NUMERICAL RANGE OF LIE PRODUCT OF OPERATORS

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Abstract. Denote by $W(A)$ the numerical range of a bounded linear operator A, and $[A,B]=AB-BA$ the Lie product of two operators A and B. Let H,K be complex Hilbert spaces of dimension ≥ 2 and $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ be a map whose range contains all operators of rank ≤ 1. It is shown that Φ satisfies that $W([\Phi(A),\Phi(B)])=W([A,B])$ for any $A,B\in\mathcal{B}(H)$ if and only if $\dim H=\dim K$, there exist $\varepsilon\in\{1,-1\}$, a functional $h:\mathcal{B}(H)\to\mathbb{C}$, a unitary operator $U\in\mathcal{B}(H,K)$, and a set $\mathcal{S}$ of operators in $\mathcal{B}(H)$, that consists of operators of the form $aP+bI$ for an orthogonal projection P on H if the dimension of H is at least 3, such that

$$\Phi(A)=\begin{cases}\varepsilon UAU^*+h(A)I & \text{if } A\in\mathcal{B}(H)\setminus\mathcal{S},\\ -\varepsilon UAU^*+h(A)I & \text{if } A\in\mathcal{S},\end{cases}$$

or

$$\Phi(A)=\begin{cases}i\varepsilon UA^tU^*+h(A)I & \text{if } A\in\mathcal{B}(H)\setminus\mathcal{S},\\ -i\varepsilon UA^tU^*+h(A)I & \text{if } A\in\mathcal{S},\end{cases}$$

where A^t is the transpose of A with respect to an orthonormal basis of H. The proof of this result depends on the classifications of operators A or operator pairs A,B with some symmetric properties of $W([A,B])$ that are of independent interest.

1. Introduction

Let A be a bounded linear operator acting on a complex Hilbert space H. The numerical range of A is the set $W(A)=\{\langle Ax,x\rangle : x\in H, \|x\|=1\}$, and the numerical radius of A is $w(A)=\sup\{|\lambda| : \lambda\in W(A)\}$. The numerical range and numerical radius are useful tools in studying matrices and operators, and have applications in many different areas; for example, see [10, 11, 13].

There has been considerable interest in studying maps on operators leaving invariant the numerical range or the numerical radius of different kinds of product $A\circ B$ of operators A and B. Such maps are called numerical range preservers and numerical radius preservers, respectively. Very often, such maps have the form

$$A\mapsto \xi UAU^* \text{ or } \quad A\mapsto \xi UA^tU^*, \tag{1.1}$$

for some complex unit ξ and unitary U.

Early results related to the study can be found in [19]. Here we describe some recent development. In [16], surjective maps Φ on the algebra $\mathcal{B}(H)$ of all bounded linear operators

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acting on H satisfying, respectively, $W(\Phi(A)\Phi(B)) = W(AB)$ and $W(\Phi(B)\Phi(A)\Phi(B)) = W(BAB)$ were characterized. In [8], a characterization was obtained of the numerical range preserver for Jordan product $A \circ B = AB + BA$ of operators. In [1], the authors gave a characterization of maps on matrix algebras which preserve the numerical radius of the usual product $A \circ B = AB$ of matrices, and in [4], a characterization was obtained for maps on standard operator algebras $\mathcal{A}$ on a Hilbert space H (including $\mathcal{B}(H)$) which preserve the numerical radius of operator products. Maps preserving the numerical radius of Jordan semi-triple products $A \circ B = ABA$ of matrices were characterized in [6]. The same problem on $\mathcal{B}_s(H)$, the real Jordan algebra of all self-adjoint operators in $\mathcal{B}(H)$, was solved in [12]. Moreover, the problem for the case of indefinite skew products of operators was obtained in [14]. In [21] the authors characterized maps preserving the numerical range of generalized products of operators between standard operator algebras on Hilbert space H. The maps on self-adjoint operators preserving the numerical range of ξ-Lie product of operators $[A, B]_\xi = AB - \xi BA$ were characterized in [7] when $\xi \neq 1$. In [5], maps preserving the numerical range of skew Lie products of operators $A \circ B = AB - BA^*$ on $\mathcal{B}(H)$ and additive maps preserving the numerical radius of skew Lie products on factor von Neumann algebras are characterized. In [20], the authors obtained a characterization of maps preserving the norm of Lie products of matrices on $n \times n$ matrix algebra $M_n(\mathbb{C})$ with $n \geq 3$, where the norm is any unitary invariant norm. In particular, their result implies the following.

Theorem LPS. *Let $n \geq 3$ and $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a surjective map. Then*

$$w([\phi(A), \phi(B)]) = w([A, B]) \quad \textit{for all } A, B \in M_n(\mathbb{C})$$

if and only if there exists a unitary matrix $U \in M_n(\mathbb{C})$ such that

$$\phi(A) = \mu_A U A^\dagger U^* + \nu_A I \quad \textit{for all } A \in M_n(\mathbb{C}),$$

where $\mu_A, \nu_A \in \mathbb{C}$ depend on A with $|\mu_A| = 1$, $A^\dagger = A, \bar{A}, A^t$ or A^*.

Their proof in [20] based on a result (Theorem PS below) in [24] on commutativity preserving maps on matrices. Denote by $\mathcal{C}_n$ the subset of all $n \times n$ complex matrices $A \in M_n(\mathbb{C})$ with the property that all Jordan cells in the Jordan canonical form of A are of the size 1×1 or 2×2. A map $\theta : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is called a regular locally polynomial map if for any A there exists a polynomial p_A such that $\theta(A) = p_A(A)$ and $\{p_A(A)\}' = \{A\}'$, where $\{A\}'$ stands for the commutant of A.

Theorem PS. *Let $n \geq 3$ and let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a bijective map preserving commutativity in both directions. Then there exist an invertible matrix $T \in M_n(\mathbb{C})$, an automorphism τ of $\mathbb{C}$ and a regular locally polynomial map $A \mapsto p_A(A)$ such that either $\phi(A) = Tp_A(A)T^{-1}$ for all $A \in \mathcal{C}_n$ or $\phi(A) = Tp_A(A^t_\tau)T^{-1}$ for all $A \in \mathcal{C}_n$.*

In this paper, we will characterize maps between algebras of operators on complex Hilbert spaces of dimensions ≥ 2 that preserve the numerical range (or the closure of the numerical range) of Lie product. Note that, in [8, 16, 21], to characterize the maps that preserve the numerical range of AB or $AB + BA$ or the generalized product of operators, one gets information about the maps by taking the trace of the products whenever the products are of finite rank. However, for Lie product, this technique does not work anymore because $\mathrm{Tr}([A,B]) = 0$ whenever $[A,B]$ is of finite rank.

Denote by $\overline{W}(A)$ the closure of the numerical range of A. The following is our main result.

Theorem 1.1. *Let H, K be complex Hilbert spaces of dimension ≥ 2 and $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a map of which the range contains all operators of rank ≤ 1. Then the following statements are equivalent.*

(1) *Φ satisfies that $W([\Phi(A), \Phi(B)]) = W([A,B])$ for any $A, B \in \mathcal{B}(H)$.*

(2) *Φ satisfies that $\overline{W}([\Phi(A), \Phi(B)]) = \overline{W}([A,B])$ for any $A, B \in \mathcal{B}(H)$.*

(3) *$\dim H = \dim K$, and there exist $\varepsilon \in \{1, -1\}$, a functional $h : \mathcal{B}(H) \to \mathbb{C}$, a unitary operator $U \in \mathcal{B}(H, K)$, and a set $\mathcal{S}$ of operators in $\mathcal{B}(H)$, which consists of operators of the form $aP + bI$ for an orthogonal projection P on H if $\dim H \geq 3$, such that either*

$$\Phi(A) = \begin{cases} \varepsilon UAU^* + h(A)I & \text{if } A \in \mathcal{B}(H) \setminus \mathcal{S}, \\ -\varepsilon UAU^* + h(A)I & \text{if } A \in \mathcal{S}, \end{cases}$$

or

$$\Phi(A) = \begin{cases} i\varepsilon UA^tU^* + h(A)I & \text{if } A \in \mathcal{B}(H) \setminus \mathcal{S}, \\ -i\varepsilon UA^tU^* + h(A)I & \text{if } A \in \mathcal{S}, \end{cases}$$

where A^t is the transpose of A with respect to an orthonormal basis of H.

At a first glance, for $\dim H \geq 3$, one may think that the finite dimensional case of Theorem 1.1 is an easy consequence of Theorem LPS above. However, much effort is needed to determine the structure of the functional $A \mapsto \mu_A$ to arrive at the conclusion on ε in our theorem. When $\dim H = 2$, the choice of the set $\mathcal{S}$ is less restrictive and our proof requires a different treatment. Furthermore, it is worth pointing out that a difficulty in proving Theorem 1.1 comes from the lack of a corresponding result as Theorem PS in the infinite dimensional case. Therefore, alternative approach is needed.

Note also that the assumption "the range of Φ contains all operators of rank ≤ 1" in Theorem 1.1 cannot be removed simply. For example, if H is infinite dimensional, we can identify H with $H \oplus H$ and define a map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H \oplus H)$ by $\Phi(A) = A \oplus A$ for all $A \in \mathcal{B}(H)$. Then Φ preserves the numerical range of Lie products as $W(A) = W(A \oplus A)$ for all A. However, Φ is not of the form in Theorem 1.1. It would be interesting to see the above assumption can be removed in the finite dimensional case and the structure of Φ still can be characterized.

If we assume that the map Φ is continuous (under any operator topology such as the norm topology, strong operator topology and weak operator topology), then we get the following.

Corollary 1.2. *Let H, K be complex Hilbert spaces of dimension ≥ 2 and $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a continuous map of the range containing all operators of rank ≤ 1. Then the following statements are equivalent.*

(1) *Φ satisfies that $W([\Phi(A), \Phi(B)]) = W([A,B])$ for any $A, B \in \mathcal{B}(H)$.*

(2) *Φ satisfies that $\overline{W}([\Phi(A), \Phi(B)]) = \overline{W}([A,B])$ for any $A, B \in \mathcal{B}(H)$.*

(3) *There exist a unitary operator* $U \in \mathcal{B}(H,K)$, *a scalar* $\varepsilon \in \{-1,1\}$ *and a continuous functional* $h : \mathcal{B}(H) \to \mathbb{C}$ *such that either*

$$\Phi(A) = \varepsilon UAU^* + h(A)I \quad \text{for all} \quad A \in \mathcal{B}(H);$$

or

$$\Phi(A) = i\varepsilon UA^tU^* + h(A)I \quad \text{for all} \quad A \in \mathcal{B}(H).$$

The proof of the main theorem depends on the classifications of operators A or operator pairs A, B with some symmetric properties of $W([A,B])$ that are of independent interest. These results will be presented in Section 2. The proof of the main theorem will be given in Section 3.

2. Operators with symmetry on the numerical range of their Lie products

In this section, we obtain classifications of operators or operator pairs with symmetry on the numerical range of their Lie products. The results are useful in the proof of the main theorem and are of independent interest. We begin with the following lemma; see [2, 3, 9, 28].

Lemma 2.1. *For any complex matrix* $M \in M_n(\mathbb{C})$, *if the boundary of its numerical range* $\partial W(M)$ *contains an elliptic arc of length* > 0, *then the elliptic disc is contained in* $W(M)$ *and the two foci of the ellipse are eigenvalues of* M. *Particularly, if* $W(M)$ *is an elliptic disc, then its foci are eigenvalues of* M*; if* $W(M)$ *is a circular disc centered at 0, then 0 is an eigenvalue of* M *of multiplicity* ≥ 2 *with its geometric multiplicity strictly less than its algebraic multiplicity.*

Denote by $\mathcal{P} = \mathcal{P}(H)$ the set of all operators of projections P in $\mathcal{B}(H)$ and $\mathcal{P}_1 = \mathcal{P}_1(H)$ be the set of all rank-1 projections in $\mathcal{P}(H)$. Let $\mathcal{F}_k(H)$ be the set of all bounded linear operators of rank $\leq k$. The following proposition gives a characterization of projections perturbed by a scalar, that is, the quadratic normal operators, in terms of the numerical range of Lie product. Note that, if $\dim H = 2$, for every pair of operators $T, S \in \mathcal{B}(H)$, $W([T,S])$ is an elliptic disc centered at 0 (may be degenerate) and thus we always have $W([T,S]) = -W([T,S])$. So the following proposition does not hold for the case $\dim H = 2$.

Proposition 2.2. *Let* H *be a complex Hilbert space with* $\dim H \geq 3$ *and* $A \in \mathcal{B}(H)$. *Then the following conditions are equivalent.*

(1) $A \in \mathbb{C}\mathcal{P}(H) + \mathbb{C}I$, *i.e.,* $A = \alpha P + \beta I$ *for some* $\alpha, \beta \in \mathbb{C}$ *and* $P^* = P = P^2 \in \mathcal{B}(H)$.

(2) *For any* $B \in \mathcal{B}(H)$, $W([A,B]) = -W([A,B])$.

(3) *For any* $B \in \mathcal{F}_1(H)$, $W([A,B]) = -W([A,B])$.

Proof. (1)⇒(2).

Let $A \in \mathbb{C}\mathcal{P}(H) + \mathbb{C}I$. As the case $A = \alpha I$ is obvious, we may assume that, there exists a space decomposition $H = H_1 \oplus H_2$ such that $A = \begin{pmatrix} \alpha I_{H_1} & 0 \\ 0 & \beta I_{H_2} \end{pmatrix}$ with $\dim H_i > 0,\ i = 1, 2$, and $\alpha \neq \beta$. For any $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \in \mathcal{B}(H_1 \oplus H_2)$, $[A,B] = (\alpha - \beta) \begin{pmatrix} 0 & B_{12} \\ -B_{21} & 0 \end{pmatrix}$.

Let $U = \begin{pmatrix} I_{H_1} & 0 \\ 0 & -I_{H_2} \end{pmatrix}$; then U is unitary and $U[A,B]U^* = -[A,B]$. So, we always have $W([A,B]) = -W([A,B])$, that is, (2) is true.

(2)⇒(3) is obvious.

(3)⇒(1). We consider two cases.

Case 1. A is quadratic, that is, there exist scalars α, β such that

$$(A - \alpha I)(A - \beta I) = 0.$$

If $A = \alpha I$ or $A = \beta I$, (1) holds. So, in the sequel we assume that $A \notin \mathbb{C}I$.

If A is not normal, that is, not a combination of some projection and the identity I, then there exists a space decomposition $H = H_1 \oplus H_2 \oplus H_3 \oplus H_4$ with $\dim H_1 = \dim H_2 \neq 0$ such that

$$A = \begin{pmatrix} \alpha I_{H_1} & D \\ 0 & \beta I_{H_2} \end{pmatrix} \oplus \alpha I_{H_3} \oplus \beta I_{H_4},$$

where $D \in \mathcal{B}(H_2, H_1)$ is injective and has dense range; see [26]. Under unitary similarity we may assume that $H_1 = H_2 = H_0$ and $D > 0$, that is, $\langle Dx, x\rangle > 0$ for all nonzero $x \in H_0$. If D has an eigenvalue d_{11} with unit eigenvector x, take $d_{21} = 0$; if D has no eigenvalue, for any $\varepsilon > 0$, take a unit vector $x = x_\varepsilon$ so that $\|D\| < \sqrt{\langle Dx, x\rangle^2 + \varepsilon^2}$. Take a unit vector y so that $x \perp y$ and $Dx = d_{11}x + d_{21}y$ with $d_{11} > 0$ and $d_{21} > 0$. Then $d_{11}^2 + d_{21}^2 \leq \|D\|^2 < d_{11}^2 + \varepsilon^2$ and $d_{21} < \varepsilon$. Thus D can be written as $D = \begin{pmatrix} d_{11} & d_{21} & 0 \\ d_{21} & d_{22} & D_{23} \\ 0 & D_{23}^* & D_{33} \end{pmatrix}$ and

$$A = \begin{pmatrix} \alpha & 0 & 0 & d_{11} & d_{21} & 0 \\ 0 & \alpha & 0 & d_{21} & d_{22} & D_{23} \\ 0 & 0 & \alpha I & 0 & D_{23}^* & D_{33} \\ 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta I \end{pmatrix}. \tag{2.1}$$

Let

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.2}$$

It follows that

$$X=[A,B]=\begin{pmatrix} d_{11} & 0 & 0 & 0 & 0 & 0\\ d_{21} & 0 & 0 & -d_{11} & -d_{21} & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ \beta-\alpha & 0 & 0 & -d_{11} & -d_{21} & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus

$$X+X^*=G_1+G_2=\begin{pmatrix} 2d_{11} & 0 & 0 & \bar\beta-\bar\alpha & 0 & 0\\ 0 & 0 & 0 & -d_{11} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ \beta-\alpha & -d_{11} & 0 & -2d_{11} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}+d_{21}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & -1 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $d_{21}=0$, $X+X^*=G_1$ is a rank-3 self-adjoint operator of trace 0 and so $W(X+X^*)\neq -W(X+X^*)$.

Assume $d_{21}\neq 0$. Solving the characteristic equation, we have

$$\sigma(G_1)=\{\lambda_1=2\sqrt[3]{r}\cos\theta,\ \lambda_2=2\sqrt[3]{r}\cos(\theta+\frac{2\pi}{3}),\ \lambda_3=2\sqrt[3]{r}\cos(\theta+\frac{4\pi}{3})\},$$

where $r=\sqrt{(\frac{5d_{11}^2+|\alpha-\beta|^2}{3})^3}>0$ and $\theta=\frac{1}{3}\arccos(-\frac{d_{11}^3}{r})$. As $0<\frac{d_{11}^3}{r}<1$, we have $\frac{\pi}{6}<\theta<\frac{\pi}{3}$, which implies that $\lambda_1>0$, $\lambda_2<0$, $\lambda_3>0$ and $\lambda_1+\lambda_2+\lambda_3=0$. So $W(G_1)=[\lambda_2,\max\{\lambda_1,\lambda_3\}]$ is not symmetric to 0. Furthermore,

$$|\lambda_2|>\max\{\lambda_1,\lambda_3\}\geq\lambda_1>2\sqrt[3]{r}\cos\frac{\pi}{3}>\sqrt{\frac{5}{3}}d_{11}. \tag{2.3}$$

Note that $W(G_2)=[-\frac{1}{2}(3+\sqrt{5})d_{21},\frac{1}{2}(3+\sqrt{5})d_{21}]$. Since $d_{11}>\|D\|-\varepsilon$ and $0\leq d_{21}<\varepsilon$, by Eq.(2.3), we see that

$$W(G_2)\subset[-\sqrt{\frac{5}{3}}d_{11},\sqrt{\frac{5}{3}}d_{11}]\subset W(G_1)$$

for sufficient small $\varepsilon>0$ and hence, $W(X+X^*)\neq -W(X+X^*)$.

Thus, we have proved that $W([A,B])\neq -W([A,B])$. As a result, if A is an quadratic operator, then (3) implies that A is of the form $A=\alpha I_{H_1}\oplus\beta I_{H_2}$, that is, A is a normal quadratic operator and (1) holds.

Case 2. A is not a quadratic operator. We have to show that A does not satisfy (3).

By Kaplansky's Theorem [17], there is $x\in H$ such that $[x,Ax,A^2x]$ =span$\{x,Ax,A^2x\}$ has dimension 3. Take an orthonormal basis $\{e_1,e_2,e_3\}$ of $[x,Ax,A^2x]$ with $e_1\in[x]$ and $e_2\in[x,Ax]$. Then, with respect to the space decomposition $H=[e_1]\oplus[e_2]\oplus[e_3]\oplus\{e_1,e_2,e_3\}^\perp$, A has the matrix representation of the form

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & A_{14}\\ a_{21} & a_{22} & a_{23} & A_{24}\\ 0 & a_{32} & a_{33} & A_{34}\\ 0 & 0 & A_{43} & A_{44} \end{pmatrix}$$

with $a_{21} > 0$, and $a_{32} > 0$.

If $A_{24} \neq 0$, then there exists unit vector $e_4 \in \{e_1, e_2, e_3\}^\perp$ and positive number a_{24} such that $A_{24} = a_{24} e_2 \otimes e_4$. With respect to the space decomposition $H = [e_1] \oplus [e_2] \oplus [e_3] \oplus [e_4] \oplus \{e_1, \ldots, e_4\}^\perp$, we have

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & A_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 \\ 0 & a_{32} & a_{33} & a_{34} & A_{35} \\ 0 & 0 & a_{43} & a_{44} & A_{45} \\ 0 & 0 & A_{53} & A_{54} & A_{55} \end{pmatrix}.$$

Let

$$B = e_1 \otimes e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

so that

$$[A, B] = \begin{pmatrix} -a_{21} & a_{11} - a_{22} & -a_{23} & -a_{24} & 0 \\ 0 & a_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$[A, B] + [A, B]^* = \begin{pmatrix} -2a_{21} & a_{11} - a_{22} & -a_{23} & -a_{24} & 0 \\ \bar{a_{11}} - \bar{a_{22}} & 2a_{21} & 0 & 0 & 0 \\ -\bar{a_{23}} & 0 & 0 & 0 & 0 \\ -\bar{a_{24}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is of rank three, and hence $W([A, B]) \neq -W([A, B])$.

If $A_{24} = 0$, then, taking the same B as above gives

$$[A, B] + [A, B]^* = \begin{pmatrix} -2a_{21} & a_{11} - a_{22} & -a_{23} & 0 & 0 \\ \bar{a_{11}} - \bar{a_{22}} & 2a_{21} & 0 & 0 & 0 \\ -\bar{a_{23}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which is of rank three and $W([A, B]) \neq -W([A, B])$ if $a_{23} \neq 0$.

So we may assume further that $a_{23} = 0$ and then A may be written in the form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & A_{15} \\ a_{21} & a_{22} & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 \\ 0 & 0 & a_{43} & a_{44} & A_{45} \\ 0 & 0 & A_{53} & A_{54} & A_{55} \end{pmatrix}.$$

In this case one picks

$$B = (e_1 + e_2) \otimes e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

so that

$$[A, B] = \begin{pmatrix} -a_{21} & a_{11} + a_{12} - a_{22} & 0 & 0 & 0 \\ -a_{21} & a_{21} & 0 & 0 & 0 \\ 0 & a_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$[A, B] + [A, B]^* = \begin{pmatrix} -2a_{21} & a_{11} + a_{12} - a_{22} - a_{21} & 0 & 0 & 0 \\ \bar{a}_{11} + \bar{a}_{12} - \bar{a}_{22} - a_{21} & 2a_{21} & a_{32} & 0 & 0 \\ 0 & a_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is of rank three as $a_{21} > 0$ and $a_{32} > 0$, and consequently, we have again $W([A, B]) \neq -W([A, B])$.

This completes the proof of the implication (3)⇒(1). □

The following result gives a characterization of combinations of rank-1 projections and I in terms of the numerical range of Lie product.

Proposition 2.3. *Let H be a complex Hilbert space with $\dim H \geq 3$ and $A \in \mathcal{B}(H)$. Then the following conditions are equivalent.*

(1) *$A \in \mathbb{C}\mathcal{P}_1(H) + \mathbb{C}I$, i.e., $A = \alpha x \otimes x + \beta I$ for some $\alpha, \beta \in \mathbb{C}$ and a unit vector $x \in H$.*

(2) *For any $B \in \mathcal{B}(H)$, $W([A, B])$ is an elliptic disc centered at 0.*

(3) *For any $B \in \mathcal{F}_1(H)$, $W([A, B])$ is an elliptic disc centered at 0.*

Proof. (1)⇒(2). Assume $A = \alpha x \otimes x + \beta I$ with $x \in H$ a unit vector and $\alpha, \beta \in \mathbb{C}$. Then, for any $B \in \mathcal{B}(H)$, $[A, B] = \alpha(x \otimes B^* x - Bx \otimes x)$. It follows that $[A, B]$ can be represented as

$$[A, B] = \begin{pmatrix} 0 & a & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{pmatrix} \oplus 0$$

for some nonnegative numbers a, b, c. Clearly, $abc = 0$ implies that $W([A, B])$ is an ellipse centered at the origin 0 (including the degeneration case). Assume $abc \neq 0$.

Let $X = \begin{pmatrix} 0 & a & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{pmatrix}$. Clearly

$$\sigma(X) = \{\lambda_1, \lambda_2, \lambda_3\} = \{0, -\sqrt{ab}, \sqrt{ab}\},$$

$$d = \text{Tr}(X^*X) - \sum_{j=1}^{3} |\lambda_j|^2 = a^2 + b^2 + c^2 - 2ab = (a-b)^2 + c^2 > 0$$

and

$$\lambda = \text{Tr}(X) + \frac{1}{d}(\sum_{j=1}^{3} |\lambda_j|^2 \lambda_j - \text{Tr}(X^*X^2)) = 0 + \frac{1}{d}(0+0) = 0,$$

which coincides with the eigenvalue $\lambda_1 = 0$ of X. By Theorem 2.3 and 2.4 of Ref [18], $W(X)$ is an elliptic disc with foci $\{-\sqrt{ab}, \sqrt{ab}\}$, which is centered at 0. So, (2) is true.

(2)⇒(3) is clear.

(3)⇒(1). Applying Proposition 2.2, we see that $A = \alpha P + \lambda I$ for some projection P and scalars α, λ. Without loss of generality, we may assume that $\alpha \neq 0$ and $P \neq 0, I$. Assume, on the contrary, that $A \notin \mathbb{C}\mathcal{P}_1 + \mathbb{C}I$; then $\dim H \geq 4$ and there exists a space decomposition $H = H_1 \oplus H_2 \oplus H_3$ such that

$$A = A_1 \oplus \alpha I_{H_2} \oplus \beta I_{H_3},$$

where $\alpha \neq \beta$ and

$$A_1 = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix} = \begin{pmatrix} \alpha I_2 & 0_2 \\ 0_2 & \beta I_2 \end{pmatrix}.$$

Let $B = B_1 \oplus 0 \oplus 0$ with

$$B_1 = \sqrt{2}(\alpha - \beta)^{-1} \begin{pmatrix} -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ -1 & 1 & 2 & 0 \end{pmatrix};$$

we see that $\text{rank}(B) = 1$,

$$[A_1, B_1] = \sqrt{2} \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}.$$

Notice that

$$\sigma([A_1, B_1]) = \{-2, 0, 0, 2\},$$
$$\sigma(\text{Re}([A_1, B_1])) = \{-(\sqrt{2}+1), -(\sqrt{2}-1), \sqrt{2}-1, \sqrt{2}+1\}$$

and

$$\sigma(\text{Im}([A_1, B_1])) = \{-1, -1, 1, 1\}.$$

If $W([A_1, B_1])$ is an eliptic disc centered at 0, then the ellipse has the foci $\{-2, 2\}$ and the length of major and minor axes equal to $2(\sqrt{2}+1)$ and 2 respectively. Notice also that for any ellipotic disc, the sum of the square of the length of minor axis and the square of the distance between the foci is equal to the square of the length of major axis. As

$$2^2 + (2 - (-2))^2 = 20 < 4(3 + 2\sqrt{2}) = (2(\sqrt{2}+1))^2,$$

we see that $W([A_1, B_1])$ cannot be an elliptic disc centered at 0. Since $0 \in W([A_1, B_1])$, we have $W([A, B]) = W([A_1, B_1])$. So the numerical range of $[A, B]$ cannot be an elliptic disc with center 0 and A does not satisfy (3). □

Next, we consider $A, B \in \mathcal{B}(H)$ such that $W([A, C]) = W([B, C])$ for all $C \in \mathcal{B}(H)$.

Proposition 2.4. *Let H be a complex Hilbert space with* $\dim H \geq 2$ *and $A, B \in \mathcal{B}(H)$. Then the following statements are equivalent.*

(1) $\sigma([A, x \otimes x]) = \sigma([B, x \otimes x])$ *for every unit vector $x \in H$.*

(2) $W([A, x \otimes x]) = W([B, x \otimes x])$ *for any unit vector $x \in H$.*

(3) $A + B$ or $A - B$ is a scalar operator.

Proof. (3)⇒(1) is obvious as $\sigma([A, x \otimes x]) = -\sigma([A, x \otimes x])$. Let us check (1)⇒(3).

(1) implies that $\sigma([A, x \otimes x]^2) = \sigma([B, x \otimes x]^2)$ and thus $\text{Tr}([A, x \otimes x]^2) = \text{Tr}([B, x \otimes x]^2)$, that is,

$$\langle Ax, x\rangle^2 - \langle A^2x, x\rangle = \langle Bx, x\rangle^2 - \langle B^2x, x\rangle \tag{2.3}$$

holds for each unit vector x. Let y, z be two orthogonal unit vectors and let $x = \frac{\sqrt{2}}{2}(e^{i\xi}y + z)$ for $\xi \in [-\pi, \pi]$. Substituting x in Eq.(2.3) gives

$$\begin{aligned} & \langle A(e^{i\xi}y + z), (e^{i\xi}y + z)\rangle^2 - 2\langle A^2(e^{i\xi}y + z), (e^{i\xi}y + z)\rangle \\ = \; & \langle B(e^{i\xi}y + z), (e^{i\xi}y + z)\rangle^2 - 2\langle B^2(e^{i\xi}y + z), (e^{i\xi}y + z)\rangle. \end{aligned} \tag{2.4}$$

Comparing the coefficients of $e^{2i\xi}$ in the expansions of Fourier series on both sides of Eq.(2.4) shows that

$$\langle Ay, z\rangle^2 = \langle By, z\rangle^2 \quad \text{holds for all orthogonal unit vectors } y, z \in H. \tag{2.5}$$

Thus, for any orthogonal vectors $x, f \in H$,

$$\langle Ax, f\rangle = 0 \Leftrightarrow \langle Bx, f\rangle = 0.$$

So, for any $x \in H$ and $f \in [Ax, x]^\perp$, we have $\langle Bx, f\rangle = 0$. This entails that $Bx \in [Ax, x]$, the subspace spanned by Ax and x. Thus, for any $x \in H$, there exist $\alpha_x, \beta_x \in \mathbb{C}$ such that $Bx = \alpha_x Ax + \beta_x x$. By Eq.(2.5), we have $\langle Ax, f\rangle^2 = \langle \alpha_x Ax, f\rangle^2 = \alpha_x^2\langle Ax, f\rangle^2$ holds for all $f \in [x]^\perp$, which implies that $\alpha_x = \pm 1$. It follows from $|\beta_x|\|x\| \leq \|Bx\| + \|\alpha_x Ax\| \leq (\|B\| + \|A\|)\|x\|$ that $|\beta_x| \leq \|B\| + \|A\|$. Therefore, B is a regular local linear combination of A and I. Then, by [15], B is a linear combination of A and I. So $B = \alpha A + \beta I$ with $\alpha \in \{-1, 1\}$ and $\beta \in \mathbb{C}$. Thus, condition (3) holds.

Next, we turn to the equivalence of (2) and (3).

Assume that $\dim H = 2$; then $W([A, x \otimes x]) = W([B, x \otimes x])$ is an elliptic disc centered at 0 and hence have the same foci. It follows from Lemma 2.1 that $\sigma([A, x \otimes x]) = \sigma([B, x \otimes x])$ for all $x \in H$. By (1)⇔(3) just proved above, (2) and (3) are equivalent.

Now assume that $\dim H \geq 3$. The implication (3)⇒(2) follows from Propositions 2.2 or 2.3. We consider (2)⇒(3). By Proposition 2.3, for any unit vector x, $W([A, x \otimes x])$ is an elliptic disc centered at the origin. Let $-\alpha, \alpha$ be the foci of this elliptic disc; then, by Lemma 2.1, $\sigma([A, x \otimes x]) = \{0, -\alpha, \alpha\}$. So, (2) implies that $\sigma([A, x \otimes x]) = \sigma([B, x \otimes x])$ holds for any unit vector $x \in H$. By the implication (1)⇒(3) established above, we see that (3) holds. □

Corollary 2.5. *Let H be a complex Hilbert space with* $\dim H \geq 3$*,* $A, B \in \mathcal{B}(H)$ *and* $A \notin \mathbb{C}\mathcal{P}(H) + \mathbb{C}I$*. Then the following conditions are equivalent.*

(1) $W([A, C]) = W([B, C])$ *for any* $C \in \mathcal{B}(H)$.

(2) $W([A, C]) = W([B, C])$ *for any* $C \in \mathcal{B}(H)$ *with* $\operatorname{rank} C \leq 2$.

(3) $A - B$ *is scalar operator.*

Proof. Obviously, (3)⇒(1)⇒(2).

(2)⇒(3). By Proposition 2.4, we have $B = \alpha A + \beta I$ for some $\alpha \in \{-1, 1\}$ and $\beta \in \mathbb{C}$. As $A \notin \mathbb{C}\mathcal{P} + \mathbb{C}I$, Proposition 2.2 ensures that $B = A + \beta I$, as desired. □

3. Proofs of Theorem 1.1

The implications (3)⇒(1) and (3)⇒(2) are true by Proposition 2.2. Let us prove (1)⇒(3). (2)⇒(3) can be proved similarly by notice that $\overline{W}(A) = W(A)$ if A is of finite rank.

Assume that $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a map preserving the numerical range of Lie product of operators, of which the range contains all operators of rank ≤ 1.

We complete the proof by considering two different cases: $\dim H \geq 3$ and $\dim H = 2$.

Proof of the case when $\dim H \geq 3$.

In this case, we must have $\dim K \geq 3$. If not, $W([\Phi(A), \Phi(B)])$ is an elliptic disc for every pair $A, B \in \mathcal{B}(H)$. But then, if $\{e_1, e_2, e_3\} \in H$ is an orthonormal set, $A = e_1 \otimes e_2 + ie_2 \otimes e_3$ and $B = e_2 \otimes e_1 + e_3 \otimes e_2$, $W([\Phi(A), \Phi(B)]) = W([A, B])$ is a triangle (including interior) with vertexes $1, i - 1, -i$, which is a contradiction.

Claim 1. $\Phi(\mathbb{C}I) \subseteq \mathbb{C}I$ and if $A \in \mathcal{B}(H)$ satisfies $\Phi(A) = \lambda I \in \mathcal{B}(K)$, then $A \in \mathbb{C}I$.

As $W([\Phi(\lambda I), \Phi(A)]) = W([\lambda I, A]) = \{0\}$ holds for any $A \in \mathcal{B}(H)$ and the range of Φ contains all operators of rank ≤ 1, we must have $\Phi(\lambda I) \in \mathbb{C}I$. If λI lies in the range of Φ, then there is some A such that $\Phi(A) = \lambda I$ and hence $W([A, B]) = W([\lambda I, \Phi(B)]) = \{0\}$ for every $B \in \mathcal{B}(H)$. It follows that $A \in \mathbb{C}I$.

Claim 2. $\Phi(\mathcal{P}_1(H) + \mathbb{C}I) \subseteq \mathbb{C}\mathcal{P}_1(K) + \mathbb{C}I$ and, if $T \in \mathcal{B}(K)$ is a rank one projection, then there exist a rank one projection A and a scalar λ such that $\Phi(A + \lambda I) = T$.

If $A \in \mathcal{P}_1(H) + \mathbb{C}I$, then by Proposition 2.3, for any $B \in \mathcal{B}(H)$, $W([A, B])$ is an elliptic disc centered at 0. Thus $W([\Phi(A), \Phi(B)])$ is an elliptic disc centered at 0, too. Since the range of Φ contains all operators of rank one, by Proposition 2.3 again, we see that $\Phi(A) \in \mathbb{C}\mathcal{P}_1(K) + \mathbb{C}I$.

If $T \in \mathcal{P}_1(K)$, then $T \in \operatorname{ran}(\Phi)$ and thus $T = \Phi(A)$ for some $A \in \mathcal{B}(H)$. Applying Proposition 2.3, one sees that, for any $B \in \mathcal{B}(H)$, $W([A, B]) = W([T, \Phi(B)])$ is an elliptic disc centered at 0. Again Proposition 2.3 ensures that $A \in \mathbb{C}\mathcal{P}_1(H) + \mathbb{C}I$.

Claim 3. There exist a unitary operator $U \in \mathcal{B}(H, K)$, a function $\mu : \mathcal{P}_1(H) \to \mathbb{C} \setminus \{0\}$ and a function $\nu : \mathcal{P}_1(H) \to \mathbb{C}$ such that either

(i) $\Phi(x \otimes x) = \mu(x \otimes x)Ux \otimes xU^* + \nu(x \otimes x)I$ for every $x \otimes x \in \mathcal{P}_1(H)$; or

(ii) $\Phi(x \otimes x) = \mu(x \otimes x)U(x \otimes x)^t U^* + \nu(x \otimes x)I$ for every $x \otimes x \in \mathcal{P}_1(H)$, where A^t is the the transpose of A with respect to an arbitrarily fixed orthnormal basis of H. Particularly, we have $\dim K = \dim H$.

By Claim 2, for any rank-1 projection $x\otimes x$ on H, $\Phi(x\otimes x)=\mu(x\otimes x)y_x\otimes y_x+\nu(x\otimes x)I$ for some unit vector $y_x\in K$ and scalars $\mu(x\otimes x),\nu(x\otimes x)$. It is clear that $\mu(x\otimes x)\neq 0$ by Claim 1 and that $y_x\otimes y_x$ is uniquely determined as $\dim K\geq 3$. Let $\Psi:\mathcal{P}_1(H)\to\mathcal{P}_1(K)$ be the map defined by $\Psi(x\otimes x)=y_x\otimes y_x$ for every $x\otimes x\in\mathcal{P}_1(H)$. By Claim 2, Ψ is bijective. Also, Ψ preserves commutativity in both directions since Φ does. So Ψ preserves orthogonality in both directions. By a reformulation of Uhlhorn extension of Wigners theorem [27], Ψ is induced by a unitary or an antiunitary operator, that is, there exists a unitary operator $U\in\mathcal{B}(H,K)$ such that either $\Psi(x\otimes x)=Ux\otimes xU^*$ for all $x\otimes x\in\mathcal{P}_1(H)$; or, with respect to an arbitrarily given orthonormal basis, $\Psi(x\otimes x)=U(x\otimes x)^tU^*$ for all $x\otimes x\in\mathcal{P}_1(H)$ (Also, ref. [22, 25]). In addition, the existence of the unitary operator $U:H\to K$ ensures that H and K have the same dimension. Now, Claim 3 follows.

Claim 4. There exist a sign function $\varepsilon:\mathcal{B}(H)\to\{-1,1\}$ and a functional $h:\mathcal{B}(H)\to\mathbb{C}$ such that either

(1°) $\Phi(A)=\varepsilon(A)UAU^*+h(A)I$ for all $A\in\mathcal{B}(H)$; or

(2°) $\Phi(A)=i\varepsilon(A)UA^tU^*+h(A)I$ for all $A\in\mathcal{B}(H)$.

Recall that Φ has the form (i) or the form (ii) in Claim 3. Without loss of generality we may assume in the sequel that $U=I$.

We first assume that Φ takes the form (i) in Claim 3. Thus $\Phi(x\otimes x)=\mu(x\otimes x)x\otimes x+\nu(x\otimes x)I$ for every unit vector x. For any unit vectors $x,y\in H$, as

$$\mu(x\otimes x)\mu(y\otimes y)W([x\otimes x,y\otimes y])=W([x\otimes x,y\otimes y]),$$

we get $\mu(x\otimes x)\mu(y\otimes y)=\pm 1$ by Proposition 2.2. This entails that $\mu(x\otimes x)\in\{1,-1\}$. Let $\eta:\mathcal{B}(H)\to\{-1,1\}$ be the map defined by $\eta(x\otimes x)=\mu(x\otimes x)$ and $\eta(A)=1$ otherwise, and let $\Phi_1=\eta\Phi$. By Proposition 2.2, Φ_1 still preserves the numerical range of Lie product of operators, and

$$\Phi_1(x\otimes x)=x\otimes x+h_1(x\otimes x)I$$

for every $x\otimes x\in\mathcal{P}_1$, where $h_1(x\otimes x)=\eta(x\otimes x)\nu(x\otimes x)$. Thus, for any $A\in\mathcal{B}(H)$, we have

$$W([\Phi_1(A),x\otimes x])=W([A,x\otimes x])$$

for all $x\otimes x\in\mathcal{P}_1$. Therefore, by Proposition 2.4,

$$\Phi_1(A)=\varepsilon_1(A)A+h_1(A)I$$

for some $\varepsilon_1(A)\in\{-1,1\}$ and $h_1(A)\in\mathbb{C}$. It follows that, there exist maps $\varepsilon:\mathcal{B}(H)\to\{-1,1\}$ and $h:\mathcal{B}(H)\to\mathbb{C}$ such that $\Phi(A)=\varepsilon(A)A+h(A)I$ for all A, that is, (1°) holds.

Suppose that Φ has the form (ii) in Claim 3. Then $\Phi(x\otimes x)=\mu(x\otimes x)(x\otimes x)^t+\nu(x\otimes x)I$ for every unit vector x. Let Φ_2 be defined by $\Phi_2(A)=i\Phi(A)^t$ for each A. Then for any $A,B\in\mathcal{B}(H)$,

$$\begin{aligned} & W([\Phi_2(A),\Phi_2(B)])=W([i\Phi(A)^t,i\Phi(B)^t]) \\ = & -W([\Phi(A)^t,\Phi(B)^t])=W([\Phi(A),\Phi(B)]^t) \\ = & W([\Phi(A),\Phi(B)])=W([A,B]). \end{aligned}$$

Also note that $\Phi_2(x\otimes x) = \mu(x\otimes x)x\otimes x+\nu(x\otimes x)I$ for some scalars $\mu(x\otimes x), \nu(x\otimes x)$. So Φ_2 has the form (i) in Claim 3. By what just proved above, Φ_2 has the form $\Phi_2(A) = -\varepsilon(A)A+h_1(A)I$ for all A. It follows that

$$\Phi(A) = -i\Phi_2(A)^t = i\varepsilon(A)A^t + h(A)I$$

holds for all $A \in \mathcal{B}(H)$, that is, Φ takes the form (2°) in Claim 4.

Claim 5. ε is a constant function when restricted on the set $\mathcal{B}(H) \setminus (\mathbb{C}\mathcal{P} + \mathbb{C}I)$.

We have to show that if A, B are not combinations of some projection and I, then $\varepsilon(A) = \varepsilon(B)$.

Note that, by Claim 4, we always have

$$\varepsilon(A)\varepsilon(B)W([A, B]) = W([A, B]). \tag{3.1}$$

We first assert that, if $A = x\otimes f$, $B = y\otimes x$ are rank-1 nilpotent operators such that $\langle y, x\rangle = 0$ and $\langle y, f\rangle \neq 0$, then $\varepsilon(A) = \varepsilon(B)$. In fact, $[A, B] = [x \otimes f, y \otimes x] = \langle y, f\rangle x \otimes x - \|x\|^2 y \otimes f$ which has a matrix representation of the form

$$[A, B] = \begin{pmatrix} -\alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & \beta & 0 \end{pmatrix} \oplus 0 = X \oplus 0$$

with nonzero scalars α, β. It is clear that $W([A, B]) = W(X)$ and $W(X)$ is either an elliptic disc with foci $\{0, \alpha\}$, or a "cone-like" figure which is the convex hull of $\{-\alpha\}$ and the ellipse E with foci $\{0, \alpha\}$. Hence $W([A, B]) \neq -W([A, B])$. By Eq.(3.1), this entails that $\varepsilon(A)\varepsilon(B) = 1$, that is, $\varepsilon(A) = \varepsilon(B)$, the assertion is true.

If $x \otimes f$ and $x \otimes u$ are rank-1 nilpotent and if $u \perp f$, then, the assertion above ensures that

$$\begin{aligned} \varepsilon(x \otimes f) = {} & \varepsilon((f + u) \otimes x) = \varepsilon(x \otimes u) = \varepsilon(u \otimes (x + f)) \\ = {} & \varepsilon(f \otimes u) = \varepsilon((x + u) \otimes f) = \varepsilon(f \otimes x). \end{aligned}$$

So we have

$$\varepsilon(x \otimes f) = \varepsilon(x \otimes u) = \varepsilon(f \otimes x) \tag{3.2}$$

whenever $\{x, f, u\}$ is orthogonal.

Next we show that,

$$\varepsilon(x \otimes f) = \varepsilon(x \otimes u) \quad \text{for any nonzero vectors } f, u \in [x]^\perp. \tag{3.3}$$

If $u \perp f$, use Eq.(3.2). If $u = \lambda f$ for some nonzero scalar λ, taking $v \in [x, f]^\perp$ gives

$$\varepsilon(x \otimes f) = \varepsilon(x \otimes v) = \varepsilon(x \otimes u).$$

If u is linearly independent to f and $u \not\perp f$, then there exists a unit vector v such that $v \in [x, f]^\perp$ and $u = \alpha f + \beta v$ with nonzero scalars α, β, which gives, by Eq.(3.2),

$$\varepsilon(x \otimes u) = \varepsilon(f \otimes x) = \varepsilon(x \otimes f),$$

finishing the proof of Eq.(3.3).

Thirdly, we claim that $\varepsilon(A) = \varepsilon(B)$ holds for any rank-1 nilpotent operators A, B.

In fact, for any two rank-1 nilpotent operators $A = x \otimes f$ and $B = y \otimes g$, taking unit vector $u \in [x, y]^\perp$, one gets by Eq.(3.3) that

$$\varepsilon(x \otimes f) = \varepsilon(x \otimes u) = \varepsilon(y \otimes u) = \varepsilon(y \otimes g),$$

as desired.

Now, we prove that, for any two operators $A, B \in \mathcal{F}_1 \setminus (\mathbb{C}\mathcal{P} + \mathbb{C}I)$, we have $\varepsilon(A) = \varepsilon(B)$.

Since every rank-1 operator is quadratic, by the proof of Proposition 2.2 (see the argument in Case 1), one sees that, for any $A, B \in \mathcal{F}_1 \setminus (\mathbb{C}\mathcal{P} + \mathbb{C}I)$, there exist rank-1 nilpotent operators $x \otimes f, y \otimes g$ so that $W([A, x \otimes f]) \neq -W([A, x \otimes f])$ and $W([B, y \otimes g]) \neq -W([B, y \otimes g])$. Therefore, by Eq.(3.1) and the fact proved above, we obtain that

$$\varepsilon(A) = \varepsilon(x \otimes f) = \varepsilon(y \otimes g) = \varepsilon(B).$$

Finally, for any $A, B \in \mathcal{B}(H) \setminus (\mathbb{C}\mathcal{P} + \mathbb{C}I)$, by Proposition 2.2, there exist rank-1 operators $E, F \in \mathcal{F}_1 \setminus (\mathbb{C}\mathcal{P} + \mathbb{C}I)$ such that $W([A, E]) \neq -W([A, E])$ and $W([B, F]) \neq -W([B, F])$. As $\varepsilon(E) = \varepsilon(F)$ which is proved just above, we get again

$$\varepsilon(A) = \varepsilon(E) = \varepsilon(F) = \varepsilon(B).$$

this completes the proof of Claim 5.

The proof of Theorem 1.1 completes when $\dim H \geq 3$. □

Proof of the case when $\dim H = 2$.

Clearly, (1) ⇔ (2).

Assume (3) holds. For simplicity, assume that $U = I_2$. If Φ has the form in (3) and if $C = [A, B]$, then $[\Phi(A), \Phi(B)] \in \{\pm C, \pm C^t\}$. Since $W(C)$ is an elliptic disk centered at the origin, we see that $W(\pm C) = W(\pm C^t)$. Thus, $W([A, B]) = W([\Phi(A), \Phi(B)])$. So (1) is true.

Now, suppose (1) holds. First we show that $\dim H = \dim K$. If not, then $\dim K \geq 3$. By Proposition 2.2, we may take rank-1 operators $B_1, B_2 \in \mathcal{B}(K)$ such that $W([B_1, B_2]) \neq -W([B_1, B_2])$. Since the range of Φ contains all rank-1 operators, there exist $A_1, A_2 \in \mathcal{B}(H)$ such that $(\Phi(A_1), \Phi(A_2)) = (B_1, B_2)$. Note that $W([A_1, A_2])$ is an elliptic disk centered at 0. So we have $W([B_1, B_2]) = W([A_1, A_2]) = -W([A_1, A_2]) = -W([B_1, B_2])$, which is impossible. So $\dim K = \dim H = 2$ and we can assume that $\mathcal{B}(H) = \mathcal{B}(K) = M_2(\mathbb{C})$.

We show that Φ has the asserted form. We may modify the functional $h(A)$ in the map Φ so that $\Phi(A)$ has trace 0 for all $A \in M_2(\mathbb{C})$. Then we can focus on the set M_2^0 of trace zero matrices in $M_2(\mathbb{C})$.

Consider the matrices

$$X = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the following holds:

(1) $\{X, Y, Z\}$ is an orthonormal basis for M_2^0 using the inner product $\langle A, B\rangle = \operatorname{tr} AB^*$.
(2) $XY = \frac{i}{\sqrt{2}}Z = -YX, \quad YZ = \frac{i}{\sqrt{2}}X = -ZY, \quad ZX = \frac{i}{\sqrt{2}}Y = -XZ.$
(3) $W([X, Y]) = W([Y, Z]) = W([Z, X]) = i[-1, 1]$.

(4) If $A = a_1X + a_2Y + a_3Z$ and $B = b_1X + b_2Y + b_3Z$ in M_2^0, then

$$[A, B] = \sqrt{2}i(c_1X + c_2Y + c_3Z),$$

where

$$c_1 = a_2b_3 - a_3b_2, \quad c_2 = -(a_1b_3 - a_3b_1), \quad c_3 = a_1b_2 - a_2b_1.$$

In other words, $(c_1, c_2, c_3)^t = (a_1, a_2, a_3)^t \times (b_1, b_2, b_3)^t$, the cross product in $\mathbb{C}^3$.

(5) Every unitary similarity map $a_1X + a_2Y + a_3Z = A \mapsto UAU^* = b_1X + b_2Y + b_3Z$ on M_2^0 corresponds to a real special orthogonal transformation $T \in M_3(\mathbb{C})$ such that $T(a_1, a_2, a_3)^t = (b_1, b_2, b_3)^t$.

Claim. There exist a unitary $U \in M_2(\mathbb{C})$ and $\varepsilon \in \{1, -1\}$ such that either

(a) $\Phi(A) = \varepsilon UAU^*$ for all $A \in \{X, Y, Z\}$; or (b) $\Phi(A) = i\varepsilon UA^tU^*$ for all $A \in \{X, Y, Z\}$.

Proof. Assume that the image of X, Y, Z under Φ are respectively

$$X_1 = a_{11}X + a_{21}Y + a_{31}Z, \quad Y_1 = a_{12}X + a_{22}Y + a_{32}Z, \quad Z_1 = a_{13}X + a_{23}Y + a_{33}Z.$$

Let $T = (a_{pq}) \in M_3(\mathbb{C})$. We will show that either T is a real orthogonal matrix or iT is a real orthogonal matrix. In the former case, Φ will satisfy (a); in the latter case, Φ will satisfy (b).

Note that the hypothesis and conclusion will not be affected by changing T to PTQ for any real orthogonal matrices $P, Q \in M_3(\mathbb{C})$. It just corresponds to changing Φ to a map of the form

$$A \mapsto \varepsilon_P U_P \Phi(\varepsilon_Q U_Q A U_Q^*) U_P^*$$

for some unitary $U_P, U_Q \in M_2(\mathbb{C})$ and $\varepsilon_P, \varepsilon_Q \in \{1, -1\}$ depending on P and Q.

We use the above fact to simplify the structure of T as follows. Suppose $T = T_1 + iT_2$ where T_1, T_2 are real matrices. By the singular value decomposition of real matrices, let P, Q be real orthogonal such that $PT_2Q = \mathrm{diag}(s_1, s_2, s_3)$ with $s_1 \geq s_2 \geq s_3 \geq 0$. Now, replace T by PTQ so that $T = T_1 + iT_2$ with $T_2 = \mathrm{diag}(s_1, s_2, s_3)$. We will show that either

(a′) $T_2 = 0$ and T_1 is real orthogonal; or (b′) $T_1 = 0$ and T_2 is real orthogonal.

Then conclusions (a) and (b) will follow respectively.

To achieve our goal, let $B = (b_{pq}) = ((-1)^{p+q} \det(T(p, q)))$, where $T(p, q)$ is obtained from $T = (a_{pq})$ by deleting its pth row and qth column. Then

$$[Y_1, Z_1] = \sqrt{2}i(b_{11}X + b_{21}Y + b_{31}Z),$$

$$[Z_1, X_1] = \sqrt{2}i(b_{12}X + b_{22}Y + b_{32}Z),$$

$$[X_1, Y_1] = \sqrt{2}i(b_{13}X + b_{23}Y + b_{33}Z).$$

Because $W([X_1, Y_1]) = W([X, Y]) = i[-1, 1]$, we see that $B_1 = [Y_1, Z_1]$ is skew-Hermitian with eigenvalues $i, -i$. Thus, $ib_{31} \in i[-1, 1]$, $b_{11} - ib_{21} = \overline{(b_{11} + ib_{21})}$. If $b_{11} = x_1 + iy_1, b_{12} = x_2 + iy_2$ with $x_1, x_2, y_1, y_2 \in \mathbb{R}$, the above equality implies that $y_1 = y_2 = 0$, i.e., $b_{11}, b_{21} \in \mathbb{R}$. Moreover, $1 = \det(B_1)$ implies that $|b_{11}|^2 + |b_{12}|^2 + |b_{13}|^2 = 1$. The same arguments apply to B_2 and B_3. Hence,

$$B = (b_{pq})$$

is a real matrix such that every column has length 1.

Recall that $T = T_1 + iT_2$ with $T_2 = \mathrm{diag}(s_1, s_2, s_3)$. Let A_1, A_2, A_3 be the three columns of T. We consider 2 cases.

Case 1. If $s_1 = 0$, then T is real. We may further replace T by PTQ for suitable real orthogonal matrices P and Q such that $T = \mathrm{diag}(d_1, d_2, d_3)$ with $d_1 \geq d_2 \geq d_3 \geq 0$. The resulting matrix still has columns such that the cross product of any two of them is a unit vector in $\mathbb{R}^3$. Thus, $d_1 = d_2 = d_3 = 1$.

Case 2. Suppose $s_1 > 0$. Then $a_{32} = a_{23} = 0$ because $A_1 \times A_2$ and $A_1 \times A_3$ are real vectors. If $s_2 = s_3 = 0$, then we also have $a_{22}, a_{33} = 0$ so that $0 = A_2 \times A_3 = B_1$, which is impossible. If $s_2 > 0$, then $a_{31} = a_{13} = 0$ because $A_1 \times A_2$ and $A_2 \times A_3$ are real vectors. Let $a_{11} = r_1 e^{it} \notin \mathbb{R}$. Then $a_{33} = r_3 e^{-it} \notin \mathbb{R}$ because $A_1 \times A_3$ is a real vector and a_{33} is the only nonzero entry in A_3. As a result, $a_{12} = 0$ because $A_2 \times A_3$ is real. Now, a_{22} is the only nonzero entry in A_2 and $A_1 \times A_2$ is a real vector. So, $a_{22} = r_2 e^{-it}$. Now, the fact $A_2 \times A_3$ is real implies $a_{22}, a_{33} \in i\mathbb{R}$. Thus, $a_{22} = is_2$ and $a_{33} = is_3$. It then follows that a_{11} is the only nonzero entry in A_1 and $a_{11} = is_1$. Hence, $T = i\mathrm{diag}(s_1, s_2, s_3)$. Since $A_1 \times A_2, A_2 \times A_3, A_1 \times A_3$ are unit vectors, we see that $s_1 = s_2 = s_3 = 1$. The proof of Claim is finished.

Now, suppose condition (a) in the **Claim** holds. We will prove that Φ has the first form in Theorem 1.1. We may change Φ by a map of the form

$$A \mapsto \varepsilon U A U^*$$

for some suitable unitary $U \in M_2(\mathbb{C})$ and $\varepsilon \in \{1, -1\}$, and assume that Φ fixes X, Y, Z. Suppose $A = \sqrt{2}E_{12}$ and $\Phi(A) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. Then

$$[X, A] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad [\Phi(X), \Phi(A)] = \begin{pmatrix} c-b & -2a \\ 2a & b-c \end{pmatrix},$$

$$[Y, A] = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad [\Phi(Y), \Phi(A)] = i \begin{pmatrix} b+c & 2a \\ 2a & -b-c \end{pmatrix},$$

$$[Z, A] = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad [\Phi(Z), \Phi(A)] = \begin{pmatrix} 0 & 2b \\ 2c & 0 \end{pmatrix}.$$

Because $W([C, A]) = W([\Phi(C), \Phi(A)])$ for all $C \in \{X, Y, Z\}$, we see that

$$\Phi(A) \in \{A, -A, A^t, -A^t\}.$$

If $\Phi(A) = -A$, we may change the set $\mathcal{S}$ and assume also that $\Phi(A) = A$. If $\Phi(A) = A^t$, we can replace Φ by the map $C \mapsto DCD^t$ with $D = E_{12} + E_{21}$. Then $\Phi(A) = A$, $\Phi(X) = X$, $\Phi(Y) = -Y$ and $\Phi(Z) = Z$. Again, we can adjust the set $\mathcal{S}$ and assume that Φ fixes A, X, Y, Z. We can do similar adjustment and assume that Φ fixes A, X, Y, Z if $\Phi(A) = -A^t$.

Now, consider $B = \sqrt{2}E_{21}$. By the fact that $W([C, B]) = W([\Phi(C), \Phi(B)]) = W([C, \Phi(B)])$ for $C \in \{A, X, Y, Z\}$, we see that $\Phi(B) \in \{-B, B\}$. We can adjust the set $\mathcal{S}$ if necessary and assume that $\Phi(B) = B$. Now, by the fact that $\Phi(C) = C$ and $W([C, R]) = W([\Phi(C), \Phi(R)]) =$

$W([C,\Phi(R)])$ for all $C\in\{A,B,X,Y,Z\}$, we conclude that $\Phi(R)\in\{-R,R\}$ for every $R\in M_2^0$. Thus (a) holds implies that Φ has the first form in (3) of Theorem 1.1.

If (b) holds, we can use a similar argument to show that Φ has the second form in (3) of Theorem 1.1. □

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