining $A, B \in V$ if and only if $A \sim B$. We call $G = ({\mathbb{F}_q}^{m\times n}, \sim)$ a *matrix graph*, which is also called a *bilinear forms graph* in graph theory. This graph has a lot of interesting properties. For example, it is easy to check that $G$ is a regular graph with diameter equal to $\min\{m, n\}$; it is Eulerian if and only if $q$ is odd. We

E-mail addresses: lipingmath@163.com (L.-P. Huang), mathzejun@gmail.com (Z. Huang), ckli@math.wm.edu (C.K. Li), raymond.sze@polyu.edu.hk (N.S. Sze)

will give an easy constructive proof to show that G is Hamiltonian. Furthermore, we determine the independence number, chromatic number and clique number of $G$; see Section 2.

Note that in graph theory literature, it is common to write $G = (V, E)$ with $V$ as the vertex set, and $E$ as the edge set consisting of all the unordered pairs of vertices $u$ and $v$ that are adjacent.

Recall that for two given graphs $G = (V, \sim)$ and $G' = (V', \sim')$, a map $\phi : V \to V'$ is a *graph homomorphism* if

$$\phi(a) \sim' \phi(b) \text{ in } G' \quad \text{whenever} \quad a \sim b \text{ in } G.$$

A graph homomorphism is called a *graph endomorphism* if $G = G'$. Thus, the fundamental theorem of geometry of ${\mathbb{F}_q}^{m\times n}$ can be formulated in terms of graph endomorphisms on $({\mathbb{F}_q}^{m\times n}, \sim)$. In [16, 17], the author characterized the graph endomorphisms on symmetric matrix graphs and hermitian matrix graphs over a finite field. In Section 3, we will characterize graph endomorphisms on matrix graphs using results in Section 2.

It is worth pointing out that in addition to the connection to the geometry of matrices, matrix graphs also have nice combinatorial and algebraic porperties, and are useful in the study of the group theory, design theory, association schemes, and coding theory, etc. For example, see [20, 21, 22, 24].

We will always assume that $2 \leqslant m \leqslant n$ in our discussion unless specified otherwise. For the case $m > n \geqslant 2$, one can consider the transposes of matrices to get similar results. The basic definitions and results in graph theory can be found in [1, 4, 6, 5]. We denote by $\mathbb{F}^n$ the set of $n \times 1$ vectors over $\mathbb{F}$, and denote by $|X|$ the cardinality of a set $X$.

## 2 Graph properties

A graph $G = (V, \sim)$ is connected if any two distinct vertices $a, b \in V$ are connected by a path, i.e., a sequence of vertices $v_0, \ldots, v_m$ such that $a = v_0$, $b = v_m$, and $v_i \sim v_{i+1}$ for $i = 0, 1, \ldots, m-1$. The number of edges in the shortest path joining two distinct vertices $a, b \in V$ is the *distance* between $a$ and $b$, which is denoted by $d(a, b)$. For a connected graph, the longest distance between two vertices is the *diameter* of $G$. The following observation are well known (cf. [11, Lemma 3.3]).

**Proposition 2.1.** *The graph $G = ({\mathbb{F}_q}^{m\times n}, \sim)$ $(m, n \geqslant 2)$ is connected, where $A \sim B \Leftrightarrow \operatorname{rank}(A - B) = 1$ for all $A, B \in {\mathbb{F}_q}^{m\times n}$. The distance between two matrices $A, B \in {\mathbb{F}_q}^{m\times n}$ is given by*

$$d(A, B) = \operatorname{rank}(A - B).$$

*Consequently, the diameter of $G$ equals* $\min\{m, n\}$.

The *degree* of a vertex $v$ in a graph $G$ is the number of vertices in $G$ that are adjacent to $v$, and is denoted by $\deg(v)$. A graph $G$ is $r$-regular if every vertex of $G$ has degree $r$. The following result is known (cf. [2, Theorem 9.5.2]): The graph $G = \left({\mathbb{F}_q}^{m\times n}, \sim\right)$ is an $r$-regular graph with $r = \frac{(q^m-1)(q^n-1)}{q-1}$.

A *vertex-cut* of a graph $G$ is a set $S$ of vertices of $G$ such that removing the vertices in $S$ and the edges incident to them from $G$ results in a disconnected graph. A vertex-cut of $G$ with minimum cardinality is called a *minimum vertex-cut* of $G$ and this minimum cardinality is called the *connectivity* of $G$ and is denoted by $\kappa(G)$.

A connected graph $G$ with diameter $d$ is *distance-regular* if for any vertices $u$ and $v$ of $G$ and any integers $i, j = 0, 1, \ldots, d$, the number of vertices at distance $i$ from $u$ and distance $j$ from $v$ depends only on $i, j$, and the graph distance between $u$ and $v$, and is independent of the choice of $u$ and $v$. For $G = ({\mathbb{F}_q}^{m\times n}, \sim)$, It is well known that the graph $G$ is distance-regular. Therefore, by Theorem 1 of [3] and $G$ is an $r$-regular graph with $r = (q^m - 1)(q^n - 1)/(q - 1)$, we have $\kappa(G) = (q^m - 1)(q^n - 1)/(q - 1)$.

Let $G$ be a nontrivial connected graph. A circuit $C$ of $G$ that contains every edge of $G$ (necessarily exactly once) is an *Eulerian circuit*. A connected graph $G$ is called *Eulerian* if $G$ contains an Eulerian circuit. It is well known (see for example [4, Theorem 3.1]) that a connected graph $G$ is Eulerian if and only if every vertex of $G$ has even degree. Since $G$ is an $r$-regular graph with $r = (q^m - 1)(q^n - 1)/(q - 1)$, $G = ({\mathbb{F}_q}^{m\times n}, \sim)$ is an Eulerian graph if and only if $q$ is odd.

A cycle in a graph $G$ that contains every vertex of $G$ is called a *Hamiltonian cycle* of $G$. A graph that contains a Hamiltonian cycle is called a *Hamiltonian graph*.

**Theorem 2.2.** *The matrix graph $G = ({\mathbb{F}_q}^{m\times n}, \sim)$ is Hamiltonian.*

**Remark 2.3.** *Since the graph $G = ({\mathbb{F}_q}^{m\times n}, \sim)$ is connected, one can use Corollary 3.2 of [15] to obtain this result. Here we give a constructive proof based only on the definition of the graph.*

*Proof.* Let $\mathbb{F}_q = \{0, 1, x_3, \ldots, x_q\}$, and $A_0 = 0_{m,n}$. Keeping all other rows to be zero, we can change the first row of $A_0$ from $(0, \ldots, 0)$ to $(x_q, \ldots, x_q)$ in $q^n - 1$ steps by adding a rank one matrix in each step. So, in $q^n - 1$ steps (i.e., using $q^n - 1$ edges in the graph), we get all the matrices with arbitrary first row and other rows equal $(0, \ldots, 0)$. In other words, we have a path with $q^n - 1$ edges joining $q^n$ vertices corresponding to matrices with arbitrary first row and other rows equal $(0, \ldots, 0)$.

Now, we extend the path constructed in the preceding paragraph as follows. Change the second row to $(0, \cdots, 0, 1)$ by adding a rank one matrix (one more edge). In the next $q^n - 1$ steps (edges), we change the first row from $(x_q, \ldots, x_q)$ back to $(0, \ldots, 0)$. So, in $1 + (q^n - 1) = q^n$ steps, we get all the matrices with arbitrary first row, second row equal $(0, \ldots, 0, 1)$, and all other rows equal to $(0, \ldots, 0)$.

Next, we change the second row to $(0, \ldots, 0, x_3)$ if $q > 2$, and then change the first row from $(0, \ldots, 0)$ to $(x_q, \ldots, x_q)$. So, in another $q^n$ steps, we get all the matrices with arbitrary first row, second row equal $(0, \ldots, 0, x_3)$, and all other rows equal to $(0, \ldots, 0)$.

We may keep changing the second row till we get $(x_q, \ldots, x_q)$ so that every change of the second row always followed by a change of the first row from $(0, \ldots, 0)$ to $(x_q, \ldots, x_q)$, or a change from $(x_q, \ldots, x_q)$ to $(0, \ldots, 0)$. Then in $q^{2n} - 1$ steps, we will get all the matrices with arbitrary first two rows, and other rows equal to $(0, \ldots, 0)$. Moreover, in the latest step, the first row is either $(0, \ldots, 0)$ or $(x_q, \ldots, x_q)$, and the second row is $(x_q, \ldots, x_q)$, which is a rank one matrix. If $m = 2$, we can change this matrix to the zero matrix and complete the Hamiltonian cycle.

Suppose $m > 2$. We can work on the third row, fourth row, and so forth. After $q^{mn} - 1$ steps, we go through all the matrices in ${\mathbb{F}_q}^{m\times n}$ and end with a matrix $A$. Moreover, the last row of $A$ is $(x_q, \ldots, x_q)$, and all other rows of $A$ have the form $(0, \ldots, 0)$ or $(x_q, \ldots, x_q)$. Thus, the matrix $A$ has rank one, and is adjacent to the zero matrix $A_0$. So, we may change the $A$ to the zero matrix $A_0$ by adding a rank one matrix.

Hence, in $q^{mn}$ steps, we obtain a cycle of $G$ which contains every vertex of $G$. Thus $G$ is Hamiltonian. $\square$

Let $K_r$ be the *complete graph* of $r$ vertices. Let $G = (V, \sim)$. A subset $A \subseteq V$ of $r$ vertices is called an *$r$-clique* if the induced subgraph $G[A]$ is a complete subgraph $K_r$, i.e., any two distinct vertices in the subgraph are adjacent. A *clique* $A$ is maximal if there is no clique of $G$ which properly contains $A$ as a subset. A clique is *maximum* if there is no clique of $G$ of larger cardinality. In the geometry of matrices, we use the term *maximal set* to indicate a *maximal clique*. Let $\omega(G)$ be the number of vertices in a *maximum clique* of $G$, and it is called the *clique number* of $G$.

Denote by $GL_n(\mathbb{F}_q)$ the set of $n \times n$ invertible matrices over $\mathbb{F}_q$ and ${}^tA$ the transpose matrix of $A \in {\mathbb{F}_q}^{m\times n}$. Let $E_{ij}^{m\times n}$ ($E_{ij}$ for short) be the $m \times n$ matrix whose $(i, j)$-entry is 1 and all other entries are 0's, and let $e_i$ be the $i$-th column of $I$. In ${\mathbb{F}_q}^{m\times n}$ $(m, n \geqslant 2)$, let

$$\mathcal{M}_i = \left\{ \sum_{j=1}^{n} x_j E_{ij} : x_j \in \mathbb{F}_q \right\},\ i = 1, \ldots, m, \tag{2.1}$$

$$\mathcal{N}_j = \left\{ \sum_{i=1}^{m} y_i E_{ij} : y_i \in \mathbb{F}_q \right\},\ j = 1, \ldots, n. \tag{2.2}$$

Let $S_1$, $S_2$ be two subsets of ${\mathbb{F}_q}^{m\times n}$ and $A \in {\mathbb{F}_q}^{m\times n}$. We denote by

$$S_1 + A = \{X + A : X \in S_1\}, \quad AS_1 = \{AX : X \in S_1\}, \quad S_1A = \{XA : X \in S_1\}.$$

We first state the following lemma; see [9, 11, 20].

**Lemma 2.4.** *In ${\mathbb{F}_q}^{m\times n}$ $(m, n \geqslant 2)$, all $\mathcal{M}_i$'s, $\mathcal{N}_j$'s, $i = 1, \ldots, m, j = 1, \ldots, n$, are maximal sets (i.e. maximal cliques). Moreover, any maximal set is of one of the following forms.*

*Type one.* $\mathcal{M} = P\mathcal{M}_1 + A$, *where $P \in GL_m(\mathbb{F}_q)$ and $A \in {\mathbb{F}_q}^{m\times n}$ are fixed.*

*Type two.* $\mathcal{M} = \mathcal{N}_1 Q + A$, *where $Q \in GL_n(\mathbb{F}_q)$ and $A \in {\mathbb{F}_q}^{m\times n}$ are fixed.*

**Remark 2.5.** *When $m < n$ (resp. $m > n$), every maximal set of the type one (resp. type two) is a maximum clique of the matrix graph $G = \left({\mathbb{F}_q}^{m\times n}, \sim\right)$ while every maximal set of type two (resp. type one) is a maximal clique but is not a maximum clique. When $m = n$, every maximal set is a maximum clique.*

In a graph $G = (V, \sim)$, an *independent set* (or *stable set*) of $V$ is a subset $X$ of vertices such that no two of which are adjacent. Let $\alpha(G)$ be the number of vertices in an independent set of maximum cardinality, and it is called the *independence number* (or *stability number*) of $G$.

**Theorem 2.6.** *When $2 \leqslant m \leqslant n$, the independence number of $G = \left(\mathbb{F}_q{}^{m\times n}, \sim\right)$ is*

$$\alpha(G) = q^{n(m-1)}. \tag{2.3}$$

*Moreover, let $L = E_{21} + E_{32} + \cdots + E_{m,m-1} + E_{1m} \in \mathbb{F}_q{}^{m\times m}$, and $P$ be a matrix in $\mathbb{F}_q{}^{n\times n}$ such that $f(x) = \det(xI - P)$ is an irreducible polynomial in $\mathbb{F}_q[x]$. Then the set*

$$\tilde{\mathcal{S}} = \{LA + AP : A \in \mathbb{F}_q{}^{m\times n} \text{ with the last row equal to } 0\} \tag{2.4}$$

*is an independent set of $\mathbb{F}_q{}^{m\times n}$ with $|\tilde{\mathcal{S}}| = q^{n(m-1)}$.*

**Remark 2.7.** *It is well-known that there exists an irreducible polynomial $f(x) \in \mathbb{F}_q[x]$ with* $\deg(f(x)) = n$ *(cf.* [22, Theorem 7.7]*), and there exists a $P \in GL_n(\mathbb{F}_q)$ such that $f(x) =$* $\det(xI - P)$*, for example, let $P$ be the companion matrix of $f(x)$. Moreover, if $f(x) \in \mathbb{F}_q[x]$ is irreducible and $\alpha$ is a root of $f(x)$ in an extension field of $\mathbb{F}_q$, then $\alpha$ is a root of another polynomial $h(x) \in \mathbb{F}_q[x]$ if and only if $f(x)$ divides $h(x)$ (cf.* [14, Lemma 2.12]*).*

*Proof.* Let $\mathcal{S}$ be an independent set of $\mathbb{F}_q{}^{m\times n}$. Note that the first $m-1$ rows (columns) of those matrices in $\mathcal{S}$ must be distinct. Else, there are two matrices $A_1, A_2 \in \mathcal{S}$ such that their first $m-1$ rows are identical so that $A_1 \sim A_2$. Thus, we have $|\mathcal{S}| \leqslant q^{(m-1)n}$. We will prove that

(1) $\tilde{\mathcal{S}}$ has $q^{(m-1)n}$ elements, and (2) $\tilde{\mathcal{S}}$ is an independent set of $\mathbb{F}_q{}^{m\times n}$.

To prove (1), it suffices to show that $LA_1 + A_1P \neq LA_2 + A_2P$ for any two matrices $A_1, A_2 \in \mathbb{F}_q^{m\times n}$ with zero last row. Equivalently, we need to show that $LA + AP \neq 0$ for any nonzero $A \in \mathbb{F}_q^{m\times n}$ with zero last row. Since $\det(xI - P)$ is irreducible, $P$ is invertible. If $LA = -AP$, then the first row of $-AP$ is zero so that the first row of $A$ must be zero. It follows that the second row of $LA$ is zero, and so is that of $-AP$, and hence the second row of $A$ is zero. But then the third row of $LA$ is zero and so is that of $-AP$, and hence the third row of $A$ is zero. Repeating this argument, we see that $A$ is the zero matrix. Thus, assertion (1) holds.

Next, we establish (2). Suppose there are two elements $X, Y \in \tilde{\mathcal{S}}$ such that $\operatorname{rank}(X-Y) = 1$. Then there is $A \in \mathbb{F}_q{}^{m\times n}$ with last row equal to zero such that $X - Y = LA + AP$. Suppose the $k$-th row of $A$ is its first nonzero row, which is denoted by $w$. Then the first $k-1$ rows of $AP$ and the first $k$ rows of $LA$ are zero. Moreover, the first nonzero row of $X - Y$ is its $k$-th row, which is $wP$. Since $\operatorname{rank}(X - Y) = 1$, there exist $u_{k+1}, \ldots, u_m \in \mathbb{F}_q$ such that the $i$-th row of $X - Y$ is $u_i wP$ for $i = k+1, \ldots, m$. Therefore,

$$X - Y = LA + AP = u\,{}^t e_k AP,$$

where $u = {}^t[0, \ldots, 0, u_k, u_{k+1}, \ldots, u_m]$ with $u_k = 1$. Multiplying ${}^tL$ to both sides of the equation, and rearranging, we see that

$$A = {}^tL(u\,{}^t e_k - I_m)AP.$$

Consequently, removing the first $k-1$ zero rows and the last $\ell (\geq 1)$ zero rows of $A$ to get $\tilde{A}$, we have

$$\tilde{A} = \tilde{U}\tilde{A}P,$$

and

$$\tilde{U} = \sum_{j=1}^{N} u_{k+j} E_{j1} - \sum_{j=1}^{N-1} E_{j,j+1} = \begin{bmatrix} u_{k+1} & -1 & 0 & \cdots & 0 \\ u_{k+2} & 0 & -1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ u_{m-\ell} & 0 & 0 & \cdots & -1 \\ u_{m-\ell+1} & 0 & 0 & \cdots & 0 \end{bmatrix} \in {\mathbb{F}_q}^{N\times N}$$

with $N = m-k-\ell+1$. Here we use the fact that $u_k = 1$.

Now by the theory of matrix equation (cf. [7, Lemma 4.3.1]), the linear system $\tilde{A} = \tilde{U}\tilde{A}P$ has a non-trivial solution $\tilde{A} \in {\mathbb{F}_q}^{N\times n}$ implies that

$$I - \tilde{U} \otimes {}^tP = I - \begin{bmatrix} u_{k+1}\,{}^tP & -\,{}^tP & 0 & \cdots & 0 \\ u_{k+2}\,{}^tP & 0 & -\,{}^tP & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ u_{m-\ell}\,{}^tP & 0 & 0 & \cdots & -\,{}^tP \\ u_{m-\ell+1}\,{}^tP & 0 & 0 & \cdots & 0 \end{bmatrix}$$

has determinant 0. Taking Schur complement, we see that

$$0 = \det\left( I_n - u_{k+1}\,{}^tP + [\,{}^tP\ 0\ \cdots\ 0] \begin{bmatrix} I & {}^tP & 0 & \cdots & 0 \\ 0 & I & {}^tP & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & {}^tP \\ 0 & 0 & 0 & \cdots & I \end{bmatrix}^{-1} \begin{bmatrix} u_{k+2}\,{}^tP \\ \vdots \\ \vdots \\ u_{m-\ell+1}\,{}^tP \end{bmatrix} \right).$$

Thus we have

$$0 = \det\left( I - \sum_{j=1}^{N} \beta_j ({}^tP)^j \right)$$

with $\beta_i = (-1)^i u_{k+i}$ for $i = 1, \ldots, N$. Hence, an eigenvalue $\lambda$ of ${}^tP$ in an extension field of $\mathbb{F}_q$, which is a zero of

$$\det(xI - {}^tP) = \det(xI - P) = f(x),$$

is a zero of the polynomial $g(x) = 1 - \sum_{j=1}^{N} \beta_j x^j$ of degree $N = m-k-\ell+1 < n$ (because $m \le n$ and $k, \ell \ge 1$). On the other hand, $f(x)$ is irreducible implies that $f(x)$ divides $g(x)$, which is a contradiction. Then we have proved that $\tilde{\mathcal{S}}$ is an independent set of ${\mathbb{F}_q}^{m\times n}$. $\square$

**Example 2.8.** *Let $f(x) = x^3 + x^2 + 1$ be an irreducible polynomial in $\mathbb{F}_2[x]$ such that $f(x) = \det(xI - P)$, where $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Assume $L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and*

$$\tilde{\mathcal{S}} = \{LA + AP : A \in {\mathbb{F}_2}^{2\times 3} \textit{ with the last row equal to } 0\}.$$

*Then*

$$\tilde{\mathcal{S}} = \left\{0, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right\}$$

*is an independent set of* ${\mathbb{F}_2}^{2\times 3}$.

A *c-coloring* is a partition of the vertices $V = X_1 \cup X_2 \cup \cdots \cup X_c$ such that $X_i \cap X_j = \emptyset$ for all $i \neq j$ and each $X_i$ is an independent set. In such a case, the members of $X_i$, are "painted" with the color $i$ and adjacent vertices will receive different colors. We say that $G$ is *c-colorable*. Let $\chi(G)$ be the smallest possible $c$ for which there exists a $c$-coloring of $G$, which is called the *chromatic number* of $G$. It is well-known that

$$\chi(G) \geqslant \omega(G), \quad \text{and} \quad \chi(G) \geqslant \frac{|V|}{\alpha(G)}. \tag{2.5}$$

The chromatic number and the independence number of a graph are two important quantities. In general, it is difficult to determine the chromatic number or the clique number of a graph. For $G = \left({\mathbb{F}_q}^{m\times n}, \sim\right)$ these quantities can be shown to be equal to $q^{\max\{m,n\}}$.

**Theorem 2.9.** *Let $m, n \geqslant 2$ be integers and $G = \left({\mathbb{F}_q}^{m\times n}, \sim\right)$. Then the chromatic number and the clique number of $G$ are the same, and*

$$\chi(G) = \omega(G) = q^k, \quad \textit{where } k = \max\{m,n\}. \tag{2.6}$$

*Proof.* Without loss of generality, we assume that $2 \leqslant m \leqslant n$ (when $m > n \geqslant 2$, using the transposes of matrices we can transform into the situation $2 \leqslant m < n$). Then $k = \max\{m,n\} = n$. By the geometry of matrices (cf. [9, 11, 20] and Lemma 2.4), every maximum clique of $G$ is of the form $P\mathcal{M}_1 + A$ when $n > m$ and is of the forms $P\mathcal{M}_1 + A$ or $\mathcal{N}_1 Q + A$ when $m = n$, where $P, Q \in GL_m(\mathbb{F}_q)$ and $A \in {\mathbb{F}_q}^{m\times n}$ are fixed. Thus $\omega(G) = |\mathcal{M}_1| = q^n$. By (2.5), we have $\chi(G) \geqslant q^n$.

By Theorem 2.6, let $\mathcal{T} = \left\{T_1, \ldots, T_{q^{n(m-1)}}\right\}$ be an independent set of $G$ which has maximum cardinality. Then $\operatorname{rank}(T_i - T_j) \geqslant 2$ for all $i \neq j$. Let $\mathcal{S}_i = \mathcal{T} + X_i$, where $X_i \in \mathcal{M}_1$, $i = 1, \ldots, q^n$. Then $\mathcal{S}_1, \ldots, \mathcal{S}_{q^n}$ are $q^n$ independent sets of $G$ which has maximum cardinality, and $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$ for all $i \neq j$. Since $|\mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots \cup \mathcal{S}_{q^n}| = q^{mn}$, $\{\mathcal{S}_1, \ldots, \mathcal{S}_{q^n}\}$ is truly a partition of ${\mathbb{F}_q}^{m\times n}$, which means that the map $c : {\mathbb{F}_q}^{m\times n} \to \mathbb{F}_q$, given by $c|_{\mathcal{S}_i} \equiv X_i$, is well defined. Therefore, it is a $q^n$-coloring of $G$. By the definition of chromatic number, we get $\chi(G) \leqslant q^n$. Thus, $\chi(G) = \omega(G) = q^n$. $\square$

## 3 Graph endomorphisms on finite matrix graphs

In this section, we characterize the graph endomorphisms of the matrix graph $G = \left({\mathbb{F}_q}^{m\times n}, \sim\right)$. For finite graphs, a bijective graph homomorphism is a graph *isomorphism*, and a bijective graph endomorphism is a graph *automorphism*. In the language of the geometry of matrices or the preserver problems, we determine the adjacency preserving maps on ${\mathbb{F}_q}^{m\times n}$.

For $G = \left({\mathbb{F}_q}^{m\times n}, \sim\right)$, recall that a *maximal set* is equivalent to a maximal clique; an adjacency preserving map (resp. adjacency preserving bijective map) on ${\mathbb{F}_q}^{m\times n}$ is the same as a graph endomorphism (resp. graph automorphism) on $G$. The following result is useful in our discussion.

**Lemma 3.1.** [6, Lemma 1.4.1] *The chromatic number of a graph $G$ is the smallest integer $r$ such that there is a graph homomorphism from $G$ to $K_r$.*

A finite graph $G$ is called a *core* if every graph homomorphism from $G$ to itself is a graph automorphism. A subgraph $\Gamma$ of finite graph $G$ is called a *core* of $G$ if it is a core and there exists some graph homomorphism $\varphi : G \to \Gamma$. Every finite graph $G$ has a core, which is an induced subgraph and is unique up to isomorphism (cf. [6, Lemma 6.2.2]).

**Theorem 3.2.** *Let $m, n$ be integers $\geqslant 2$ and $G = \left({\mathbb{F}_q}^{m\times n}, \sim\right)$. Then a subgraph of $G$ is a core of $G$ if and only if it is a maximum clique of $G$. In particular, $G$ itself is not a core.*

*Proof.* Let $k = \max\{m, n\}$. Suppose that $\mathcal{M}$ is a maximum clique of $G$. Then clearly $\mathcal{M}$ is a core and $|\mathcal{M}| = \omega(G) = q^k$ (cf. Lemma 2.4). By Theorem 2.9, $\chi(G) = q^k = \omega(G)$. Since $\mathcal{M}$ is a complete subgraph $K_{q^k}$, Lemma 3.1 implies that there is a graph homomorphism $\psi$ from $G$ to $\mathcal{M}$. Thus, $\mathcal{M}$ is a core of $G$, and $G$ is not a core. On the other hand, by [6, Lemma 6.2.2], any two cores of $G$ are graph isomorphic. Hence every core of $G$ is a maximum clique of $G$. □

**Corollary 3.3.** *Let $m, n \geqslant 2$ be integers and $G = \left({\mathbb{F}_q}^{m\times n}, \sim\right)$. If $\mathcal{M}$ is a maximum clique of $G$, then there exists a graph homomorphism (or adjacency preserving map) $\varphi$ from $G$ to $\mathcal{M}$, and $\varphi$ is a proper colouring of $G$, i.e., $\varphi$ partitions ${\mathbb{F}_q}^{m\times n}$ into $|\mathcal{M}|$ classes such that no adjacent vertices are in the same class.*

For $A = [a_{ij}] \in {\mathbb{F}_q}^{m\times n}$ and a map $\sigma : \mathbb{F}_q \to \mathbb{F}_q$, we write $A^\sigma = [a_{ij}^\sigma]$. The Hua's theorem on the geometry of rectangular matrices over a division ring [8, 23, 9] can be viewed as the algebraic description of graph automorphisms of the matrix graph $G = \left({\mathbb{F}_q}^{m\times n}, \sim\right)$:

**Hua's Theorem** *Let $m, n \geqslant 2$ be integers and $G = \left({\mathbb{F}_q}^{m\times n}, \sim\right)$. Then $\varphi$ is a graph automorphism of $G$, i,e., $\varphi : {\mathbb{F}_q}^{m\times n} \to {\mathbb{F}_q}^{m\times n}$ is a bijective map such that* $\operatorname{rank}(X - Y) = 1$ *implies that* $\operatorname{rank}(\varphi(X) - \varphi(Y)) = 1$ *for all $X, Y \in {\mathbb{F}_q}^{m\times n}$, if and only if there is an automorphism $\sigma$ of $\mathbb{F}_q$, invertible matrices $P \in M_m(\mathbb{F}_q)$ and $Q \in M_n(\mathbb{F}_q)$ such that one of the following holds.*

(i) *$\varphi$ is of the form*

$$\varphi(X) = PX^\sigma Q + \varphi(0) \quad \text{for all} \quad X \in {\mathbb{F}_q}^{m\times n}. \tag{3.1}$$

(ii) *$m = n$ and $\varphi$ is of the form*

$$\varphi(X) = P\,{}^tX^\sigma Q + \varphi(0) \quad \text{for all} \quad X \in {\mathbb{F}_q}^{m\times n}. \tag{3.2}$$

For general graph homomorphism, we will prove the following.

**Theorem 3.4.** *Let $m, n \geqslant 2$ be integers and $G = \left(\mathbb{F}_q{}^{m\times n}, \sim\right)$. Then $\varphi$ is a graph endomorphism of $G$ if and only if one of the following holds.*

(a) *The image $\varphi(\mathbb{F}_q{}^{m\times n})$ is a maximum clique of $G$, and $\varphi$ is a $q^k$-coloring where $k = \max\{m, n\}$, i.e., the partition of $\mathbb{F}_q{}^{m\times n}$*

$$\bigcup_{i=1}^{q^k} \left\{\varphi^{-1}[B_i] : B_i \in \varphi(\mathbb{F}_q{}^{m\times n})\right\},$$

*where $\varphi^{-1}[B_i] = \{X \in \mathbb{F}_q{}^{m\times n} : \varphi(X) = B_i\}$ is the inverse image of $B_i$.*

(b) *The image $\varphi(\mathbb{F}_q{}^{m\times n})$ is not a maximum clique of $G$, and $\varphi$ is a graph automorphism.*

In the language of the geometry of matrices, the result can be restated as follows.

**Theorem 3.5.** *Let $m, n \geqslant 2$ be integers and $\varphi : \mathbb{F}_q{}^{m\times n} \to \mathbb{F}_q{}^{m\times n}$ a map. Then $\varphi$ has the property that* $\operatorname{rank}(X-Y) = 1$ *implies* $\operatorname{rank}(\varphi(X)-\varphi(Y)) = 1$ *if and only if one of the following holds.*

(a) *Any two matrices in $\varphi(\mathbb{F}_q{}^{m\times n})$ are adjacent; there is a maximum clique $\mathcal{M}$ of $G$ such that $\varphi(\mathbb{F}_q{}^{m\times n}) = \mathcal{M}$, and $\varphi$ is a proper colouring of $G = \left(\mathbb{F}_q{}^{m\times n}, \sim\right)$.*

(b) *There are two non-adjacent matrices in $\varphi(\mathbb{F}_q{}^{m\times n})$; $\varphi$ is bijective. Moreover, $\varphi$ is of the form* (3.1)*, or $\varphi$ is of the form* (3.2) *provided $m = n$.*

One may see [10, 12] for other formulations of graph isomorphisms between matrix graphs. To prove our main theorem, we establish some auxiliary results which are of independent interest.

**Remark 3.6.** *In [19], the author studied Hua's fundamental theorem of geometry of matrices on EAS division ring. Suppose that $D$ is an EAS division ring such that $D \neq \mathbb{F}_2$ and $D \neq \mathbb{F}_3$, $m, p, q \geqslant n \geqslant 3$. He characterizes the adjacency preserving map $\phi : D^{m\times n} \to D^{p\times q}$ under the assumption that $\phi(0) = 0$ and there exists $A_0$ such that* $\operatorname{rank}\phi(A_0) = n$ *(cf. Theorem 4.2 and Corollary 4.6 of [19]). None of these assumptions is needed in our result. Moreover, the result of [19] does not treat the degenerate case whereas ours give the complete description for matrices over finite fields.*

**Lemma 3.7.** *Let $\mathcal{M}$ and $\mathcal{M}'$ be two distinct type one* (*resp. type two*) *maximal sets of $\mathbb{F}_q{}^{m\times n}$* ($m, n \geqslant 2$) *as defined in Lemma 2.4 such that $\mathcal{M} \cap \mathcal{M}' \neq \emptyset$. Then $\mathcal{M} \cap \mathcal{M}' = \{A\}$ for some $A \in \mathbb{F}_q{}^{m\times n}$ and there is an invertible matrix $P$* (*over $\mathbb{F}_q$*) *such that $\mathcal{M} = P\mathcal{M}_1 + A$ and $\mathcal{M}' = P\mathcal{M}_2 + A$* (*resp. $\mathcal{M} = \mathcal{N}_1 P + A$ and $\mathcal{M}' = \mathcal{N}_2 P + A$*)*.*

*Proof.* Suppose $\mathcal{M}$ and $\mathcal{M}'$ are type one maximal sets and $A \in \mathcal{M} \cap \mathcal{M}'$. Then there exist linearly independent vectors $x_1, x_2 \in \mathbb{F}_q{}^m$ such that

$$\mathcal{M} = \{A + x_1\,{}^t\!y : y \in \mathbb{F}_q{}^n\} \text{ and } \mathcal{M}' = \{A + x_2\,{}^t\!y : y \in \mathbb{F}_q{}^n\}.$$

Extend $\{x_1, x_2\}$ to a basis of ${\mathbb{F}_q}^m$, say, $\{x_1, x_2, \ldots, x_m\}$. Let $P = [x_1\ x_2\ \cdots\ x_m]$. Then

$$\mathcal{M} = P\mathcal{M}_1 + A \text{ and } \mathcal{M}' = P\mathcal{M}_2 + A,$$

which implies $\mathcal{M} \cap \mathcal{M}' = \{A\}$.

If $\mathcal{M}$ and $\mathcal{M}'$ are type two maximal sets, one can get the results by considering the transposes of matrices in ${\mathbb{F}_q}^{m\times n}$. $\square$

**Corollary 3.8.** [9, 11, 20] *Let $A$ and $B$ be two adjacent matrices in ${\mathbb{F}_q}^{m\times n}$ $(m, n \geqslant 2)$. Then there are exactly two maximal sets that contain both $A$ and $B$.*

**Corollary 3.9.** [9, 11, 20] *Let $\mathcal{M}$ and $\mathcal{M}'$ be two distinct maximal sets of the same type in ${\mathbb{F}_q}^{m\times n}$ $(m, n \geqslant 2)$. If $\mathcal{M} \cap \mathcal{M}' \neq \emptyset$, then $|\mathcal{M} \cap \mathcal{M}'| = 1$.*

By Corollary 3.11 and Proposition 3.14 of [20], or by Lemma 3.7 and Corollary 3.9 of [11], we have

**Lemma 3.10.** [11, 20] *Let $\mathcal{M}$ and $\mathcal{M}'$ be two maximal sets of different types in ${\mathbb{F}_q}^{m\times n}$ $(m, n \geqslant 2)$. If $\mathcal{M} \cap \mathcal{M}' \neq \emptyset$, then $|\mathcal{M} \cap \mathcal{M}'| = q$.*

Recall that $d(A, B) = \operatorname{rank}(A - B)$, and we always assume that $2 \leqslant m \leqslant n$ in our discussion unless specified otherwise.

**Lemma 3.11.** *Let $\varphi : {\mathbb{F}_q}^{m\times n} \to {\mathbb{F}_q}^{m\times n}$ be an adjacency preserving map. If $\mathcal{M}$ is a type one maximal set, then $\varphi(\mathcal{M})$ is also a maximal set and $|\varphi(\mathcal{M})| = |\mathcal{M}| = q^n$. If $\mathcal{M}'$ is a type two maximal set, then there exists a unique maximal set containing $\varphi(\mathcal{M}')$. Moreover, we have*

$$d(A, B) \geqslant d(\varphi(A), \varphi(B)) \quad \text{for all } A, B \in {\mathbb{F}_q}^{m\times n}. \tag{3.3}$$

**Lemma 3.12.** *Let $\varphi : {\mathbb{F}_q}^{m\times n} \to {\mathbb{F}_q}^{m\times n}$ be an adjacency preserving map. Let $A \in {\mathbb{F}_q}^{m\times n}$, $P \in GL_m(\mathbb{F}_q)$ and $Q \in GL_n(\mathbb{F}_q)$. Assume that there are distinct numbers $r, s \in \{1, \ldots, n\}$ such that*

$$\varphi(P\mathcal{M}_r Q + A) = \varphi(P\mathcal{M}_s Q + A) \quad \text{or} \quad \varphi(P\mathcal{N}_r Q + A) = \varphi(P\mathcal{N}_s Q + A).$$

*Then there exists a maximum clique $\mathcal{M}$ in ${\mathbb{F}_q}^{m\times n}$ such that for any $P' \in GL_m(\mathbb{F}_q)$ and $Q' \in GL_n(\mathbb{F}_q)$,*

$$\varphi(P'\mathcal{M}_i Q' + A) = \mathcal{M} \quad \text{and} \quad \varphi(P'\mathcal{N}_j Q' + A) \subseteq \mathcal{M} \tag{3.4}$$

*for all $i = 1, \ldots, m$, $j = 1, \ldots, n$. Furthermore, the inclusion in* (3.4) *becomes equality when $m = n$.*

*Proof.* *Case 1.* There are distinct numbers $r, s \in \{1, \ldots, m\}$ such that

$$\varphi(P\mathcal{M}_r Q + A) = \varphi(P\mathcal{M}_s Q + A) =: \mathcal{M}.$$

Recalling $2 \leqslant m \leqslant n$, $\mathcal{M}$ is a maximum clique in ${\mathbb{F}_q}^{m\times n}$, i.e., $\mathcal{M}$ is a maximal set with $|\mathcal{M}| = q^n$. Without loss of generality, we assume $r = 1, s = 2$, $A = 0, P = I_m, Q = I_n$ and

$$\varphi(\mathcal{M}_1) = \mathcal{M} = \varphi(\mathcal{M}_2).$$

For any given $P' \in GL_m(\mathbb{F}_q)$, $Q' \in GL_n(\mathbb{F}_q)$ and $j \in \{1, \ldots, n\}$, since $P'\mathcal{N}_jQ' \cap \mathcal{M}_1 = \{\gamma E_{1j}Q' : \gamma \in \mathbb{F}_q\} =: \mathcal{S}_1$ and $P'\mathcal{N}_jQ' \cap \mathcal{M}_2 = \{\gamma E_{2j}Q' : \gamma \in \mathbb{F}_q\} =: \mathcal{S}_2$, we have

$$\varphi(\mathcal{S}_1) \subseteq \varphi(P'\mathcal{N}_jQ') \cap \varphi(\mathcal{M}_1) \subseteq \varphi(P'\mathcal{N}_jQ') \cap \mathcal{M}$$

and

$$\varphi(\mathcal{S}_2) \subseteq \varphi(P'\mathcal{N}_jQ') \cap \varphi(\mathcal{M}_2) \subseteq \varphi(P'\mathcal{N}_jQ') \cap \mathcal{M}.$$

On the other hand, for any distinct nonzero matrices $X \in \mathcal{S}_1$ and $Y \in \mathcal{S}_2$, we have $\mathrm{rank}(X - Y) = 1$. It follows that $\varphi(X) \neq \varphi(Y)$. Therefore,

$$|\varphi(P'\mathcal{N}_jQ') \cap \mathcal{M}| \geqslant |\varphi(\mathcal{S}_1) \cup \varphi(\mathcal{S}_2)| = 2q - 1 > q. \tag{3.5}$$

Applying Corollary 3.9, Lemma 3.10 and Lemma 3.11 we get

$$\varphi(P'\mathcal{N}_jQ') \subseteq \mathcal{M},\ j = 1, \ldots, n. \tag{3.6}$$

Now replacing the roles of $\{\mathcal{M}_1, \mathcal{M}_2\}$ with $\{\mathcal{N}_1, \mathcal{N}_2\}$ and applying similar arguments as above, we can get $\varphi(P'\mathcal{M}_iQ') = \mathcal{M}$ for all $i = 3, \ldots, m$. Thus (3.4) holds.

*Case 2.* There are distinct numbers $r, s \in \{1, \ldots, n\}$ such that

$$\varphi(P\mathcal{N}_rQ + A) = \varphi(P\mathcal{N}_sQ + A) =: \mathcal{M}'.$$

Then $|\mathcal{M}'| = q^m$ since $\varphi\,|_{P\mathcal{N}_rQ+A}$ is injective. Without loss of generality, we assume $r = 1, s = 2$, $A = 0, P = I_m, Q = I_n$ and

$$\varphi(\mathcal{N}_1) = \mathcal{M}' = \varphi(\mathcal{N}_2).$$

Using the same arguments as (3.5), we can get $|\varphi(P'\mathcal{M}_iQ') \cap \mathcal{M}'| > q$ for all $i = 1, \ldots, m$. Clearly, all $\varphi(P'\mathcal{M}_iQ')$, $i = 1, \ldots, m$, are maximum cliques in ${\mathbb{F}_q}^{m\times n}$. Since $\mathcal{M}'$ is a $q^m$-clique, there is a maximal clique $\mathcal{M}$ such that $\mathcal{M}' \subseteq \mathcal{M}$. Since $|\varphi(P'\mathcal{M}_iQ') \cap \mathcal{M}| > q$, Corollary 3.9 and Lemma 3.10 imply that

$$\varphi(P'\mathcal{M}_iQ') = \mathcal{M},\ i = 1, \ldots, m.$$

Similar to the proof of (3.6), we can prove that $\varphi(P'\mathcal{N}_jQ') \subseteq \mathcal{M}$, $j = 1, \ldots, n$. Hence (3.4) holds.

When $m = n$, by $|\varphi(P'\mathcal{N}_jQ' + A)| = |\mathcal{M}| = q^n$, the inclusion in (3.4) becomes equality. This completes the proof. $\square$

**Theorem 3.13.** *Let $\varphi : {\mathbb{F}_q}^{m\times n} \to {\mathbb{F}_q}^{m\times n}$ $(m, n \geqslant 2)$ be an adjacency preserving map. Suppose that there are two distinct type one maximal sets $\mathcal{M}$ and $\mathcal{M}'$ such that $\mathcal{M} \cap \mathcal{M}' \neq \emptyset$ and $\varphi(\mathcal{M}) = \varphi(\mathcal{M}')$. Then $\varphi({\mathbb{F}_q}^{m\times n})$ is a maximum clique.*

**Remark 3.14.** *By Remark 2.5, when $m < n$ (resp. $m > n$), every maximal set of the type one (resp. type two) is a maximum clique, while every maximal set of type two (resp. type one) is not a maximum clique. When $m = n$, every maximal set is a maximum clique.*

*We only prove the case of $2 \leqslant m \leqslant n$. When $m > n \geqslant 2$, define $\psi : {\mathbb{F}_q}^{n\times m} \to {\mathbb{F}_q}^{n\times m}$ by $\psi(X) = {}^t\varphi({}^tX)$. Then ${}^t\mathcal{M}$ and ${}^t\mathcal{M}'$ are type two maximal sets with nonempty intersection such* *that $\psi({}^t\mathcal{M}) = \psi({}^t\mathcal{M}')$. By Lemma 3.12 and Lemma 3.7, there exist distinct type one maximal sets $\mathcal{P}$ and $\mathcal{P}'$ with nonempty intersection such that $\psi(\mathcal{P}) = \psi(\mathcal{P}')$. By the case $n \geq m \geq 2$ of Theorem 3.13, $\psi({\mathbb{F}_q}^{n\times m})$ is a maximum clique, so $\varphi({\mathbb{F}_q}^{n\times m})$ is a maximum clique.*

*Proof.* By Remark 3.14, we only prove the case of $2 \leqslant m \leqslant n$. From now on, we assume that $2 \leqslant m \leqslant n$. Then $n = \max\{m, n\}$.

Let $\mathcal{M}$ and $\mathcal{M}'$ be two distinct type one maximal sets such that $\mathcal{M} \cap \mathcal{M}' = \{A_1\} \neq \emptyset$ and $\varphi(\mathcal{M}) = \varphi(\mathcal{M}')$. By Lemma 3.7, $\mathcal{M} = P_1\mathcal{M}_1 + A_1$ and $\mathcal{M}' = P_1\mathcal{M}_2 + A_1$, where $P_1$ is invertible. It follows from Lemma 3.11 that $\varphi(\mathcal{M})$ and $\varphi(\mathcal{M}')$ are maximum cliques.

We only consider the case that $\varphi(\mathcal{M})$ is a type one maximal set. Note that we can replace $\varphi(A)$ with ${}^t\varphi(A)$ and a similar argument works when $\varphi(\mathcal{M})$ is a type two maximal set. Then there exists $P_2 \in GL_m(\mathbb{F}_q)$ and $A_2 \in {\mathbb{F}_q}^{m\times n}$ such that $\varphi(\mathcal{M}) = P_2\mathcal{M}_1 + A_2$. Replacing $\varphi$ with the adjacency preserving map

$$X \longmapsto P_2^{-1}\left[\varphi(P_1X + A_1) - A_2\right], \ \forall\ X \in {\mathbb{F}_q}^{m\times n},$$

we have

$$\varphi(\mathcal{M}_1) = \mathcal{M}_1 = \varphi(\mathcal{M}_2). \tag{3.7}$$

Denote by $\mathcal{R}_k = \{A \in {\mathbb{F}_q}^{m\times n} : \text{rank}(A) \leqslant k\}$. By Lemma 3.12, we have $\varphi(P\mathcal{M}_1Q) = \mathcal{M}_1$ and $\varphi(P\mathcal{N}_1Q) \subseteq \mathcal{M}_1$ for any $P \in GL_m(\mathbb{F}_q)$ and $Q \in GL_n(\mathbb{F}_q)$. It follows that

$$\varphi(\mathcal{R}_1) = \mathcal{M}_1. \tag{3.8}$$

We claim that

$$\varphi(\mathcal{R}_2) = \mathcal{M}_1. \tag{3.9}$$

To prove (3.9), it suffices to verify that $\varphi(T) \in \mathcal{M}_1$ for any $T \in \mathcal{R}_2$ with rank $(T) = 2$. Note that $T$ can be written as $T = P(E_{11} + E_{22})Q$ for some nonsingular matrices $P \in {\mathbb{F}_q}^{m\times m}, Q \in {\mathbb{F}_q}^{n\times n}$. Without loss of generality, we assume $P = I_m$ and $Q = I_n$. Then

$$T = E_{11} + E_{22}.$$

Since $\mathcal{M}_1 + E_{11} \subseteq \mathcal{R}_1$ and $\mathcal{N}_1 + E_{11} \subseteq \mathcal{R}_1$, we have

$$\varphi(\mathcal{M}_1 + E_{11}) = \mathcal{M}_1 \quad \text{and} \quad \varphi(\mathcal{N}_1 + E_{11}) \subseteq \mathcal{M}_1. \tag{3.10}$$

For any $i \in \{2, \dots, m\}$, by

$$|\varphi(\mathcal{M}_i + E_{11}) \cap \mathcal{M}_1| \geqslant |\varphi(\mathcal{M}_i + E_{11}) \cap \varphi(\mathcal{N}_1 + E_{11})| \geqslant |(\mathcal{M}_i + E_{11}) \cap (\mathcal{N}_1 + E_{11})| = q$$

we have either

$$\varphi(\mathcal{M}_i + E_{11}) = \mathcal{M}_1 \tag{3.11}$$

or $\varphi(\mathcal{M}_i + E_{11})$ is a type 2 maximal set. Similarly, for $j \in \{2, \dots, n\}$, by

$$|\varphi(\mathcal{N}_j + E_{11}) \cap \varphi(\mathcal{M}_1 + E_{11})| \geqslant |(\mathcal{N}_j + E_{11}) \cap (\mathcal{M}_1 + E_{11})| = q$$

we have either

$$\varphi(\mathcal{N}_j + E_{11}) \subseteq \mathcal{M}_1 \tag{3.12}$$

or $\varphi(\mathcal{N}_j + E_{11})$ is a type 2 maximal set.

*Case 1.* (3.11) or (3.12) holds for some $i \in \{2, \ldots, m\}$ or $j \in \{2, \ldots, n\}$. Then we have

$$\varphi(\mathcal{M}_i + E_{11}) = \mathcal{M}_1 = \varphi(\mathcal{M}_1 + E_{11})$$

or

$$\varphi(\mathcal{N}_j + E_{11}) \subseteq \mathcal{M}_1 \text{ with } \varphi(\mathcal{N}_1 + E_{11}) \subseteq \mathcal{M}_1$$

respectively. If $\varphi(\mathcal{M}_i + E_{11}) = \mathcal{M}_1 = \varphi(\mathcal{M}_1 + E_{11})$, then applying Lemma 3.12 we have $\varphi(\mathcal{M}_2 + E_{11}) = \mathcal{M}_1$. Since $T \in \mathcal{M}_2 + E_{11}$, we get

$$\varphi(T) \in \mathcal{M}_1. \tag{3.13}$$

Now we assume that $\varphi(\mathcal{N}_j + E_{11}) \subseteq \mathcal{M}_1$ with $\varphi(\mathcal{N}_1 + E_{11}) \subseteq \mathcal{M}_1$. Clearly,

$$|(\mathcal{M}_2 + E_{11}) \cap (\mathcal{N}_1 + E_{11})| = |(\mathcal{M}_2 + E_{11}) \cap (\mathcal{N}_j + E_{11})| = q.$$

Therefore, $|(\mathcal{M}_2 + E_{11}) \cap [(\mathcal{N}_1 + E_{11}) \cup (\mathcal{N}_j + E_{11})]| = 2q - 1 > q$. It follows that

$$|\varphi\left((\mathcal{M}_2 + E_{11}) \cap [(\mathcal{N}_1 + E_{11}) \cup (\mathcal{N}_j + E_{11})]\right)| > q.$$

We have

$$\begin{aligned} & \varphi\left((\mathcal{M}_2 + E_{11}) \cap [(\mathcal{N}_1 + E_{11}) \cup (\mathcal{N}_j + E_{11})]\right) \\ \subseteq\ & \varphi(\mathcal{M}_2 + E_{11}) \cap [\varphi(\mathcal{N}_1 + E_{11}) \cup \varphi(\mathcal{N}_j + E_{11})] \\ \subseteq\ & \varphi(\mathcal{M}_2 + E_{11}) \cap \mathcal{M}_1, \end{aligned}$$

thus $|\varphi(\mathcal{M}_2 + E_{11}) \cap \mathcal{M}_1| > q$. Since $\varphi(\mathcal{M}_2 + E_{11})$ is a maximum clique, Corollary 3.9 and Lemma 3.10 imply that $\varphi(\mathcal{M}_2 + E_{11}) = \mathcal{M}_1$. Similarly, we have (3.13).

*Case 2.* $\varphi(\mathcal{M}_i + E_{11})$ and $\varphi(\mathcal{N}_j + E_{11})$ are type 2 maximal sets for all $i \in \{2, \ldots, m\}$ and $j \in \{2, \ldots, n\}$. Then $m = n$. Hence every maximal set is a maximum clique with cardinality $q^n$. Consequently, $\varphi$ maps a maximal set onto a maximal set.

Since $|(\mathcal{M}_i + E_{11}) \cap (\mathcal{N}_j + E_{11})| = q$, we have $|\varphi(\mathcal{M}_i + E_{11}) \cap \varphi(\mathcal{N}_j + E_{11})| \geqslant q$ and hence

$$\varphi(\mathcal{M}_i + E_{11}) = \varphi(\mathcal{N}_j + E_{11}) =: \mathcal{N}, \ \ i, j = 2, \ldots, n. \tag{3.14}$$

Suppose $n \geqslant 3$. Then (3.14) and Lemma 3.12 imply that $\varphi(\mathcal{M}_i + E_{11}) = \mathcal{N}$ for all $i = 1, \ldots, n$. Thus $\varphi(\mathcal{M}_1) = \varphi(\mathcal{M}_1 + E_{11}) = \mathcal{N}$, a contradiction to (3.7). Therefore, we must have $m = n = 2$.

Note that $\mathcal{N}_2 + E_{12}, \mathcal{M}_1 + E_{12} \subseteq \mathcal{R}_1$. It follows from (3.8) that

$$\varphi(\mathcal{N}_2 + E_{12}) = \varphi(\mathcal{M}_1 + E_{12}) = \varphi(\mathcal{M}_1 + E_{11} + E_{12}) = \mathcal{M}_1. \tag{3.15}$$

Since $|(\mathcal{M}_2 + E_{11} + E_{12}) \cap \mathcal{R}_1| = q$ and $|(\mathcal{M}_2 + E_{11} + E_{12}) \cap (\mathcal{N}_2 + E_{11})| = q$, we get

$$\varphi(\mathcal{M}_2 + E_{11} + E_{12}) = \mathcal{M}_1 \tag{3.16}$$

or

$$\varphi(\mathcal{M}_2 + E_{11} + E_{12}) = \mathcal{N}. \tag{3.17}$$

If (3.16) holds, then by (3.15) and Lemma 3.12 we get $\varphi(\mathcal{N}_2 + E_{11} + E_{12}) = \mathcal{M}_1$. It follows that $\varphi(T) \in \mathcal{M}_1$ since $T \in \mathcal{N}_2 + E_{11} + E_{12}$.

From now on we suppose that (3.17) holds. We prove that there is a contradiction as follows.

Since $|(\mathcal{N}_1 + E_{11} + E_{12}) \cap (\mathcal{M}_i + E_{11} + E_{12})| = q$ for $i = 1, 2$, we have

$$\varphi(\mathcal{N}_1 + E_{11} + E_{12}) = \mathcal{M}_1 \tag{3.18}$$

or

$$\varphi(\mathcal{N}_1 + E_{11} + E_{12}) = \mathcal{N}. \tag{3.19}$$

If (3.19) holds, then $\varphi(\mathcal{N}_2 + E_{11} + E_{12}) = \varphi(\mathcal{N}_2 + E_{11}) = \mathcal{N}$ and Lemma 3.12 will lead to $\varphi(\mathcal{M}_1 + E_{11} + E_{12}) = \mathcal{N}$, which contradicts with (3.15). Hence we must have

$$\varphi(\mathcal{N}_1 + E_{12}) = \varphi(\mathcal{N}_1 + E_{11} + E_{12}) = \mathcal{M}_1.$$

Thus by (3.15) and Lemma 3.12, we get

$$\varphi(\mathcal{M}_2 + E_{12}) = \mathcal{M}_1. \tag{3.20}$$

By $\mathcal{N}_1 \subseteq \mathcal{R}_1$ and (3.8), $\varphi(\mathcal{N}_1) = \mathcal{M}_1$. Since $\varphi(\mathcal{M}_2 + E_{11}) = \mathcal{N}$, $|\varphi(\mathcal{M}_2 + E_{11}) \cap \varphi(\mathcal{N}_1)| = |\mathcal{N} \cap \mathcal{M}_1| = q$. Choose $R$ and invertible $Q_1$ such that $\mathcal{N} = \mathcal{N}_1 Q_1 + R$. Without loss of generality, we assume $Q_1 = I_2$. Then

$$R = \begin{bmatrix} u & r \\ 0 & 0 \end{bmatrix},$$

where $0 \neq r \in \mathbb{F}_q$ (by (3.8)). Hence,

$$\mathcal{N} = \mathcal{N}_1 + R = \left\{ \begin{bmatrix} x & r \\ y & 0 \end{bmatrix} : x, y \in \mathbb{F}_q \right\}.$$

Since $|(\mathcal{M}_2 + E_{11}) \cap \mathcal{N}_1| = |\mathcal{N} \cap \mathcal{M}_1| = q$ and $|\varphi(\mathcal{M}_2 + E_{11})| = |\mathcal{N}| = q^2$, we have

$$\varphi((\mathcal{M}_2 + E_{11}) \cap \mathcal{N}_1) = \mathcal{N} \cap \mathcal{M}_1 = (\mathcal{N}_1 + R) \cap \mathcal{M}_1 = \mathcal{N}_1 \cap \mathcal{M}_1 + R. \tag{3.21}$$

Thus, we can suppose that

$$\varphi \begin{bmatrix} 1 & 0 \\ x & 0 \end{bmatrix} = \begin{bmatrix} x^\tau & r \\ 0 & 0 \end{bmatrix} \text{ for all } x \in \mathbb{F}_q, \tag{3.22}$$

where $\tau : \mathbb{F}_q \to \mathbb{F}_q$ is a bijection. Then by $\varphi(\mathcal{M}_2 + E_{11}) = \mathcal{N}$ and (3.22), it is easy to see that

$$\varphi \begin{bmatrix} 1 & 0 \\ x & y \end{bmatrix} = \begin{bmatrix} x^* & r \\ y^* & 0 \end{bmatrix} \text{ for all } x, y \in \mathbb{F}_q. \text{ Moreover, } y \neq 0 \Leftrightarrow y^* \neq 0. \tag{3.23}$$

We assert that

$$\varphi\begin{bmatrix} 0 & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} s^\mu & r \\ 0 & 0 \end{bmatrix} \text{ for all } s \in \mathbb{F}_q, \tag{3.24}$$

where $\mu : \mathbb{F}_q \to \mathbb{F}_q$ is a bijective map. In fact, by (3.8) we can assume that $\varphi\begin{bmatrix} 0 & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} s^\mu & s^\eta \\ 0 & 0 \end{bmatrix}$ for all $s \in \mathbb{F}_q$, where $\mu$ and $\eta$ are some functions on $\mathbb{F}_q$. By (3.23) we get $\varphi\begin{bmatrix} 1 & 0 \\ s+1 & -1 \end{bmatrix} = \begin{bmatrix} (s+1)^* & r \\ (-1)^* & 0 \end{bmatrix}$ where $(-1)^* \neq 0$. Since $\begin{bmatrix} 0 & 1 \\ 0 & s \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ s+1 & -1 \end{bmatrix}$, we have $\begin{bmatrix} s^\mu & s^\eta \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} (s+1)^* & r \\ (-1)^* & 0 \end{bmatrix}$. Hence $s^\eta = r$ and $\mu$ is bijective. Thus (3.24) holds.

For any $x, y \in \mathbb{F}_q$, by (3.20) and (3.24) we can assume that

$$\varphi\begin{bmatrix} 0 & 1 \\ x & y \end{bmatrix} = \begin{bmatrix} \nu & z \\ 0 & 0 \end{bmatrix}. \text{ Moreover, } x \neq 0 \Leftrightarrow z \neq r. \tag{3.25}$$

By (3.23) and (3.25), we can let $\varphi\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & r \\ b & 0 \end{bmatrix}$ where $b \neq 0$, and $\varphi\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} c & z \\ 0 & 0 \end{bmatrix}$ where $z \neq r$. Then $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, but $\begin{bmatrix} a & r \\ b & 0 \end{bmatrix}$ and $\begin{bmatrix} c & z \\ 0 & 0 \end{bmatrix}$ are not adjacent, a contradiction. Therefore, (3.17) cannot happen. Thus we always have (3.13). Then we have proved (3.9).

Now we can use a simple induction on $k$ to verify

$$\varphi(\mathcal{R}_k) = \mathcal{M}_1 \tag{3.26}$$

for $3 \leqslant k \leqslant \min\{m, n\}$. Assume (3.26) holds for $k-1$ with $k \geqslant 3$. Then for any rank $k$ matrix $X \in {\mathbb{F}_q}^{m\times n}$, there exist nonsingular matrices $P, Q$ such that $X = P(E_{11} + \cdots + E_{kk})Q$. Since $P\mathcal{M}_iQ + X - PE_{kk}Q \subseteq \mathcal{R}_{k-1}$ for $i = 1, 2$, we have $\varphi(P\mathcal{M}_iQ + X - PE_{kk}Q) = \mathcal{M}_1$ for $i = 1, 2$. Applying Lemma 3.12 we have $\varphi(P\mathcal{M}_kQ + X - PE_{kk}Q) = \mathcal{M}_1$. It follows that $\varphi(X) \in \mathcal{M}_1$ since $X \in P\mathcal{M}_kQ + X - PE_{kk}Q$. Hence $\varphi({\mathbb{F}_q}^{m\times n})$ is a maximum clique. □

**Lemma 3.15.** *Let $\varphi : {\mathbb{F}_q}^{m\times n} \to {\mathbb{F}_q}^{m\times n}$ be an adjacency preserving map such that $\varphi({\mathbb{F}_q}^{m\times n})$ is not a maximum clique. Assume that $\mathcal{M}, \mathcal{M}'$ are two distinct type one maximal sets with $\mathcal{M} \cap \mathcal{M}' \neq \emptyset$. Then $\varphi(\mathcal{M}), \varphi(\mathcal{M}')$ are two distinct maximal sets of the same type which are maximum cliques.*

*Proof.* Let $\mathcal{M}$ and $\mathcal{M}'$ be two distinct type one maximal sets with $\mathcal{M} \cap \mathcal{M}' \neq \emptyset$. Then $\mathcal{M} = \{x_0\,{}^t\!y + A : y \in {\mathbb{F}_q}^n\}$ and $\mathcal{M}' = \{x_1\,{}^t\!y + A : y \in {\mathbb{F}_q}^n\}$ with $x_0, x_1 \in {\mathbb{F}_q}^m$ being linearly independent and $A \in {\mathbb{F}_q}^{m\times n}$. Let $\mathcal{M}'' = \{x_2\,{}^t\!y + A : y \in {\mathbb{F}_q}^n\}$, where $x_2 \in {\mathbb{F}_q}^m$ and $x_0, x_1, x_2$

are pairwise linearly independent. Then $\mathcal{M}'' \neq \mathcal{M}$, $\mathcal{M}'' \neq \mathcal{M}'$ and $\mathcal{M} \cap \mathcal{M}' \cap \mathcal{M}'' = \{A\} \neq \emptyset$. It follows from Lemma 3.11 and Theorem 3.13 that $\varphi(\mathcal{M}), \varphi(\mathcal{M}')$ and $\varphi(\mathcal{M}'')$ are three distinct maximal sets. Clearly, there are two of them being of the same type. Without loss of generality we assume that $\varphi(\mathcal{M}')$ and $\varphi(\mathcal{M}'')$ are of the same type. We prove that $\varphi(\mathcal{M}), \varphi(\mathcal{M}')$ and $\varphi(\mathcal{M}'')$ are of the same type as follows.

*Case 1.* $\varphi(\mathcal{M}')$ and $\varphi(\mathcal{M}'')$ are type one. Assume $\mathcal{N}^{(1)} = \{x\,{}^t\!y_1 + A : x \in {\mathbb{F}_q}^m\}$ and $\mathcal{N}^{(2)} = \{x\,{}^t\!y_2 + A : x \in {\mathbb{F}_q}^m\}$ with $y_1, y_2 \in {\mathbb{F}_q}^n$ being linearly independent. Then $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$ are type two maximal sets. Suppose $i \in \{1,2\}$. By Lemma 3.10 we have $|\mathcal{N}^{(i)} \cap \mathcal{M}| = |\mathcal{N}^{(i)} \cap \mathcal{M}'| = |\mathcal{N}^{(i)} \cap \mathcal{M}''| = q$. Let ${\mathcal{N}^{(i)}}^*$ be a maximal set containing $\varphi(\mathcal{N}^{(i)})$. It is easy to see that $|{\mathcal{N}^{(i)}}^* \cap \varphi(\mathcal{M})| \geqslant q$, $|{\mathcal{N}^{(i)}}^* \cap \varphi(\mathcal{M}')| \geqslant q$ and $|{\mathcal{N}^{(i)}}^* \cap \varphi(\mathcal{M}'')| \geqslant q$. Since $\varphi(\mathcal{M}') \neq \varphi(\mathcal{M}'')$, by Corollary 3.9, ${\mathcal{N}^{(1)}}^*$ and ${\mathcal{N}^{(2)}}^*$ must be type two. Since both $\mathcal{N}^{(i)}$ and ${\mathcal{N}^{(i)}}^*$ are type two maximal sets, we get $|\mathcal{N}^{(i)}| = |{\mathcal{N}^{(i)}}^*|$, $i = 1, 2$. Since $\varphi$ is an adjacency preserving map and $\varphi(\mathcal{N}^{(i)}) \subseteq {\mathcal{N}^{(i)}}^*$, we have $|\varphi(\mathcal{N}^{(i)})| = |\mathcal{N}^{(i)}| = |{\mathcal{N}^{(i)}}^*|$, and hence $\varphi(\mathcal{N}^{(i)}) = {\mathcal{N}^{(i)}}^*$, $i = 1, 2$. Now by Lemma 3.12 and Theorem 3.13, we must have ${\mathcal{N}^{(1)}}^* \neq {\mathcal{N}^{(2)}}^*$. Otherwise, ${\mathcal{N}^{(1)}}^* = {\mathcal{N}^{(2)}}^*$ implies that $\varphi({\mathbb{F}_q}^{m\times n})$ is a maximum clique, a contradiction to the conditions. By $|{\mathcal{N}^{(i)}}^* \cap \varphi(\mathcal{M})| \geqslant q$ and Corollary 3.9, $\varphi(\mathcal{M})$ must be type one. Therefore, $\varphi(\mathcal{M}), \varphi(\mathcal{M}')$ and $\varphi(\mathcal{M}'')$ are of the same type.

*Case 2.* $\varphi(\mathcal{M}')$ and $\varphi(\mathcal{M}'')$ are type two. Then Lemma 3.11 implies that $m = n$. Let $\psi(X) = {}^t\varphi(X)$. Then $\psi$ is also an adjacency preserving map from ${\mathbb{F}_q}^{n\times n}$ to itself such that $\psi({\mathbb{F}_q}^{n\times n})$ is not a maximum clique. Since $\psi(\mathcal{M}')$ and $\psi(\mathcal{M}'')$ are type one, by Case 1, $\psi(\mathcal{M}), \psi(\mathcal{M}')$ and $\psi(\mathcal{M}'')$ are of the same type. Consequently, $\varphi(\mathcal{M}), \varphi(\mathcal{M}')$ and $\varphi(\mathcal{M}'')$ are of the same type. $\square$

**Lemma 3.16.** *Let $\varphi : {\mathbb{F}_q}^{m\times n} \to {\mathbb{F}_q}^{m\times n}$ be an adjacency preserving map such that $\varphi({\mathbb{F}_q}^{m\times n})$ is not a maximum clique. Then*

$$\varphi(A) \neq \varphi(B) \quad \textit{for any} \quad A, B \in {\mathbb{F}_q}^{m\times n} \textit{ with } d(A,B) = 2. \tag{3.27}$$

*Proof.* Without loss of generality we assume that $\varphi(0) = 0$.

We first claim that $\varphi(X) \neq 0$ for all $X \in {\mathbb{F}_q}^{m\times n}$ with $d(X, 0) = 2$. To prove the claim, notice that there exists an invertible matrix $P$ such that $X = P\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix}$ with linearly independent $\alpha_1$ and $\alpha_2$ in ${\mathbb{F}_q}^n$. Let $\mathcal{M} = \left\{ P\begin{bmatrix} x + \alpha_1 \\ -x + \alpha_2 \\ 0 \end{bmatrix} : x \in {\mathbb{F}_q}^n \right\}$. Then $\mathcal{M}$ is a type one maximal set containing $X$ and $\mathcal{M} \cap P\mathcal{M}_1 \neq \emptyset$. By Lemma 3.15, $\varphi(\mathcal{M})$ and $\varphi(P\mathcal{M}_1)$ are two distinct maximal sets of the same type such that their cardinality is $q^n$, and $\varphi(\mathcal{M}) \cap \varphi(P\mathcal{M}_1) \neq \emptyset$. Let $C = P\begin{bmatrix} \alpha_2 + \alpha_1 \\ 0 \\ 0 \end{bmatrix}$. Since $\mathcal{M} \cap P\mathcal{M}_1 = \{C\}$ and $X \sim C$, $\varphi(X) \sim \varphi(C) \sim \varphi(0) = 0$. If $\varphi(X) = 0$, then $\varphi(\mathcal{M}) \cap \varphi(P\mathcal{M}_1) = \{\varphi(C), \varphi(X)\}$, which contradicts to Corollary 3.9. Thus, $\varphi(X) \neq 0$ and the claim holds.

Now, let $A, B \in \mathbb{F}_q{}^{m\times n}$ with $d(A,B) = 2$. Define $\psi(X) = \varphi(X + A) - \varphi(A)$ for all $X \in \mathbb{F}_q{}^{m\times n}$. Then $\psi : \mathbb{F}_q{}^{m\times n} \to \mathbb{F}_q{}^{m\times n}$ is also an adjacency preserving map such that $\psi(\mathbb{F}_q{}^{m\times n})$ is not any maximum clique and $\psi(0) = 0$. By the above claim, $\varphi(B) - \varphi(A) = \psi(B - A) \neq 0$. Thus (3.27) holds. $\square$

For $2 \leqslant k \leqslant m$, we let

$$\mathcal{L}_k = \left\{ \begin{bmatrix} X \\ 0 \end{bmatrix} : X \in \mathbb{F}_q{}^{k\times n} \text{ and } 0 \in \mathbb{F}_q{}^{(m-k)\times n} \right\}. \tag{3.28}$$

**Lemma 3.17.** *Let $\varphi : \mathbb{F}_q{}^{m\times n} \to \mathbb{F}_q{}^{m\times n}$ be an adjacency preserving map such that $\varphi(\mathbb{F}_q{}^{m\times n})$ is not a maximum clique and $\varphi(0) = 0$. Then*

(i) *When $m \neq n$, there is a $P \in GL_m(\mathbb{F}_q)$ such that*

$$\varphi(\mathcal{M}_i) = P\mathcal{M}_i, \ \ i = 1, 2, \tag{3.29}$$

*and*

$$\varphi(\mathcal{L}_2) = P\mathcal{L}_2. \tag{3.30}$$

(ii) *When $m = n$, either $\varphi$ or the map $X \mapsto {}^t\varphi(X)$ satisfies* (3.29)-(3.30).

*Moreover, the map $\varphi\,|_{\mathcal{L}_2}: \mathcal{L}_2 \longrightarrow P\mathcal{L}_2$ or the map ${}^t\varphi\,|_{\mathcal{L}_2}: \mathcal{L}_2 \longrightarrow P\mathcal{L}_2$ is bijective.*

*Proof.* By Lemmas 3.7 and 3.15, when $m \neq n$, there is $P \in GL_m(\mathbb{F}_q)$ such that (3.29) holds. When $m = n$, either $\varphi$ or the map $X \mapsto {}^t\varphi(X)$ satisfies (3.29). Without loss of generality, we assume the former case holds. It remains to show that $\varphi(\mathcal{L}_2) = P\mathcal{L}_2$. Clearly, $\varphi(\mathcal{M}_1 \cup \mathcal{M}_2) \subseteq P\mathcal{L}_2$.

We first claim that for any type one maximal set $\mathcal{M}$, if $\mathcal{M} \cap P\mathcal{L}_2$ contains at least two elements, then $\mathcal{M} \subseteq P\mathcal{L}_2$. To see this, by Lemma 2.4,

$$\mathcal{M} = P'\mathcal{M}_1 + A = \left\{ P \begin{bmatrix} p_1x + \alpha_1 \\ p_2x + \alpha_2 \\ \vdots \\ p_mx + \alpha_m \end{bmatrix} : x \in \mathbb{F}_q{}^n \right\},$$

for some $p_k \in \mathbb{F}_q$, $\alpha_k \in \mathbb{F}_q{}^n$ and invertible $P' \in \mathbb{F}_q{}^{m\times m}$. Suppose there are two distinct elements in $\mathcal{M} \cap P\mathcal{L}_2$. Then there exist two distinct $x_1, x_2 \in \mathbb{F}_q{}^n$ such that

$$\begin{bmatrix} p_3x_1 + \alpha_3 \\ \vdots \\ p_mx_1 + \alpha_m \end{bmatrix} = \begin{bmatrix} p_3x_2 + \alpha_3 \\ \vdots \\ p_mx_2 + \alpha_m \end{bmatrix} = 0.$$

It follows that $p_3 = \cdots = p_m = 0$ and $\alpha_3 = \cdots = \alpha_m = 0$. Therefore, $\mathcal{M} \subseteq P\mathcal{L}_2$.

Now let $A = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix} \in \mathcal{L}_2$ with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. We prove $\varphi(A) \in P\mathcal{L}_2$ as follows.

*Case 1.* Suppose $\operatorname{rank}(A) = 2$. Set

$$\mathcal{M} = \left\{ \begin{bmatrix} x + \alpha_1 \\ -x + \alpha_2 \\ 0 \end{bmatrix} : x \in \mathbb{F}_q{}^n \right\}.$$

Then $\mathcal{M}$ is a type one maximal set containing $A$ and $|\mathcal{M} \cap \mathcal{M}_i| = 1$, $i = 1, 2$. By Lemma 3.15, Corollary 3.9 and (3.29), $\varphi(\mathcal{M})$ is a type one maximal set and $|\varphi(\mathcal{M}) \cap P\mathcal{M}_i| = 1$, $i = 1, 2$. Suppose $\varphi(\mathcal{M}) \cap P\mathcal{M}_i = \{B_i\}$ with $i = 1, 2$. Since $0 \notin \varphi(\mathcal{M})$ and $P\mathcal{M}_1 \cap P\mathcal{M}_2 = \{0\}$, $B_1$ and $B_2$ are two distinct elements in $\varphi(\mathcal{M}) \cap P\mathcal{L}_2$ and by the claim, $\varphi(\mathcal{M}) \subseteq P\mathcal{L}_2$. Thus, $\varphi(A) \in P\mathcal{L}_2$.

*Case 2.* Suppose $\operatorname{rank}(A) = 1$. Set $\mathcal{M} = \mathcal{M}_1 + A$. Then $\mathcal{M}$ is a type one maximal set containing $A$ and $|\mathcal{M} \cap \mathcal{M}_2| = 1$. By Lemma 3.15, Corollary 3.9 and (3.29), $\varphi(\mathcal{M})$ is a type one maximal set and $|\varphi(\mathcal{M}) \cap P\mathcal{M}_2| = 1$. Notice that $\mathcal{M}$ contains at least two distinct rank two matrices, say $A_1$ and $A_2$. By Case 1, $\varphi(A_1)$ and $\varphi(A_2)$ are in $P\mathcal{L}_2$. So $\varphi(\mathcal{M}) \cap P\mathcal{L}_2$ contains at least two distinct elements. By the claim, $\varphi(\mathcal{M}) \subseteq P\mathcal{L}_2$, and hence, $\varphi(A) \in P\mathcal{L}_2$.

From the two cases, we conclude that $\varphi(\mathcal{L}_2) \subseteq P\mathcal{L}_2$. By Lemma 3.16, the map $\varphi \mid_{\mathcal{L}_2}: \mathcal{L}_2 \longrightarrow P\mathcal{L}_2$ is injective, thus $|\mathcal{L}_2| = |P\mathcal{L}_2|$ implies that it is bijective. Then $\varphi(\mathcal{L}_2) = P\mathcal{L}_2$. □

**Lemma 3.18.** *Let $3 \leqslant m \leqslant n$ and $\varphi : \mathbb{F}_q{}^{m\times n} \to \mathbb{F}_q{}^{m\times n}$ be an adjacency preserving map such that $\varphi(\mathbb{F}_q{}^{m\times n})$ is not a maximum clique and $\varphi(0) = 0$. For any $i$ with $1 \leq i < m$, if $\varphi(\mathcal{M}_i) = \mathcal{M}_i$ and $\varphi(\mathcal{M}_{i+1}) = \mathcal{M}_{i+1}$, then*

$$\varphi(\mathcal{M}_i + \mathcal{M}_{i+1}) = \mathcal{M}_i + \mathcal{M}_{i+1}, \tag{3.31}$$

*where $\mathcal{M}_i + \mathcal{M}_{i+1}$ denotes the set $\{A + B : A \in \mathcal{M}_i, B \in \mathcal{M}_{i+1}\}$. Moreover, the restricted map $\varphi|_{\mathcal{M}_i+\mathcal{M}_{i+1}}$ is an adjacency preserving bijective map.*

*Proof.* Clearly, $\mathcal{M}_1 + \mathcal{M}_2 = \mathcal{L}_2$. For $2 \leqslant i \leqslant m - 1$, we have

$$\mathcal{M}_i + \mathcal{M}_{i+1} = \left\{ \begin{bmatrix} 0_{i-1} \\ X \\ 0 \end{bmatrix} : X \in \mathbb{F}_q{}^{2\times n} \right\},$$

where $0_{i-1}$ is the $(i-1) \times n$ zero matrix. There exists a permutation matrix $Q$ such that $Q\,{}^tQ = I_m$ and $\mathcal{M}_i = Q\mathcal{M}_1$, $\mathcal{M}_{i+1} = Q\mathcal{M}_2$ and $\mathcal{M}_i + \mathcal{M}_{i+1} = Q\mathcal{L}_2$. Let $\psi(X) = Q^{-1}\varphi(QX)$ for all $X \in \mathbb{F}_q{}^{m\times n}$. Then $\psi$ is also an adjacency preserving map from $\mathbb{F}_q{}^{m\times n}$ to itself such that $\psi(\mathbb{F}_q{}^{m\times n})$ is not any maximum clique and $\psi(0) = 0$. Moreover, we have $\psi(\mathcal{M}_1) = \mathcal{M}_1$, $\psi(\mathcal{M}_2) = \mathcal{M}_2$ and $\psi(\mathcal{L}_2) = \mathcal{L}_2$.

Applying similar arguments as in the proof of Lemma 3.17, we can prove that $\psi(\mathcal{L}_2) = \mathcal{L}_2$ and the restricted map $\psi|_{\mathcal{L}_2} : \mathcal{L}_2 \longrightarrow \mathcal{L}_2$ is bijective. It follows that (3.31) holds and the restricted map $\varphi|_{\mathcal{M}_i+\mathcal{M}_{i+1}}$ is an adjacency preserving bijective map. □

We are now ready to present the proof of our main result.

**Proof of Theorem 3.5.** Without loss of generality, we assume that $2 \leqslant m \leqslant n$ (by Remark 3.14, when $m > n \geqslant 2$, using the transposes of matrices we can transform into the situation $2 \leqslant m < n$). Then $n = \max\{m, n\}$.

If (a) or (b) holds, then clearly $\varphi$ preserves adjacency.

Conversely, suppose $\varphi$ preserves adjacency. Suppose (a) does not hold. Then $\varphi({\mathbb{F}_q}^{m\times n})$ is not a maximum clique. Replacing the map $\varphi$ by $X \mapsto \varphi(X) - \varphi(0)$, we have $\varphi(0) = 0$. By Lemma 3.17, and further replacing the map $\varphi$ by $X \mapsto {}^t\varphi(X)$, if necessary, we can assume that there is $P \in GL_m(\mathbb{F}_q)$ such that

$$\varphi(\mathcal{M}_i) = P\mathcal{M}_i,\ i = 1, 2, \tag{3.32}$$

$$\varphi(\mathcal{L}_2) = P\mathcal{L}_2. \tag{3.33}$$

Moreover, the map $\varphi\ |_{\mathcal{L}_2}\colon \mathcal{L}_2 \longrightarrow P\mathcal{L}_2$ is an adjacency preserving bijective map. Thus, by Hua's Theorem, we have that

$$\varphi\begin{bmatrix} X \\ 0 \end{bmatrix} = P\begin{bmatrix} P_1X^\sigma Q_1 \\ 0 \end{bmatrix} \quad \text{for all } X \in {\mathbb{F}_q}^{2\times n}, \tag{3.34}$$

where $P_1 \in GL_2(\mathbb{F}_q)$, $Q_1 \in GL_n(\mathbb{F}_q)$, and $\sigma$ is an automorphism of $\mathbb{F}_q$.

Modify the adjacency preserving map $\varphi(X)$ by $X \mapsto \left[(P_1^{-1} \oplus I_{m-2})P^{-1}\varphi(X)Q_1^{-1}\right]^{\sigma^{-1}}$. We obtain

$$\varphi\begin{bmatrix} X \\ 0 \end{bmatrix} = \begin{bmatrix} X \\ 0 \end{bmatrix} \tag{3.35}$$

for all $X \in {\mathbb{F}_q}^{2\times n}$. If $m = 2$, then this proof ends. From now on we assume $m > 2$.

For any $A = \left[\begin{smallmatrix} x_1 \\ 0 \end{smallmatrix}\right] \in \mathcal{M}_1$ with ${}^tx_1 \in {\mathbb{F}_q}^n$, let $\psi_A(X) = \varphi(X+A) - \varphi(A)$ for all $X \in {\mathbb{F}_q}^{m\times n}$. Then $\psi_A$ is an adjacency preserving map such that $\psi_A(0) = 0$. By (3.35), it is clear that

$$\psi_A\begin{bmatrix} X \\ 0 \end{bmatrix} = \begin{bmatrix} X \\ 0 \end{bmatrix}, \text{ for all } X \in {\mathbb{F}_q}^{2\times n}. \tag{3.36}$$

In particular, $\psi_A(\mathcal{M}_i) = \mathcal{M}_i$, $i = 1, 2$. By (3.36) and Corollary 3.8, we have

$$\psi_A(\mathcal{N}_1) = \mathcal{N}_1. \tag{3.37}$$

By (3.37), let $\psi_A(E_{31}) = \sum_{i=1}^m a_{i1}E_{i1}$. We show $\sum_{i=3}^m a_{i1}E_{i1} \neq 0$. Otherwise, $\psi_A(E_{31}) = a_{11}E_{11} + a_{21}E_{21}$, but $E_{31} \sim a_{11}E_{11} + a_{21}E_{21}$ and hence (3.36) implies that $\psi_A(E_{31}) \sim a_{11}E_{11} + a_{21}E_{21} = \psi_A(E_{31})$, a contradiction. Applying appropriate elementary transformation of rows of matrix, we can assume that (3.35)-(3.37) hold and $\psi_A(E_{31}) = E_{31}$. By Corollary 3.8, there are two and only two maximal sets containing $E_{31}$ and 0, they are $\mathcal{N}_1$ and $\mathcal{M}_3$. It follows from (3.37) and Lemma 3.15 that

$$\psi_A(\mathcal{M}_3) = \mathcal{M}_3. \tag{3.38}$$

Therefore, by Lemma 3.18, we have

$$\psi_A(\mathcal{M}_2 + \mathcal{M}_3) = \mathcal{M}_2 + \mathcal{M}_3. \tag{3.39}$$

Moreover, the induced map from (3.39) is an adjacency preserving bijective map. Using Hua's Theorem, one has

$$\psi_A \begin{bmatrix} 0_1 \\ X \\ 0 \end{bmatrix} = \begin{bmatrix} 0_1 \\ P_A X^{\sigma_A} Q_A \\ 0 \end{bmatrix}, \text{ for all } X \in {\mathbb{F}_q}^{2\times n}, \tag{3.40}$$

where $0_1$ is the $1 \times n$ zero matrix, $P_A \in GL_2(\mathbb{F}_q)$, $Q_A \in GL_n(\mathbb{F}_q)$, and $\sigma_A$ is an automorphism of $\mathbb{F}_q$. Denote

$$P_A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

For any $X = \begin{bmatrix} y \\ z \end{bmatrix} \in {\mathbb{F}_q}^{2\times n}$, we have $P_A X^{\sigma_A} Q_A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y^{\sigma_A} Q_A \\ z^{\sigma_A} Q_A \end{bmatrix}$. When $z = 0$, it follows from (3.36) that

$$P_A X^{\sigma_A} Q_A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y^{\sigma_A} Q_A \\ 0 \end{bmatrix} = \begin{bmatrix} a y^{\sigma_A} Q_A \\ c y^{\sigma_A} Q_A \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix}$$

for all ${}^t y \in {\mathbb{F}_q}^n$. Therefore, we get $c = 0$ and $y^{\sigma_A} Q_A = a^{-1} y$ for all ${}^t y \in {\mathbb{F}_q}^n$. Choosing $y = 0$, we have

$$P_A X^{\sigma_A} Q_A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 \\ z^{\sigma_A} Q_A \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 \\ a^{-1} z \end{bmatrix} = \begin{bmatrix} b a^{-1} z \\ d a^{-1} z \end{bmatrix}$$

for all ${}^t z \in {\mathbb{F}_q}^n$. It follows from (3.38) that $b = 0$ and $d \neq 0$. Let $f_A = d a^{-1}$. Then

$$P_A X^{\sigma_A} Q_A = \begin{bmatrix} y \\ f_A z \end{bmatrix}, \text{ for all } X = \begin{bmatrix} y \\ z \end{bmatrix} \in {\mathbb{F}_q}^{2\times n}.$$

Next we deduce that $f_A = f_B$ for any distinct $A = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, B = \begin{bmatrix} x_2 \\ 0 \end{bmatrix} \in \mathcal{M}_1$ with ${}^t x_1, {}^t x_2 \in {\mathbb{F}_q}^n$. Recalling the definition of $\psi_A$, for any ${}^t y, {}^t z \in {\mathbb{F}_q}^n$, we have

$$\varphi \begin{bmatrix} x_1 \\ y \\ z \\ 0 \end{bmatrix} = \psi_A \begin{bmatrix} 0 \\ y \\ z \\ 0 \end{bmatrix} + \varphi(A) = \begin{bmatrix} x_1 \\ y \\ f_A z \\ 0 \end{bmatrix}, \quad \varphi \begin{bmatrix} x_2 \\ y \\ z \\ 0 \end{bmatrix} = \psi_B \begin{bmatrix} 0 \\ y \\ z \\ 0 \end{bmatrix} + \varphi(B) = \begin{bmatrix} x_2 \\ y \\ f_B z \\ 0 \end{bmatrix}.$$

Since

$$\operatorname{rank} \left( \varphi \begin{bmatrix} x_1 \\ y \\ z \\ 0 \end{bmatrix} - \varphi \begin{bmatrix} x_2 \\ y \\ z \\ 0 \end{bmatrix} \right) = \operatorname{rank} \left( \begin{bmatrix} x_1 \\ y \\ f_A z \\ 0 \end{bmatrix} - \begin{bmatrix} x_2 \\ y \\ f_B z \\ 0 \end{bmatrix} \right) = 1$$

for all ${}^tz \in {\mathbb{F}_q}^n$, we must have $f_A = f_B$ when $x_1 \neq x_2$. Replacing the map $\varphi(Z)$ with $(I_2 \oplus f_A^{-1} \oplus I_{m-3})\varphi(Z)$ if necessary, we can assume $f_A = 1$. Therefore, we obtain that

$$\varphi \begin{bmatrix} X \\ 0 \end{bmatrix} = \begin{bmatrix} X \\ 0 \end{bmatrix} \text{ for all } X \in {\mathbb{F}_q}^{3\times n}. \tag{3.41}$$

Inductively, assume that $4 \leqslant k \leqslant m$ and $\varphi \begin{bmatrix} X \\ 0 \end{bmatrix} = \begin{bmatrix} X \\ 0 \end{bmatrix}$ for all $X \in {\mathbb{F}_q}^{(k-1)\times n}$. By the method above and (3.3), we can prove that $\varphi \begin{bmatrix} X \\ 0 \end{bmatrix} = \begin{bmatrix} X \\ 0 \end{bmatrix}$ for all $X \in {\mathbb{F}_q}^{k\times n}$. Therefore, we can prove $\varphi(X) = X$ for all $X \in {\mathbb{F}_q}^{m\times n}$. The proof is complete. $\square$

**Acknowledgment**

The research of L.P. Huang was supported by a National Natural Science Foundation of China (Project 11371072) and a Scientific Research Fund of Hunan Provincial Education Department (Grant No. 10A002). Li and Sze were partially supported by an Hong Kong RGC grant PolyU 502411; this grant supported the post-doctoral fellowship of Z. Huang at the Hong Kong Polytechnic University. Li was also supported by a USA NSF grant; he was a visiting professor of the University of Hong Kong in the spring of 2012, an honorary professor of Taiyuan University of Technology (100 Talent Program scholar), and an honorary professor of the Shanghai University. Finally, the authors wish to thank the referee for valuable comments and suggestions and bring our attention to references [2, 14, 15, 19].

# References

[1] J. A. Bondy, U. S. R. Murty, Graph Theory, Springer, Springer, Berlin, 2008.

[2] A. E. Brouwer, A. M. Cohen, A. Neumaier, Distance-regular graphs. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 18. Springer-Verlag, Berlin, 1989.

[3] A. E. Brouwer, J. H. Koolen, The vertex-connectivity of a distance-regular graph, European Journal of Combinatorics, 30 (2009) 668-673.

[4] G. Chartrand, P. Zhang, Chromatic Graph Theory, Taylor & Francis Group, Boca Raton, London, New York, 2009.

[5] R. Diestel, Graph Theory, Seconbd Edition, Sringer-Verlag Heidelberg, New York, 2000.

[6] C. Godsil and G. Royle, Algebtaic Graph Theory, Springer, New York, 2001.

[7] R. A. Horn, C. R. Johnson, Topics in matrix analysis. Cambridge University Press, Cambridge, 1991.

[8] L. K. Hua, A theorem on matrices over a sfield and its applications, Acta Math. Sinica 1 (1951) 109-163.

[9] L. P. Huang, Geometry of Matrices over Ring, Beijing, Science Press, 2006.

[10] L. P. Huang, Good distance graphs and the geometry of matrices, Linear Algebra Appl. 433 (2010) 221-232.

[11] L. P. Huang, Extending Hua's theorem on the geometry of matrices to Bezout domains, Linear Algebra Appl., 436 (2012) 2446-2473.

[12] L. P. Huang, X. Xiao, Good distance graph on rectangular matrices over a Bezout domain, Proceedings of the 9th International Conference on Matrix Theory and Its Applications, Volume 2, Edited by E.-X. Jiang, pp. 243-246, World Academic Union (World Academic Press), 2010.

[13] W. L. Huang, Z. X. Wan, Adjacency preserving mappings of rectangular matrices, Beiträge zur Algebra und Geometrie (Contributions to Algebra and Geometry) 45(2) (2004) 435-446.

[14] R. Lidl, H. Niederreiter, Finite Fields (foreword by P.M. Cohn), Second edition, Cambridge University Press, Cambridge, 1997.

[15] D. Marušič, Hamiltonian circuits in Cayley graphs. Discrete Math. 46 (1983), no. 1, 49-54.

[16] M. Orel, Adjacency preservers, symmetric matrices, and cores, J. Algebr Comb., 35 (2011) 633-647.

[17] M. Orel, A note on adjacency preservers on hermitian matrices over finite fields, Finite Fields and Their Applications 15 (2009) 441-449.

[18] P. Šemrl, On Hua's fundamental theorem of the geometry of matrices, J. Algebra 272 (2004) 801-837.

[19] P. Šemrl, The optimal version of Hua's fundamental theorem of geometry of matrices, Mem. Amer. Math. Soc., to appear.

[20] Z. X. Wan, Geometry of Matrices, World Scientific, Singapore, 1996.

[21] Z. X. Wan, Geometry of Classical Groups over Finite Fields, Second Edition, Beijing, New York, Science Press, 2002.

[22] Z. X. Wan, Lectures on Finite Fields and Galois Rings, World Scientific, New Jersey, London, Singapore, Hong Kong, 2003.

[23] Z. X. Wan, Design Theory, Higher Education Press, Beijing, 2009.

[24] Y. X. Wang, Y. J. Huo and C. L. Ma, Association Schemes of Matrices, Science Press, Beijing, 2006. (in Chinese.)