Linear Operators Preserving Decomposable Numerical Radii on Orthonormal Tensors

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Abstract

Let $1 \le m \le n$, and let $\chi : H \to \mathbb{C}$ be a degree 1 character on a subgroup H of the symmetric group of degree m. The generalized matrix function on an $m \times m$ matrix $B = (b_{ij})$ associated with χ is defined by $d_\chi(B) = \sum_{\sigma \in H} \chi(\sigma) \prod_{j=1}^m b_{j,\sigma(j)}$, and the decomposable numerical radius of an $n \times n$ matrix A on orthonormal tensors associated with χ is defined by

$$r_\chi^\perp(A) = \max\{|d_\chi(X^*AX)| : X \text{ is an } n \times m \text{ matrix such that } X^*X = I_m\}.$$

We study those linear operators L on $n \times n$ complex matrices that satisfy $r_\chi^\perp(L(A)) = r_\chi^\perp(A)$ for all $A \in M_n$. In particular, it is shown that if $1 \le m < n$, such an operator must be of the form

$$A \mapsto \xi U^* A U \quad \text{or} \quad A \mapsto \xi U^* A^t U$$

for some unitary matrix U and some $\xi \in \mathbb{C}$ with $|\xi| = 1$.

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1 Introduction

Let M_n be the algebra of $n \times n$ complex matrices. Suppose $1 \le m \le n$ and $\chi : H \to \mathbb{C}$ is a degree 1 character on a subgroup H of the symmetric group S_m of degree m. The generalized matrix function of $B = (b_{ij}) \in M_m$ associated with χ is defined by

$$d_\chi(B) = \sum_{\sigma \in H} \chi(\sigma) \prod_{j=1}^m b_{j,\sigma(j)}.$$

For instance, $d_\chi(B) = \mathrm{per}\,(B)$, the permanent of B, when χ is the principal character on $H = S_m$; $d_\chi(B) = \det(B)$, the determinant of B, when χ is the alternate character on $H = S_m$.

Define the decomposable numerical range of $A \in M_n$ on orthonormal tensors associated with χ by

$$W_\chi^\perp(A) = \{d_\chi(X^*AX) : X \text{ is an } n \times m \text{ matrix such that } X^*X = I_m\}$$

and the decomposable numerical radius of A by

$$r_\chi^\perp(A) = \max\{|z| : z \in W_\chi^\perp(A)\}.$$

When $m = 1$, these reduce to the classical numerical range and numerical radius of A, denoted in the sequel by $W(A)$ and $r(A)$. The decomposable numerical range can be viewed as the image of the quadratic form $x^* \mapsto (K(A)x^*, x^*)$, defined by the induced matrix $K(A)$ associated with χ, on the decomposable unit tensors $x^* = x_1 * \cdots * x_m$ such that $\{x_1, \ldots, x_m\}$ is an orthonormal set in $\mathbb{C}^n$. One may see [16, 19] for some general background.

The classical numerical range and numerical radius are useful tools for studying matrices and operators (see e.g. [6, 7, 8, 11]). Likewise, the decomposable numerical range and radius are useful tools for studying induced matrices acting on symmetry classes of tensors (see [16, 19] and their references).

There has been considerable interest in studying linear preservers of $W_\chi^\perp$ or $r_\chi^\perp$, i.e., those linear operators L on M_n satisfying

$$W_\chi^\perp(L(A)) = W_\chi^\perp(A) \text{ for all } A \in M_n, \quad \text{or} \quad r_\chi^\perp(L(A)) = r_\chi^\perp(A) \text{ for all } A \in M_n.$$

The results on $W_\chi^\perp$ preservers are quite complete. We summarize them in the following.

(1) Frobenius [5] proved that when $m = n$ and χ is the alternate character on S_n, i.e., $W_\chi^\perp(A) = \{\det(A)\}$, a linear preserver of $W_\chi^\perp$ on M_n must be of the form $A \mapsto MAN$ or $A \mapsto MA^tN$ for some $M, N \in M_n$ with $\det(MN) = 1$.

(2) Pellegrini [21] proved that when $m = 1$, a linear preserver of the numerical range on M_n must be of the form $A \mapsto U^*AU$ or $A \mapsto U^*A^tU$ for some unitary U.

(3) Marcus and Filippenko [17] proved that when χ is the alternate character on $H = S_m$ with $m < n$, a linear preserver of $W_\chi^\perp$ on M_n must be of the form

$$A \mapsto \xi U^*AU \quad \text{or} \quad A \mapsto \xi U^*A^tU \tag{1.1}$$

for some unitary $U \in M_n$ and $\xi \in \mathbb{C}$ with $\xi^m = 1$.

(4) Hu and Tam [9, 10] (see [4] for the correction of the statement in [9, Theorem 6]) proved that when χ is the principal character on $H < S_m$, a linear operator L on M_n is a preserver of $W_\chi^\perp$ if and only if there exist a unitary matrix $U \in M_n$ and $\xi \in \mathbb{C}$ with $\xi^m = 1$ such that
(i) L is of the form described in (1.1), or
(ii) $m = n = 2$, $H = S_2$ and L is of the form

$$A \mapsto \xi[U^*AU + (\pm i - 1)(\operatorname{tr} A)I/2] \quad \text{or} \quad A \mapsto \xi[U^*A^tU + (\pm i - 1)(\operatorname{tr} A)I/2].$$

(5) Very recently, these authors [15] proved that in all the remaining cases, a linear preserver of $W_\chi^\perp$ on M_n must be of the form described in (1.1).

While the structure of the linear preservers of $W_\chi^\perp$ is completely determined, the linear preserver problems on $r_\chi^\perp$ are not so well studied. The only known results are for the cases when $m = 1$ [13] and when χ is the alternate character on $H = S_m$ [14] (see also [24]). In these cases, the linear preservers of $r_\chi^\perp$ are always unit multiples of $W_\chi^\perp$ preservers. In fact, this phenomenon always occurs in linear preserver problems involving generalized numerical ranges and radii. Furthermore, even if the linear preservers of the generalized numerical range are determined, it always requires different (and usually more difficult) techniques to characterize the linear preservers of the generalized numerical radius (see e.g. [21]).

In this paper, we study the linear preservers of $r_\chi^\perp$ and confirm that when $1 \le m < n$ the linear preservers of $r_\chi^\perp$ are indeed unit multiples of $W_\chi^\perp$ preservers. We believe that the same result is true when $m = n$, but we are not able to prove it at present.

Our paper is organized as follows. In Section 2, we present some preliminary results on matrix inequalities and equalities involving $r_\chi^\perp$. Furthermore, we describe some results on unitary groups and a group theory scheme we are going to use. In Section 3, we prove that linear preservers of $r_\chi^\perp$ form a group G, and we study the largest connected Lie group in G. In Section 4, we present the characterization of the linear preservers of $r_\chi^\perp$ for the cases $m < n$ and $m = n = 2$, and some partial results for the remaining cases.

We use M_n' to denote the set of matrices in M_n with zero trace. The standard basis of M_n is denoted by $\{E_{11}, E_{12}, \dots, E_{nn}\}$. We always assume that $M_{n\times k}$ is equipped with the inner product $\langle A, B\rangle = \operatorname{tr}(AB^*)$. This includes the special case $M_{n\times 1} = \mathbb{C}^n$, where the inner product reduces to $\langle x, y\rangle = \operatorname{tr}(xy^*) = y^*x$.

We shall use $s_1(A) \ge \cdots \ge s_n(A)$ to denote the singular values of $A \in M_n$. When $H = S_m$, we denote by $\varepsilon : S_m \to \mathbf{C}$ the alternating character of S_m.

We always assume that $n \ge 2$. Since the results on the classical numerical radius and the classical numerical range are known, we exclude them and consider only the cases when $m \ge 2$ in this paper.

2 Preliminaries

2.1 Matrix Inequalities and Equalities

In this subsection, we collect some inequalities and equalities involving $r_\chi^\perp$ for future use. Recall that we always assume that $m \ge 2$.

Proposition 2.1 (see [1]) *Let $A \in M_n$.*

(a) *When χ is the principal character, $r_\chi^\perp(A) = 0$ if and only if $A = 0$.*

(b) *When χ is not the principal character, if $\operatorname{rank}(A) = 1$ then $r_\chi^\perp(A) = 0$.*

Proposition 2.2 (see [2, Theorem 1] and [3, Theorem 1]) *Suppose $A, B \in M_n$ are positive semi-definite matrices with eigenvalues $a_1 \ge \cdots \ge a_n$ and $b_1 \ge \cdots \ge b_n$, respectively, such* *that* $b_j \geq a_j$ *for all* $j = 1, \ldots, m$. *Then* $r_\chi^\perp(A) \leq r_\chi^\perp(B)$. *Suppose* $\mathrm{rank}(B) \geq m$. *The equality holds if and only if the* m *largest eigenvalues of* A *are the same as those of* B.

Proposition 2.3 *Let* $m \leq n$ *and* $A \in M_n$. *Then*

$$|\det(A)|^{m/n} \leq \binom{n}{m}^{-1} E_m(s_1(A), \ldots, s_n(A)) \leq r_\varepsilon^\perp(A) \leq r_\chi^\perp(A).$$

Suppose $K(A) \neq 0$ *and* $\chi \neq \varepsilon$ *(on* S_m*).*
(a) *When* $2 \leq m < n$, A *is a multiple of a unitary matrix if and only if* $|\det(A)|^{m/n} = r_\chi^\perp(A)$.
(b) *If* $m = n$ *and* A *is not normal, then* $|\det(A)| < r_\chi^\perp(A)$.

Proof. The first inequality is easy. The next two inequalities are proved in [2, Lemma 6 and Theorem 5]. When $K(A) \neq 0$, we have $r_\chi^\perp(A) \neq 0$. Thus, (a) follows from [2, Theorem 4 and Theorem 10], and (b) follows from [2, Lemma 10]. □

Proposition 2.4 *Let* $A \in M_n$. *Then* $r_\chi^\perp(A) \leq \|A\|^m$. *The equality holds if and only if* A *is unitarily similar to* $\mathrm{diag}\,(\alpha_1, \ldots, \alpha_m) \oplus A_2$ *with* $|\alpha_1| = \cdots = |\alpha_m| = \|A\|$.

Proof. Follows from [3, Theorem 2]. □

Proposition 2.5 (see [10]) *Suppose* $m = n = 2$, *and* χ *is the principal character. Let* $A \in M_2$ *be a normal matrix with eigenvalues* λ_1 *and* λ_2. *Then*

$$r_\chi^\perp(A) = \begin{cases} \max\{|\lambda_1\lambda_2|, |\lambda_1 + \lambda_2|^2/4\} & \textit{if } H = \{e\}, \\ \max\{|\lambda_1\lambda_2|, |\lambda_1^2 + \lambda_2^2|/2\} & \textit{if } H = S_2. \end{cases}$$

We prove an additional result that will be used in the future discussion.

Proposition 2.6 *Let* A *be a rank one matrix with Frobenius norm equal to* 1 *and let* χ *be the principal character. Then* $r_\chi^\perp(A) = |H|m^{-m}$.

Proof. By hypotheses, one can write $A = xy^*$ for some unit vectors x, $y \in \mathbb{C}^n$. There exists a unitary matrix $U \in M_n$ and real numbers t_j such that $UAU^* = uv^*$, where $u^* = x^*U^* = (1/\sqrt{m}, \ldots, 1/\sqrt{m}, 0, \ldots, 0)$ and $v^* = y^*U^* = (e^{it_1}/\sqrt{m}, \ldots, e^{it_m}/\sqrt{m}, 0, \ldots, 0)$ are unit vectors such that $x^*y = u^*v$. If X is obtained from U^* by deleting its last $n - m$ columns, then

$$r_\chi^\perp(A) = r_\chi^\perp(UAU^*) \geq |d_\chi(X^*AX)| = m^{-m}|H|$$

Conversely, for any $n \times m$ matrix X with $X^*X = I_m$, the vectors $x^*X^* = (u_1, \ldots, u_m)$ and $y^*X^* = (v_1, \ldots, v_m)$ have lengths less than or equal to one. Thus, $\prod_{j=1}^m |u_jv_j| \leq (1/m)^m$ by elementary calculus. It follows that

$$d_\chi(X^*AX) = |d_\chi\left((u_1, \ldots, u_m)^*(v_1, \ldots, v_m)\right)| = \prod_{j=1}^m |u_jv_j||H| \leq (1/m)^m|H|.$$

We get the result. $\square$

2.2 Group Theory Background

Let us describe a group scheme to study our problem. Suppose $PSU(n)$ is the group of operators on M_n of the form
$$A \mapsto U^*AU$$
for some unitary $U \in M_n$. In the next section, we shall prove that the linear preservers of $r_\chi^\perp$ form a group G. Clearly, every element in $PSU(n)$ is a linear preserver of $r_\chi^\perp$. Hence $PSU(n)$ is a subgroup of G. Let G_0 be the largest connected Lie subgroup in G. Using the results in [22] (see also [4]) and those Section 3, we show that $G_0 = PSU(n)$, and then completely determine G. Such a scheme of studying linear preserver problems has been used by several authors (see [4, 22] and their references).

In the following, we introduce some notations and list several group theory results that will be used in our proof.

Let $GL(n)$ and $SL(n), U(n)$ and $SU(n)$ be the general linear group, special linear group, unitary group, and special unitary group of linear operators acting on $\mathbb{C}^n$.

Suppose G_1 and G_2 are subgroups of $GL(n)$. Denote by $G_1 * G_2$ the group of operators of the form $A \mapsto UAV$ for some $U \in G_1$ and $V \in G_2$. Furthermore, let $PSL(n)$ be the group of invertible operators on M_n of the form $A \mapsto S^{-1}AS$ for some $S \in SL(n)$.

Next, let $GL(n^2)$, $SL(n^2)$ and $SU(n^2)$ be the general linear group, special linear group and special unitary group of operators acting on M_n, respectively. Moreover, let $SO(n^2)$ be the special orthogonal group with respect to the bilinear form $(A, B) = \mathrm{tr}\,(AB)$ on M_n.

Third, let $GL(n^2-1)$ be the subgroup of $GL(n^2)$ consisting of operators that fix the identity and map M_n' onto itself, and let

$SL(n^2-1) = SL(n^2) \cap GL(n^2-1)$,

$SU(n^2-1) = SU(n^2) \cap GL(n^2-1)$,

$SO(n^2-1) = SO(n^2) \cap GL(n^2-1)$,

$\mathbf{T}$: the group of operators acting as scalar on M_n' and span $\{I\}$,

$\mathbf{T}_1$: the intersection of $\mathbf{T}$ and $SL(M_n)$,

$\mathbf{U}_1$: the intersection of $\mathbf{T}_1$ and $SU(M_n)$,

$\mathbf{R}_1$: the collection of operators in $\mathbf{T}_1$ with positive eigenvalues,

$\mathbf{P}$: the group of operators of the form $A \mapsto A + (\mathrm{tr}\,A)C$ for some $C \in M_n'$,

$\mathbf{Q}$: the group of operators of the form $A \mapsto A + (\mathrm{tr}\,AC)I$ for some $C \in M_n'$.

Next, let $SU(n^2-1, 1)$ be the subgroup of $SU(n^2)$ containing the operators that preserve the non-degenerate hermitian form
$$\mathrm{tr}\,(AB^*) - \mathrm{tr}\,(A)\mathrm{tr}\,(B^*),$$
let $SO(n^2-1, 1)$ be the subgroup of $SL(n^2)$ containing the operators that preserve the non-degenerate symmetric bilinear form
$$\mathrm{tr}\,(AB) - \mathrm{tr}\,(A)\mathrm{tr}\,(B),$$

and let $GL(n^2, \mathrm{I\!R})$ be the subgroup of $GL(n^2)$ mapping the real linear space of Hermitian matrices onto itself. Set

$SL(n^2, \mathrm{I\!R}) = SL(n^2) \cap GL(n^2, \mathrm{I\!R})$,
$SL(n^2-1, \mathrm{I\!R}) = SL(n^2-1) \cap GL(n^2, \mathrm{I\!R})$,
$SO(n^2, \mathrm{I\!R}) = SO(n^2) \cap GL(n^2, \mathrm{I\!R})$,
$SO(n^2-1, \mathrm{I\!R}) = SO(n^2-1) \cap GL(n^2, \mathrm{I\!R})$,
$SO(n^2-1, 1, \mathrm{I\!R}) = SO(n^2-1, 1) \cap GL(n^2, \mathrm{I\!R})$,
$\mathbf{P}_0 = \mathbf{P} \cap GL(n^2, \mathrm{I\!R})$,
$\mathbf{Q}_0 = \mathbf{Q} \cap GL(n^2, \mathrm{I\!R})$.

When $n = 4$, we have three special subgroups in $SL(M_4')$, namely,

Λ: an embedding of $SL(6)/\langle -1 \rangle$ in $SL(4^2-1)$,
Λ_0: an embedding of $SU(6)/\langle -1 \rangle$ in $SU(4^2-1)$,
Λ_1: the intersection of Λ and $GL(4^2, \mathrm{I\!R})$.

Notice that Λ_0 and Λ_1 are subgroups of Λ. We refer the readers to [22] for a concrete construction and some discussion of these groups.

Finally, for $S \in SL(n)$ let $S \otimes \bar{S}$ denote the operator of the form $X \mapsto SXS^*$. With all the above notations, we are ready to state the following result [22, Theorem 2] (see also [4]).

Proposition 2.7 *Let G be a proper connected Lie subgroup of $SL(n^2)$ containing $PSU(n)$. If G is reducible, then either $G = G_1\mathbf{R}, G_1\mathbf{PR}, G_1\mathbf{QR}$, where G_1 is one of the following:*

$$PSU(n),\ PSL(n),\ SO(n^2-1, \mathrm{I\!R}),\ SO(n^2-1),$$

$$SU(n^2-1),\ SL(n^2-1, \mathrm{I\!R}),\ SL(n^2-1),$$

$$\text{or}\ \ \Lambda_0,\ \Lambda_1,\ \Lambda\ (n=4),$$

and $\mathbf{R}$ is a connected Lie subgroup of $\mathbf{T}_1$ (which may be trivial), or G is a $\mathbf{U}_1$-conjugate of one of the groups: $G_1\mathbf{P}_0, G_1\mathbf{Q}_0, G_1\mathbf{P}_0\mathbf{R}_1, G_1\mathbf{Q}_0\mathbf{R}_1$, where

$$G_1 = PSU(n),\ SO(n^2-1, \mathrm{I\!R}),\ SL(n^2-1, \mathrm{I\!R}),\quad \text{or}\ \ \Lambda_0,\ \Lambda_1\ (n=4).$$

If G is irreducible, then G is a $\mathbf{T}$-conjugate of one of the groups:

$$SL(n^2, \mathrm{I\!R}),\ SU(n^2),\ SU(n^2-1,1),\ SO(n^2),\ SO(n^2, \mathrm{I\!R}),\ SO(n^2-1,1,\mathrm{I\!R})^0,$$

$$\{S \otimes \bar{S} : S \in SL(n)\},\quad \text{or}\ \ G_2 * G_3\ \ \text{with}\ \ G_2, G_3 \in \{SL(n), SU(n)\}.$$

Denote by τ the transposition operator, i.e., $\tau(A) = A^t$, on M_n. We have the following result (see [22, Theorems 3 and 4 (ii)]).

Proposition 2.8 *Let G be a subgroup of $GL(n^2)$ containing $PSU(n)$ as the largest connected Lie subgroup. Then G is a subgroup of the group generated by $PSU(n)$, $\mathbf{T}$ and τ.*

3 The group of linear preservers of $r_\chi^\perp$

We continue to assume that $m \geq 2$ in the following discussion.

Proposition 3.1 *The linear preservers of $r_\chi^\perp$ form a group G in $GL(n^2)$ containing $PSU(n)$.*

Proof. First of all, we show that linear preservers of $r_\chi^\perp$ are invertible. Suppose it is not true and L is a singular linear preserver of $r_\chi^\perp$ on M_n. Let $A \in M_n$ be nonzero such that $L(A) = 0$, and let A have singular value decomposition $U(\sum_{j=1}^n a_j E_{jj})V$, where U and V are unitary, and $a_1 \geq \cdots \geq a_n$. Set $B = UV$. By Proposition 2.3

$$r_\chi^\perp(A+B) \geq |\det(A+B)|^{m/n} = \left\{\prod_{j=1}^n (1+a_j)\right\}^{m/n} > 1.$$

However,

$$1 = r_\chi^\perp(B) = r_\chi^\perp(L(B)) = r_\chi^\perp(L(A)+L(B)) = r_\chi^\perp(L(A+B)),$$

which is a contradiction.

Now, all linear preservers of $r_\chi^\perp$ are invertible. If L preserves $r_\chi^\perp$, one easily checks that L^{-1} also preserves $r_\chi^\perp$. It follows that the linear preservers of $r_\chi^\perp$ form a group G in $GL(n^2)$. It is clear that G contains $PSU(n)$. $\square$

By Proposition 3.1, we can apply the group scheme described in Section 2.2 to study linear preservers of $r_\chi^\perp$.

Let G_0 be the largest connected Lie group contained in G. Then G_0 must be one of the groups listed in Proposition 2.7. We shall establish a series of lemmas to eliminate most of the candidates on the list, and then conclude that $G_0 = PSU(n)$ if $m < n$. In each of the following lemmas, we consider a certain group $\tilde{G} \neq PSU(n)$ in Proposition 2.7 and show that there are some linear operators L in $\tilde{G}$ and $A \in M_n$ such that $r_\chi^\perp(A) \neq r_\chi^\perp(L(A))$, to conclude $\tilde{G} \not\subseteq G_0$.

Lemma 3.2 *G_0 does not contain Λ_0.*

Proof. Assume the contrary holds. By the discussion in [22, p.151], there exists $L \in \Lambda_0$ such that $L(I) = I$ and $L(A) = B$ with $A = \operatorname{diag}(1,1,-1,-1)$ and $B = 2E_{13}$. If χ is not the principal character, then $r_\chi^\perp(A) = 1 \neq 0 = r_\chi^\perp(B)$.

If χ is the principal character and $r_\chi^\perp(A) = r_\chi^\perp(L(A))$, Proposition 2.6 implies that

$$1 = r_\chi^\perp(A) = r_\chi^\perp(B) = (2/m)^m |H|.$$

Since $|H|$ divides $|S_m|$ and $m \leq n = 4$, it follows that $m = 2$ and $H = \{e\}$. Next, note that $I + iB$ and $I + e^{i\theta}B$ are unitarily similar, for any real number θ. Let X be obtained from

$I \in M_4$ by deleting the last two columns. Since $L(I + e^{i\theta}A) = I + e^{i\theta}B$, it follows that

$$\begin{aligned} 2 = 2^{m/2} &= r_\chi^\perp(I + iA) = r_\chi^\perp(L(I + iA)) = r_\chi^\perp(I + iB) = r_\chi^\perp(I + e^{i\theta}B) \\ &= r_\chi^\perp(L^{-1}(I + e^{i\theta}B)) = r_\chi^\perp(I + e^{i\theta}A) \\ &\geq |\det X^*(I + e^{i\theta}A)X| = 2 + 2\cos\theta, \end{aligned}$$

for all $\theta \in \mathbb{R}$, which is a contradiction. □

Lemma 3.3 *G_0 does not contain Λ_1.*

Proof. By the discussion in [22, p.152], there exists $L \in \Lambda_1$ such that $L(A) = B$ with $A = \operatorname{diag}(-1, 1, -1, 1)$ and

$$B = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{pmatrix}.$$

Since $B/\sqrt{2}$ and A are unitary matrices, $r_\chi^\perp(B) = 2^{m/2} \neq 1 = r_\chi^\perp(A)$, a contradiction. □

Lemma 3.4 *G_0 does not contain a $\mathbf{U}_1$-conjugate of $\mathbf{P}_0$.*

Proof. Suppose it does. Then there exists $U \in \mathbf{U}_1$ such that for any Hermitian matrix $C \in M_n'$, the operator L, defined by $L(A) = A + (\operatorname{tr} A)C$, is in $U^{-1}G_0U$. Then $ULU^{-1} \in G_0$. Let us assume that $U(A) = a(\operatorname{tr} A)I/n + b(A - (\operatorname{tr} A)I/n)$ for some $a, b \in \mathbb{C} \setminus \{0\}$. Then $U^{-1}(A) = a^{-1}(\operatorname{tr} A)I/n + b^{-1}(A - (\operatorname{tr} A)I/n)$ and

$$\begin{aligned} B &= ULU^{-1}(nE_{11}) \\ &= UL(a^{-1}I + b^{-1}(nE_{11} - I)) \\ &= U(a^{-1}I + b^{-1}(nE_{11} - I) + na^{-1}C) \\ &= I + b(a^{-1}I + b^{-1}(nE_{11} - I) + na^{-1}C - a^{-1}I) \\ &= nE_{11} + nba^{-1}C. \end{aligned}$$

We can choose $C \in M_n'$ such that

$$|\det(B)|^{m/n} = |\det(nE_{11} + nba^{-1}C)|^{m/n} > r_\chi^\perp(nE_{11}) = r_\chi^\perp(B),$$

contradicting Proposition 2.3. □

Lemma 3.5 *G_0 does not contain a $\mathbf{U}_1$-conjugate of $\mathbf{Q}_0$.*

Proof. Suppose it does. Then there exists $U \in \mathbf{U}_1$ such that for any Hermitian matrix $C \in M_n'$, the operator L, defined by $L(A) = A + (\mathrm{tr}\, AC)I$, is in $U^{-1}G_0U$. Then $ULU^{-1} \in G_0$. Let us assume that $U(A) = a(\mathrm{tr}\, A)I/n + b(A - (\mathrm{tr}\, A)I/n)$ for some $a, b \in \mathbb{C} \setminus \{0\}$. Then $U^{-1}(A) = a^{-1}(\mathrm{tr}\, A)I/n + b^{-1}(A - (\mathrm{tr}\, A)I/n)$ and

$$\begin{aligned} B &= ULU^{-1}(E_{12}) \\ &= UL(b^{-1}E_{12}) \\ &= U(b^{-1}E_{12} + \mathrm{tr}\,(b^{-1}E_{12}C)I) \\ &= a\mathrm{tr}\,(b^{-1}E_{12}C)I + b(b^{-1}E_{12} + \mathrm{tr}\,(b^{-1}E_{12}C)I - \mathrm{tr}\,(b^{-1}E_{12}C)I) \\ &= ab^{-1}\mathrm{tr}\,(E_{12}C)I + E_{12}. \end{aligned}$$

We can choose $C \in M_n'$ such that

$$|\det(B)|^{m/n} = |\det(ab^{-1}\mathrm{tr}\,(E_{12}C)I + E_{12})|^{m/n} > r_\chi^\perp(E_{12}) = r_\chi^\perp(B),$$

contradicting Proposition 2.3. □

Lemma 3.6 *G_0 does not contain a $\mathbf{T}$-conjugate of $\{S \otimes \bar{S} : S \in SL(n)\}$.*

Proof. Suppose that there does exists $T \in \mathbf{T}$ such that $T(X) = a(\mathrm{tr}\, X)I/n + b(X - (\mathrm{tr}\, X)I/n)$ and $T^{-1}G_0T = \{S \otimes \bar{S} : S \in SL(n)\}$. We may assume that $b = 1$; otherwise, replace T by T/b.

Let $A = E_{12} + \cdots + E_{n-1,n} + E_{n,1}$ and let $S = tI_{n-1} \oplus [t^{1-n}]$ for some $t > 1$. Then the operator $X \mapsto T(ST^{-1}(X)S^*)$ belongs to G_0. In particular, if

$$B = T(S(T^{-1}(A))S^*) = t^2E_{12} + \cdots + t^2E_{n-2,n-1} + t^{2-n}E_{n-1,n} + t^{2-n}E_{n,1},$$

then $r_\chi^\perp(A) = r_\chi^\perp(B)$. Since B is not a multiple of a unitary matrix, by Proposition 2.3,

$$r_\chi^\perp(A) = 1 = |\det(B)|^{m/n} < r_\chi^\perp(B),$$

a contradiction.

Lemma 3.7 *G_0 does not contain $PSL(n)$.*

Proof. If it does, then $r_\chi^\perp(A) = r_\chi^\perp(S^{-1}AS)$ for any $A \in M_n$ and any invertible $S \in M_n$. Thus, all matrices with the same set of n distinct eigenvalues will have the same $r_\chi^\perp$. Let w be a primitive root of the equation $x^n - 1 = 0$ and let $A = \mathrm{diag}\,(1, w, \ldots, w^{n-1})$. There exists an operator $L \in PSL(n)$ such that

$$L(A) = B = A + E_{12} + \cdots + E_{n-2,n-1} + E_{n-1,n}.$$

By Proposition 2.3, it follows that

$$r_\chi^\perp(B) > |\det(B)| = 1 = r_\chi^\perp(A),$$

a contradiction.

□

Lemma 3.8 *Suppose $n \geq 3$. Then G_0 does not contain $SO(n^2-1, \mathbb{R})$.*

Proof. Assume the contrary holds. Let $A = \sqrt{n}E_{12}$, $B = E_{12} + \cdots + E_{n-1,n} + E_{n,1}$, and $C = \sqrt{n/m}(E_{12} + \cdots + E_{m-1,m} + E_{m,1})$ if $m > 2$. Then A, B and C have the same Frobenius norm. Suppose $A = A_1 + iA_2$, $B = B_1 + iB_2$, and $C = C_1 + iC_2$ are the Hermitian decomposition of the three matrices. One readily checks that $A_1, A_2, B_1, B_2, C_1, C_2$ are trace zero Hermitian matrices with the same Frobenius norm and $\langle A_1, A_2\rangle = \langle B_1, B_2\rangle = \langle C_1, C_2\rangle = 0$. Thus, there exists $L \in SO(n^2-1, \mathbb{R})$ with $L(A_1) = B_1$ and $L(A_2) = B_2$. Hence, we have $L(A) = B$. Similarly, there exists $\hat{L} \in SO(n^2-1, \mathbb{R})$ with $\hat{L}(B) = C$ if $m > 2$.

Suppose χ is not the principal character. Then by Propositions 2.1 and 2.3,

$$r_\chi^\perp(A) = 0 < 1 = r_\chi^\perp(B),$$

which is a contradiction.

Next, suppose χ is the principal character. If $n > m > 2$, then by Proposition 2.3,

$$r_\chi^\perp(B) = 1 < (\sqrt{n/m})^m = r_\varepsilon^\perp(C) \leq r_\chi^\perp(C),$$

which is a contradiction.

If $m = 2$, then $|H| = 1$ or 2. Since $n \geq 3$, by Propositions 2.3 and 2.6

$$(\sqrt{n})^m |H| m^{-m} = r_\chi^\perp(A) = r_\chi^\perp(B) = 1.$$

It follows that $n = 4$ and $|H| = 1$. But then there exists $\tilde{L} \in SO(n^2-1, \mathbb{R})$ such that $\tilde{L}(E_{11} - E_{22}) = (E_{11} + E_{22} - E_{33} - E_{44})/\sqrt{2}$. However,

$$r_\chi^\perp((E_{11} + E_{22} - E_{33} - E_{44})/\sqrt{2}) = 1/2 < 1 = r_\chi^\perp(E_{11} - E_{22}),$$

which is a contradiction.

Finally, if $n = m > 2$, then Proposition 2.6 implies that $r_\chi^\perp(A) = |H|n^{-n/2}$. On the other hand, $L(A) = B$ is unitary, hence $r_\chi^\perp(L(A)) = 1$. Thus, it follows that $|H| = n^{n/2}$. Since H is a subgroup of S_n, we have $|H| = n^{n/2}$ divides $n!$, which is impossible by the following arguments. Let p be a prime number and let $a \geq 1$ be an integer such that p^a divides n while p^{a+1} does not divide n. Then $p^{an/2}$ divides $n^{n/2}$. The exponent of p in the prime factorization decomposition of $n!$ is strictly less than

$$\sum_{k=1}^{\infty} \frac{n}{p^k} = \frac{n}{p-1}\,.$$

Thus, if $n^{n/2}$ divides $n!$, then $an/2 < n/(p-1)$, hence $1 \leq a < 2/(p-1)$. This can happen only if $p = 2$ and $a = 1$. It follows that $n = 2$, a contradiction. $\square$

Lemma 3.9 *Suppose $n > m$. Then G_0 does not contain a $\mathbf{T}$-conjugate of $SU(n) * SU(n)$.*

Proof. Suppose it does and $T \in \mathbf{T}$ is such that $SU(n) * SU(n) < T^{-1}G_0T$. Let $A = E_{12} + \cdots + E_{n-2,n-1} + E_{n-1,1}$. Then A is unitarily similar to $\operatorname{diag}(0, 1, w, w^2, \ldots, w^{n-2})$, where $w = e^{2\pi i/(n-1)}$. By Proposition 2.4, $r_\chi^\perp(A) = 1$. Note that there exist permutation matrices P, Q such that $B = PAQ = E_{12} + \cdots + E_{n-2,n-1} + E_{n-1,n}$. Since $B^n = 0 \neq B^{n-1}$, the matrix B is not unitarily similar to a direct sum of matrices of smaller sizes. By Proposition 2.4,

$$r_\chi^\perp(B) < \|B\|^m = 1.$$

However, if L is defined by $L(X) = T(P(T^{-1}(X))Q)$, then $L \in G_0$ and $L(A) = B$. Since $r_\chi^\perp(B) < 1 = r_\chi^\perp(A)$, we get a contradiction. $\square$

Lemma 3.10 *Suppose $n > m$. Then $G \cap \mathbf{T}$ is the circle group, i.e., group of operators of the form $A \mapsto aA$, where $a \in \mathbb{C}$ satisfies $|a| = 1$.*

Proof. Suppose $L \in \mathbf{T}$ is a linear preserver of $r_\chi^\perp$ such that

$$L(X) = a(\operatorname{tr} X)I/n + b(X - (\operatorname{tr} X)I/n), \quad a, b \in \mathbb{C}.$$

Since $r_\chi(I) = r_\chi(L(I))$, we conclude that $|a| = 1$. We may assume that $a = 1$ and hence $L(I) = I$. Otherwise, replace L by $a^{-1}L$.

If χ is not the principal character, we have

$$L(E_{11}) = bE_{11} + (1 - b)I/n.$$

By Proposition 2.1, we have

$$0 = r_\chi^\perp(E_{11}) = r_\chi^\perp(L(E_{11})) \geq |(1 - b)/n|^m.$$

Thus $b = 1$.

If χ is the principal character, we must have $|b| = 1$ since $L(E_{12}) = bE_{12}$. Let $b = b_1 + ib_2$ with $b_1, b_2 \in \mathbb{R}$. Note that

$$L(tI \pm iE_{11}) = [t \pm i(1 - b)/n]I \pm biE_{11}$$

By [3], and for $t > 0$, we have

$$(t^2 + 1)^{m/2} = r_\chi^\perp(\sqrt{t^2 + 1}I) \geq r_\chi^\perp(\operatorname{diag}(\sqrt{t^2 + 1}, t, \ldots, t)) \geq r_\chi^\perp(tI \pm iE_{11})$$

It follows that

$$(t^2 + 1)^{m/2} \geq r_\chi^\perp(tI \pm iE_{11}) = r_\chi^\perp(L(tI \pm iE_{11})) \geq |t \pm i(1 - b)/n|^m$$

Thus, we obtain

$$(t^2 + 1)^m \geq (t \pm b_2/n)^{2m}$$

and, letting t tends to infinity, we see that this is possible only if $b_2 = 0$. Thus, $b = \pm 1$.

Suppose that $b = -1$. If n is odd, say $n = 2k+1$, let $A = -iI_k \oplus iI_k \oplus [1]$. Then $L(A) = (i + 2/n)I_k \oplus (-i + 2/n)I_k \oplus [-1 + 2/n]$. By Proposition 2.4 we have

$$r_\chi^\perp(A) = 1 < \|L(A)\|^m = r_\chi^\perp(L(A)),$$

a contradiction.

Suppose that $b = -1$ and n is even, say $n = 2k$. Let

$$A_t = -iI_{k-1} \oplus iI_{k-1} \oplus \operatorname{diag}(t + i\sqrt{1-t^2}, t - i\sqrt{1-t^2}),$$

where $t \in (0,1)$. Then $r_\chi^\perp(A_t) = 1$, $\operatorname{tr} A_t = 2t$ and

$$L(A_t) = \frac{2t}{k} I - A_t.$$

If $m \leq n-2$, then it follows from Proposition 2.4 that

$$r_\chi^\perp(A_t) = 1 < \{1 + 4t^2/k^2\}^{m/2} = \|L(A_t)\|^m = r_\chi^\perp(L(A_t)),$$

a contradiction.

Suppose that $m = n-1$. If X is obtained from I by deleting the last column, we have

$$\begin{aligned} r_\chi^\perp(L(A_t)) &\geq |\det(X^* L(A_t) X)| \\ &= \left(1 + \frac{4t^2}{k^2}\right)^{k-1} \left(1 - t^2 + \left(\frac{2t}{k} - t\right)^2\right)^{1/2} \\ &= \left(1 + \frac{4t^2}{k^2}\right)^{k-1} \left(1 - \frac{4t^2(k-1)}{k^2}\right)^{1/2} \end{aligned}$$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\varphi(y) = \left(1 + \frac{y}{k^2}\right)^{2k-2} \left(1 - \frac{y(k-1)}{k^2}\right)$$

Then the above inequality implies that

$$\left(r_\chi^\perp(L(A_t))\right)^2 \geq \varphi(4t^2)$$

and it is easy to see that

$$\varphi(0) = 1 \quad \text{and} \quad \varphi'(0) = \frac{k-1}{k^2} > 0$$

Therefore, if $t > 0$ is sufficiently small, then

$$\left(r_\chi^\perp(L(A_t))\right)^2 \geq \varphi(4t^2) > 1 = r_\chi^\perp(A_t)^2,$$

a contradiction. □

4 Characterization theorems

We continue to focus on the cases when $m \geq 2$.

Theorem 4.1 *Let $2 \leq m < n$. A linear operator L on M_n preserves $r_\chi^\perp$ if and only if there exist a unitary $U \in M_n$ and $\xi \in \mathbb{C}$ with $|\xi| = 1$ such that L is of the form*

$$A \mapsto \xi U^* A U \qquad or \qquad A \mapsto \xi U^* A^t U.$$

Proof. The ($\Leftarrow$) part of the theorem is clear. We consider the ($\Rightarrow$) part. By Proposition 3.1, linear preservers of $r_\chi^\perp$ form a group G. By Lemmas 3.2 and 3.3, we see that the largest connected Lie group G_0 in G cannot be $\Lambda_0, \Lambda_1, \Lambda$. By Lemmas 3.4 and 3.5, G_0 cannot contain any overgroups of $\mathbf{P}_0$ and $\mathbf{Q}_0$. By Lemmas 3.6 and 3.7, G_0 cannot contain any overgroups of $PSL(n)$ or $\{S \otimes \bar{S} : S \in SL(n)\}$. By Lemma 3.8, all the overgroups of $SO(n^2-1, \mathbb{R})$ are ruled out. By Lemma 3.9, G_0 cannot contain $G_2 * G_3$ with $G_2, G_3 \in \{SL(n), SU(n)\}$. Furthermore, by Lemma 3.10, G_0 cannot contain $\mathbf{R}$ or $\mathbf{R}_1$. By Propositions 2.7 and 2.8, we see that G is a subgroup of the group in $GL(n^2)$ generated by $PSU(n)$, the transposition operator τ, and a subgroup of $\mathbf{T}$. Clearly, $\tau \in G$ and the circle group is inside G. By Lemma 3.10 again, there are no other subgroup of $\mathbf{T}$ lying inside G. The result follows. $\square$

The above theorem shows that a linear preserver of $r_\chi^\perp$ is a unit multiple of a linear preserver of $W_\chi^\perp$ when $m < n$. We believe that the same is true even for $m = n$. This is known when $\chi = \varepsilon$ on S_n. Furthermore, by the result in Section 3.1 we have the following.

Proposition 4.2 *Let $2 \leq m = n$ and $\chi \neq \varepsilon$ (on S_m). Then the linear preservers of $r_\chi^\perp$ form a group G in $GL(n^2)$. Moreover, G does not contain Λ_0, Λ_1, a $\mathbf{T}$-conjugate of $\{S \otimes \bar{S} : S \in SL(n)\}$, a $\mathbf{U}_1$-conjugate of $\mathbf{P}_0$, a $\mathbf{U}_1$-conjugate of $\mathbf{Q}_0$, $PSL(n)$, and, when $n > 2$, G does not contain $SO(n^2-1, \mathbb{R})$.*

When $m = n = 2$, $|H| = 1$ or 2, and one needs only to consider the principal character χ. We have the following result.

Proposition 4.3 *Suppose $m = n = 2$ and χ is the principal character. A linear operator L on M_2 preserves $r_\chi^\perp$ if and only if there exist a unitary matrix $U \in M_2$ and $\xi \in \mathbb{C}$ with $|\xi| = 1$ such that one of the following holds.*
(i) *L is of the form $A \mapsto \xi U^* A U$ or $A \mapsto \xi U^* A^t U$.*
(ii) *$H = S_2$ and L is of the form*

$$A \mapsto \xi[U^* A U + (\pm i - 1)(\operatorname{tr} A) I/2] \qquad or \qquad A \mapsto \xi[U^* A^t U + (\pm i - 1)(\operatorname{tr} A) I/2].$$

Proof. By Proposition 4.2, we know that the linear preservers of $r_\chi^\perp$ form a group G. Let G_0 be the largest connected Lie group contained in G. We show that G_0 cannot contain the groups $SO(2^2, \mathbb{R})$ or $SU(2) * SU(2)$.

First, consider the case $H = \{e\}$. To show that G_0 does not contains $SO(2^2, \mathbb{R})$ or $SU(2) * SU(2)$, let $A = 2E_{11} + E_{22}$ and $B = 2E_{11} - E_{22}$. Then there exists an operator $L \in SO(2^2, \mathbb{R}) \cap SU(2) * SU(2)$ such that $L(A) = B$. By Proposition 2.5, we have $r_\chi^\perp(A) = 9/4 > 2 = r_\chi^\perp(B)$.

Next, consider the case $H = S_2$. If G_0 contains $SO(2^2, \mathbb{R})$, let $A = 2I + i\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. There exists $L \in SO(2^2, \mathbb{R})$ such that $L(A) = \sqrt{8}E_{11} + i\sqrt{2}E_{22}$. By Proposition 2.5, we have $r_\chi^\perp(A) = \sqrt{5} < 4 = r_\chi^\perp(L(A))$, a contradiction.

Now, suppose G_0 contains $SU(2) * SU(2)$. Then for any matrix $A \in M_2$, we have $r_\chi^\perp(A) = r_\chi^\perp(s_1E_{11} + s_2E_{22})$, where $s_1 \geq s_2 \geq 0$ are the singular values of A. Thus, using Proposition 2.5, we get $r_\chi^\perp(A) = (s_1^2 + s_2^2)/2 = \|A\|/2$. It follows that G_0 contains $SU(2^2)$, the group of linear preservers of $\|\cdot\|$. But then, $SO(2^2, \mathbb{R}) \subseteq SU(2^2) \subseteq G$, which is impossible by the result in the last paragraph.

Finally, suppose $L \in G \cap \mathbf{T}$ is such that $L(X) = (a-b)(\mathrm{tr}\, X)I/2 + bX$ with $|a| = |b| = 1$. Assume that $b = 1$. Otherwise, replace L by L/b. If $A_t = \begin{pmatrix} t & 1 \\ 1 & t \end{pmatrix}$, then $r_\chi^\perp(A_t) = r_\chi^\perp(L(A_t))$ for all $t > 0$. One easily sees that $a = 1$ if $H = \{e\}$ and $a = 1 \pm i$ if $H = S_2$. The result follows. □

Remark 4.4 By Proposition 4.2, when $m = n > 2$, proving that a linear preserver of $r_\chi^\perp$ is a unit multiple of a linear preserver of $W_\chi^\perp$ reduces to the following problems.

Problem 4.5 *If $m = n$ and $\chi \neq \varepsilon$, show that G_0 does not contain a $\mathbf{T}$-conjugate of $SU(n) * SU(n)$, i.e., extending Lemma 3.9.*

Problem 4.6 *If $m = n$, show that $G \cap \mathbf{T}$ is the circle group, i.e., extending Lemma 3.10.*

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References

[1] C.F. Chan, Some more on a conjecture of Marcus and Wang, Linear and Multilinear Algebra 25 (1989), 231-235.

[2] C.F. Chan, The generalized numerical radius associated with a positive semi-definite function, Linear and Multilinear Algebra 32 (1992), 265-281.

[3] C.F. Chan, The generalized numerical radius associated with a positive semi-definite function II, Linear and Multilinear Algebra 34 (1993), 247-259.

[4] R.M. Guralnick and C.K. Li, Invertible preservers and algebraic groups III: Preservers of unitary similarity (congruence) invariants and overgroups of some unitary groups, Linear and Multilinear Algebra 43 (1997), 257-282.

[5] G. Frobenius, Uber die Darstellung der endichen Gruppen durch Linear Substitutionen, S.B. Deutsch. Akad. Wiss. Berlin (1897), 994-1015.

[6] K.R. Gustafson and D.K.M. Rao, Numerical range: The field of values of linear operators and matrices, Universitext, Springer-Verlag, New York, 1997.

[7] P.R. Halmos, A Hilbert Space Problem Book, Second Ed., Springer-Verlag, New York, 1982.

[8] R.A. Horn and C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1991.

[9] S. A. Hu and T. Y. Tam, Operators with permanental numerical ranges on a straight line, Linear and Multilinear Algebra 29 (1991), 263-277.

[10] S. A. Hu and T. Y. Tam, On the generalized numerical ranges with principal character, Linear and Multilinear Algebra 30 (1991), 93-107.

[11] V. Istrăţescu, Introduction to Linear Operator Theory, Marcel Dekker, New York, 1981.

[12] C.K. Li, The decomposable numerical radius and numerical radius of a compound matrix, Linear Algebra Appl. 76 (1986), 45-58.

[13] C.K. Li, Linear operators preserving the numerical radius of matrices, Proc. Amer. Math. Soc. 99 (1987), 601-608.

[14] C.K. Li and N.K. Tsing, Linear operators preserving the decomposable numerical radius, Linear and Multilinear Algebra 23 (1988), 333-341.

[15] C.K. Li and A. Zaharia, Decomposable numerical range on orthonormal decomposable tensors, Linear Algebra Appl. 308 (2000), 139-152.

[16] M. Marcus, Finite Dimensional Multilinear Algebra, Part I, Marcel Dekker, New York, 1973.

[17] M. Marcus and I. Filippenko, Linear operators preserving the decomposable numerical range, Linear and Multilinear Algebra 7 (1979), 27-36.

[18] M. Marcus and M. Sandy, Conditions for the generalized numerical range to be real, Linear Algebra Appl. 71 (1985), 219-239.

[19] M. Marcus and B. Wang, Some variations on the numerical range, Linear and Multilinear Algebra 9 (1980), 111-120.

[20] V.J. Pellegrini, Numerical range preserving operators on a Banach algebra, Studia Math. 54 (1975), 143-147.

[21] S. Pierce et al, A survey of Linear Preserver Problems, Linear and Multilinear Algebra 33 (1992), 1-130.

[22] V. P. Platonov and D. Z. Doković, Subgroups of $GL(n^2, \mathbb{C})$ containing $PSU(n)$, Trans. Amer. Math. Soc. 348 (1996), 141-152.

[23] T.Y. Tam, Linear operator on matrices: the invariance of the decomposable numerical range, Linear Algebra Appl 85 (1987), 1-7.

[24] T.Y. Tam, Linear operator on matrices: the invariance of the decomposable numerical radius, Linear Algebra Appl 87 (1987), 147-154.

[25] T.Y. Tam, Linear operator on matrices: the invariance of the decomposable numerical range. II, Linear Algebra Appl 92 (1987), 197-202.