Maps preserving the joint numerical radius distance of operators

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Dedicated to Professor David Lutzer on the occasion of his retirement.

Abstract

Denote the joint numerical radius of an m-tuple of bounded operators $\mathbf{A} = (A_1, \ldots, A_m)$ by $w(\mathbf{A})$. We give a complete description of maps f satisfying $w(\mathbf{A} - \mathbf{B}) = w(f(\mathbf{A}) - f(\mathbf{B}))$ for any two m-tuples of operators $\mathbf{A} = (A_1, \ldots, A_m)$ and $\mathbf{B} = (B_1, \ldots, B_m)$. We also characterize linear isometries for the joint numerical radius, and maps preserving the joint numerical range of $\mathbf{A}$.

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1 Introduction

Let $\mathcal{B}(\mathcal{H})$ be the set of bounded linear operators acting on the Hilbert space $\mathcal{H}$ equipped with the inner product (x, y), and let $\mathcal{S}(\mathcal{H})$ be the set of self-adjoint operators in $\mathcal{B}(\mathcal{H})$. In this paper we assume $\mathcal{H}$ has finite dimension $n > 1$, and identify $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$, and $\mathcal{S}(\mathcal{H})$ with the space $\mathbf{C}^n$ of $n \times 1$ complex vectors, the set of $n \times n$ complex matrices M_n, and the set of Hermitian matrices H_n, respectively. Let $\mathcal{V}$ be $\mathcal{B}(\mathcal{H})$ or $\mathcal{S}(\mathcal{H})$. For $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{V}^m$ and any vector $x \in \mathcal{H}$ let

$$(\mathbf{A}x, x) = ((A_1 x, x), \ldots, (A_m x, x)).$$

Define the *joint numerical range* of $\mathbf{A} \in \mathcal{V}^m$ by

$$W(\mathbf{A}) = \{(\mathbf{A}x, x) : x \in \mathcal{H},\ (x, x) = 1\}$$

and the *joint numerical radius* of $\mathbf{A}$ by

$$w(\mathbf{A}) = \sup\{\ell_2(a_1, \ldots, a_m) : (a_1, \ldots, a_m) \in W(A_1, \ldots, A_m)\},$$

where $\ell_2(x_1, \ldots, x_m) = \left(\sum_{j=1}^m |x_j|^2\right)^{1/2}$ is the usual Euclidean norm.

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The joint numerical range is a generalization of the classical numerical range of $A \in \mathcal{B}(\mathcal{H})$ defined by

$$W(A) = \{(Ax, x) : x \in \mathcal{H},\ (x, x) = 1\}$$

and the joint numerical radius is a generalization of the classical numerical radius of $A \in \mathcal{B}(\mathcal{H})$ defined by

$$w(A) = \sup\{|z| : z \in W(A)\}.$$

These concepts are useful in studying the joint behaviors of several operators, and have been studied extensively; see for example [1, 4, 6, 9, 11] and their references.

The joint numerical radius, like its classical counterpart, is a norm, and as such its isometries are of interest. In Section 2, we characterize linear isometries $f : \mathcal{V}^m \to \mathcal{V}^m$ such that

$$w(\mathbf{A}) = w(f(\mathbf{A})) \quad \text{for all } \mathbf{A} \in \mathcal{V}^m.$$

Using this result, we characterize distance-preserving maps $f : \mathcal{V}^m \to \mathcal{V}^m$ (without the linearity assumption) such that

$$w(\mathbf{A} - \mathbf{B}) = w(f(\mathbf{A}) - f(\mathbf{B})) \qquad \text{for all } A, B \in \mathcal{V}^m.$$

From this, we derive a number of related results, including characterizations of additive isometries and of maps preserving the joint numerical range.

Moreover, for certain other classes of norms ν on $\mathbf{F}^m$ (where $\mathbf{F}$ is $\mathbf{R}$ or $\mathbf{C}$), we can extend our results to the ν-joint numerical radius of $\mathbf{A} \in \mathcal{V}^m$ defined by

$$w_\nu(\mathbf{A}) = \sup\{\nu(a_1, \dots, a_m) : (a_1, \dots, a_m) \in W(A_1, \dots, A_m)\}.$$

In Section 3, we consider a fairly wide class of norms on $\mathbf{F}^m$ which includes smooth norms; in Section 4, we investigate the case of oft-used symmetric norms.

2 Maps preserving the joint numerical radius distance

We first prove the result for linear isometries.

Theorem 2.1. *Let $(\mathcal{V}, \mathbf{F}) = (\mathcal{S}(\mathcal{H}), \mathbf{R})$ or $(\mathcal{B}(\mathcal{H}), \mathbf{C})$. A $\mathbf{F}$-linear map $f : \mathcal{V}^m \to \mathcal{V}^m$ satisfies*

$$w(\mathbf{A}) = w(f(\mathbf{A})) \qquad \text{for all } \mathbf{A} \in \mathcal{V}^m$$

if and only if there is a unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a linear isometry $\Gamma = (\gamma_{ij}) \in M_m(\mathbf{F})$ (that is, $\ell_2(\Gamma u) = \ell_2(u)$ for all $u \in \mathbf{F}^m$) such that f has the form

$$(A_1, \dots, A_m) \mapsto \left(\sum_{j=1}^m \gamma_{1j} U^* \psi(A_j) U, \dots, \sum_{j=1}^m \gamma_{mj} U^* \psi(A_j) U \right), \tag{1}$$

with ψ taking the form $X \mapsto X$ or $X \mapsto X^t$, where X^t is the transpose of X with respect to a fixed orthonormal basis.

We shall need the following two lemmas to prove this theorem. It will be convenient to introduce some notation. Given $X \in \mathcal{V}$ and $c = (c_1, \ldots, c_m) \in \mathbf{F}^m$, we let $c \otimes X = (c_1 X, \ldots, c_m X) \in \mathcal{V}^m$. More generally, if $\mathcal{A} \subseteq \mathbf{F}^m$ and $\mathcal{B} \subseteq \mathcal{V}$, then $\mathcal{A} \otimes \mathcal{B} = \{a \otimes b : a \in \mathcal{A}, b \in \mathcal{B}\}$. We let $\mathcal{P}$ denote the set of orthogonal rank one projections in $\mathcal{B}(\mathcal{H})$.

Lemma 2.2. *Let $(\mathcal{V}, \mathbf{F}) = (\mathcal{S}(\mathcal{H}), \mathbf{R})$ or $(\mathcal{B}(\mathcal{H}), \mathbf{C})$. Let e be a nonzero vector in $E \subseteq \mathbf{F}^m$. Suppose $g : \mathcal{V}^m \to \mathcal{V}^m$ is an injective $\mathbf{F}$-linear map such that $g(e \otimes \mathcal{P}) \subseteq E \otimes \mathcal{P}$. Then either $g(e \otimes \mathcal{V}) = \hat{e} \otimes \mathcal{V}$ for some $\hat{e} \in E$ or $g(e \otimes \mathcal{V}) \subseteq \mathbf{F}^m \otimes \hat{P}$ for some $\hat{P} \in \mathcal{P}$.*

Proof. Let $e \in E$ and let $x \in \mathcal{H}$ be a unit vector. Write $g(e \otimes xx^*) = \hat{e} \otimes \hat{x}\hat{x}^*$. Let y be any unit vector orthogonal to x and write $x(t) = (\cos t)x + (\sin t)y$. We see that

$$g(e \otimes x(t)x(t)^*) = (\cos^2 t)g(e \otimes xx^*) + (\cos t \sin t)g(e \otimes (xy^* + yx^*)) + (\sin^2 t)g(e \otimes yy^*).$$

Note that $xy^* + yx^* = [(x + y)(x + y)^* - (x - y)(x - y)^*]/2$. Thus,

$$g(e \otimes (xy^* + yx^*)) = g(e \otimes (x + y)(x + y)^*/2) - g(e \otimes (x - y)(x - y)^*/2) = a \otimes uu^* - b \otimes vv^*$$

and

$$g(e \otimes yy^*) = c \otimes ww^*$$

for some unit vectors $u, v, w \in \mathcal{H}$ and nonzero vectors $a, b, c \in E$. As a result,

$$\begin{aligned} g(e \otimes x(t)x(t)^*) &= (\cos^2 t)\hat{e} \otimes \hat{x}\hat{x}^* + (\sin^2 t)c \otimes ww^* + (\cos t \sin t)[a \otimes uu^* - b \otimes vv^*] \\ &= d(t) \otimes z(t)z(t)^* \quad (2) \end{aligned}$$

for some unit vector $z(t) \in \mathcal{H}$ and nonzero vector $d(t) \in E$. Choose an orthonormal basis for $\mathbf{F}^m$ such that $\hat{e} = (\gamma, 0, \ldots, 0)$ for some nonzero $\gamma \in \mathbf{F}$. Note that, with respect to this basis, $d_j(t + \pi) = d_j(t)$ and $d_j(0) = 0$ for all $j > 1$. There are two cases:

a) $d_j(t) = 0$ for all t and all $j > 1$.

b) $d_j(t_0) \neq 0$ for some $j > 1$ and some $t_0 \in (0, \pi)$.

For the latter case, we may suppose, without loss of generality, that $j = 2$ and let $Z = a_2 uu^* - b_2 vv^*$. Consider the 2nd coordinate of (2):

$$d_2(t)z(t)z(t)^* = c_2(\sin^2 t)ww^* + (\sin t \cos t)Z.$$

There are three possibilities:

1. rank $Z = 0$: Since $d_2(t_0) \neq 0$, $c_2 \neq 0$, whence $d_2(t) \neq 0$ for all $t \in (0, \pi)$. Thus $z(t)z(t)^* = ww^*$ for all $t \in (0, \pi)$.

2. rank $Z = 1$: Either $c_2 = 0$ and $z(t)z(t)^* = Z/\|Z\|$ for $t \neq \pi/2$, or $c_2 \neq 0$ and $Z = kww^*$ for some $k \neq 0$ (since the right side must have rank at most one). In the latter case, $z(t)z(t)^* = ww^*$ whenever $d_2(t) \neq 0$, i.e., when $\cot t \neq -c_2/k$. In both cases, $z(t)z(t)^* = Z/\|Z\|$ whenever $\cot t \neq -c_2/\|Z\|$.

3. rank $Z = 2$: This is not possible. If it was, Z would have a 2×2 compression $\hat{Z}$ of rank 2. Let $\hat{W}$ denote the corresponding 2×2 compression of $c_2 ww^*$. Then $(\sin^2 t)\hat{W} + (\sin t \cos t)\hat{Z}$ has rank 2 for sufficiently small nonzero t, contradicting its equality with a compression of $d_2(t)z(t)z(t)^*$.

Now consider the 1st coordinate of (2),

$$d_1(t)z(t)z(t)^* = \gamma(\cos^2 t)\hat{x}\hat{x}^* + (\sin^2 t)c_1 ww^* + (\sin t \cos t)(a_1 uu^* - b_1 vv^*),$$

and take the limit as $t \to 0+$ of both sides. If rank $Z = 0$, then $z(t)z(t)^* = ww^* = \hat{x}\hat{x}^*$ for all $t \in (0, \pi)$. If rank $Z = 1$, then $z(t)z(t)^* = Z/\|Z\| = \hat{x}\hat{x}^*$ when $\cot t \neq -c_2/\|Z\|$. By continuity, $z(t)z(t)^* = \hat{x}\hat{x}^*$ for all $t \in [0, \pi]$.

Thus we may conclude that either $g(e \otimes x(t)x(t)^*) = d(t) \otimes \hat{x}\hat{x}^*$ for all t in case (b), or else $g(e \otimes x(t)x(t)^*) = \alpha(t)\hat{e} \otimes z(t)z(t)^*$ for all t in case (a), where $\alpha(t)$ is an $\mathbf{F}$-valued function.

Write $P = xx^*$ and $\hat{P} = \hat{x}\hat{x}^*$. We see that for any $Q \in \mathcal{P}$, either $g(e \otimes Q) = \hat{e} \otimes \alpha R$ for some $R \in \mathcal{P}$ and $\alpha \in \mathbf{F}$, or $g(e \otimes Q) = d \otimes \hat{P}$ for some $d \in E$. Since $\mathcal{P} \setminus \{P\}$ is path-connected, so is $g(e \otimes (\mathcal{P} \setminus \{P\})) = \mathcal{A} \otimes \hat{P} \cup \hat{e} \otimes \mathcal{B}$, where $\mathcal{A} \subseteq E$, $\mathcal{B} \subseteq \mathbf{F}\mathcal{P}$. Since g is injective and no two elements of $\mathcal{P}$ are linearly dependent, $\mathbf{F}\hat{e} \notin \mathcal{A}$ and $\mathbf{F}\hat{P} \notin \mathcal{B}$, so one of $\mathcal{A}$, $\mathcal{B}$ is empty to ensure path-connectedness. It follows that $g(e \otimes \mathcal{P}) \subseteq \hat{e} \otimes \mathbf{F}\mathcal{P}$ or $g(e \otimes \mathcal{P}) \subseteq E \otimes \hat{P}$, whence $g(e \otimes \mathcal{V}) \subseteq \hat{e} \otimes \mathcal{V}$ or $g(e \otimes \mathcal{V}) \subseteq \mathbf{F}^m \otimes \hat{P}$ by linearity. In the former case, by comparing dimensions and using the injectivity of g, the set inclusion must be an equality. □

Lemma 2.3. *Let $(\mathcal{V}, \mathbf{F}) = (\mathcal{S}(\mathcal{H}), \mathbf{R})$ or $(\mathcal{B}(\mathcal{H}), \mathbf{C})$. Suppose $g : \mathcal{V}^m \to \mathcal{V}^m$ is a bijective $\mathbf{F}$-linear map such that*

a) $g(E \otimes \mathcal{P}) = E \otimes \mathcal{P}$ for some nonempty $E \subseteq \mathbf{F}^m$ such that if $v \in E$ and $|\lambda| \neq 1$, $\lambda v \notin E$,

b) $g(\mathbf{F}^m \otimes \mathcal{P}) \subseteq \mathbf{F}^m \otimes \mathcal{P}$, and

c) there exists $P \in \mathcal{P}$ such that $g(r \otimes P) = r \otimes P$ for all $r \in \mathbf{F}^m$.

Then there exists a unitary $U \in \mathcal{B}(\mathcal{H})$ such that

$$g(A_1, \ldots, A_m) = (U^*\psi(A_1)U, \ldots, U^*\psi(A_m)U)$$

for all $(A_1, \ldots, A_m) \in \mathcal{V}^m$, with ψ taking the form $X \mapsto X$ or $X \mapsto X^t$.

Proof. Let e be any nonzero vector in $\mathbf{F}^m$. Applying Lemma 2.2 with $E = \mathbf{F}^m$, we see that $g(e \otimes \mathcal{V}) = \hat{e} \otimes \mathcal{V}$ for some nonzero $\hat{e} \in \mathbf{F}^m$ or $g(e \otimes \mathcal{V}) \subseteq \mathbf{F}^m \otimes \hat{P}$ for some $\hat{P} \in \mathcal{P}$. Suppose the latter case occurs. Since $g(e \otimes P) = e \otimes P$ by hypothesis, we have $\hat{P} = P$. But then for any projection $Q \neq P$ we have $g(e \otimes Q) = r \otimes P = g(r \otimes P)$ for some $r \in \mathbf{F}^m$, whence $e \otimes Q = r \otimes P$ by the injectivity of g, a contradiction. Thus $g(e \otimes \mathcal{V}) = \hat{e} \otimes \mathcal{V}$; since $g(e \otimes P) = e \otimes P$, we have $\hat{e} = e$. Hence $g(e \otimes \mathcal{V}) = e \otimes \mathcal{V}$ for all $e \in \mathbf{F}^m$.

Writing e_j for the jth row of the identity matrix I_m, we see there exist $\mathbf{F}$-linear maps $\phi_j : \mathcal{V} \to \mathcal{V}$, $1 \leq j \leq m$, so that $g(e_j \otimes A) = e_j \otimes \phi_j(A)$, whence

$$g(A_1, \ldots, A_m) = (\phi_1(A_1), \ldots, \phi_m(A_m)).$$

Let $e = e_1 + \cdots + e_m$; then

$$(\phi_1(A), \ldots, \phi_m(A)) = g(e \otimes A) = e \otimes B$$

for some $B \in \mathcal{V}$ since $g(e \otimes \mathcal{V}) = e \otimes \mathcal{V}$. Thus $\phi_1(A) = \cdots = \phi_m(A)$ for all $A \in \mathcal{V}$, so $\phi_j = \phi$ for a common function ϕ. Now ϕ is bijective (since g is) and $\phi(\tilde{\mathcal{P}}) = \tilde{\mathcal{P}}$ by hypothesis (a), where $\tilde{\mathcal{P}} = \{\mu Q : \mu \in \mathbf{F}, |\mu| = 1, Q \in \mathcal{P}\}$ is the set of extreme points of the unit norm ball for the dual norm of the classical numerical radius. Thus ϕ^* preserves the numerical radius and has the form (see [8]) $X \mapsto \xi U^* X U$ or $X \mapsto \xi U^* X^t U$ for some unitary U and $\xi \in \mathbf{F}$ with $|\xi| = 1$. It follows that ϕ has the same form; since $\phi(P) = P$, $\xi = 1$ and the result follows. □

Proof of Theorem 2.1. Sufficiency is easy to check. For necessity, suppose $f : \mathcal{V}^m \to \mathcal{V}^m$ is a $\mathbf{F}$-linear map preserving the joint numerical radius. We define an inner product $(\mathbf{A}, \mathbf{B}) = \sum_{j=1}^m \operatorname{tr}(A_j B_j^*)$ for $\mathbf{A} = (A_1, \ldots, A_m)$, $\mathbf{B} = (B_1, \ldots, B_m)$ in $\mathcal{V}^m$ and let

$$\mathcal{E} = \{(r_1 xx^*, \ldots, r_m xx^*) : x \in \mathcal{H} \text{ and } (r_1, \ldots, r_m) \in \mathbf{F}^m \text{ are unit vectors}\}.$$

Note that $w(\mathbf{A}) = \sup\{|(\mathbf{A}, \mathbf{B})| : \mathbf{B} \in \mathcal{E}\}$, so w is the dual of a norm w^* on $\mathcal{V}^m$ whose unit norm ball is the closed convex hull of its extreme points $\mathcal{E}$. But since $\mathcal{V}^m$ is reflexive, f preserves the joint numerical radius on $\mathcal{V}^m$ if and only if it is the dual transformation of a bijective linear map g preserving the induced norm w^* on $\mathcal{V}^m$, in which case $g(\mathcal{E}) = \mathcal{E}$. We will use this condition to show that g has form (1), whence it follows that $f = g^*$ has the same form.

Fix a unit vector $x \in \mathcal{H}$. Let $\mathbf{X} = (xx^*, 0, \ldots, 0) \in \mathcal{E}$ and write

$$g(\mathbf{X}) = (s_1 yy^*, \ldots, s_m yy^*) \in \mathcal{E}.$$

Let $U \in \mathcal{B}(\mathcal{H})$ be a unitary satisfying $Uy = x$ and let $S = (s_{ij})$ be a unitary (real orthogonal if $\mathbf{F} = \mathbf{R}$) matrix whose first row is $(\bar{s}_1, \ldots, \bar{s}_m)$. Then $\tilde{g} = L_1 \circ g$ fixes $\mathbf{X}$, where

$$L_1(A_1, \ldots, A_m) = \left(\sum_{j=1}^m s_{1j} U A_j U^*, \ldots, \sum_{j=1}^m s_{mj} U A_j U^* \right).$$

Now consider $\hat{\mathbf{X}} = (0, r_2 xx^*, \ldots, r_m xx^*) \in \mathcal{E}$ and write $\tilde{g}(\hat{\mathbf{X}}) = (t_1 zz^*, \ldots, t_m zz^*) \in \mathcal{E}$. Since $a\mathbf{X} + b\hat{\mathbf{X}} \in \mathcal{E}$ for any unit vector $(a, b) \in \mathbf{R}^2$, $\tilde{g}(a\mathbf{X} + b\hat{\mathbf{X}}) = a\mathbf{X} + b\tilde{g}(\hat{\mathbf{X}}) \in \mathcal{E}$, whence $zz^* = xx^*$. Thus we can define a map $h : \mathbf{F}^m \to \mathbf{F}^m$ by $h(a_1, \ldots, a_m) = (b_1, \ldots, b_m)$ where $\tilde{g}(a_1 xx^*, \ldots, a_m xx^*) = (b_1 xx^*, \ldots, b_m xx^*)$. Since $\tilde{g}$ is a bijective linear preserver of $\mathcal{E}$, h is a linear isometry preserving the

ℓ_2-norm on $\mathbf{F}^m$. Let $T = h^{-1}$; then $\hat{g} = L \circ \tilde{g}$ fixes $(r_1xx^*, \ldots, r_mxx^*)$ for all $(r_1, \ldots, r_m) \in \mathbf{F}^m$, where

$$L(A_1, \ldots, A_m) = \left(\sum_{j=1}^{m} t_{1j}A_j, \ldots, \sum_{j=1}^{m} t_{mj}A_j \right).$$

Note that $\hat{g}$ still preserves $\mathcal{E}$, and hence $\hat{g}(\mathbf{F}^m \otimes \mathcal{P}) \subseteq \mathbf{F}^m \otimes \mathcal{P}$. Thus by Lemma 2.3, $\hat{g}$ has form (1). Since g has form (1) if and only if $\hat{g}$ does, we are done. □

Next we turn to the distance preserving maps. Note that linearity is not assumed.

Theorem 2.4. *Let $\mathcal{V} = \mathcal{S}(\mathcal{H})$ or $\mathcal{B}(\mathcal{H})$. A map $f : \mathcal{V}^m \to \mathcal{V}^m$ satisfies*

$$w(\mathbf{A} - \mathbf{B}) = w(f(\mathbf{A}) - f(\mathbf{B})) \qquad \text{for all } \mathbf{A}, \mathbf{B} \in \mathcal{V}^m$$

if and only if there is a unitary operator $U \in \mathcal{B}(\mathcal{H})$, a linear isometry $\Gamma = (\gamma_{ij}) \in M_k(\mathbf{R})$ with $\ell_2(\Gamma u) = \ell_2(u)$ for all $u \in \mathbf{R}^k$, and $R \in \mathcal{V}^m$ such that f has the form

$$(A_1, \ldots, A_m) \mapsto \left(\sum_{j=1}^{m} \gamma_{1j}U^*\psi(A_j)U, \ldots, \sum_{j=1}^{m} \gamma_{mj}U^*\psi(A_j)U \right) + R,$$

with $k = m$ if $\mathcal{V} = \mathcal{S}(\mathcal{H})$, or the form

$$(A_1 + iA_2, \ldots, A_{2m-1} + iA_{2m}) \mapsto$$
$$\left(\sum_{j=1}^{2m} U^*(\gamma_{1j}\psi(A_j) + i\gamma_{2j}\psi(A_j))U, \ldots, U^*(\gamma_{2m-1,j}\psi(A_j) + i\gamma_{2m,j}\psi(A_j))U \right) + R,$$

with $k = 2m$ if $\mathcal{V} = \mathcal{B}(\mathcal{H})$. In both cases, ψ has either the form $X \mapsto X$ or $X \mapsto X^t$.

Note that maps like $(B_1, \ldots, B_m) \mapsto (B_1^*, \ldots, B_m^*)$ are just a special case of the second form.

Proof. Sufficiency is clear. For necessity, we see that the map $A \mapsto f(A) - f(0)$ is real linear by the result in [3]. So, we can focus on the structure of real linear maps f preserving the joint numerical radius. If $\mathcal{V} = \mathcal{S}(\mathcal{H})$, then we are done. If $\mathcal{V} = \mathcal{B}(\mathcal{H})$, the result immediately follows from the real case by treating $\mathcal{B}(\mathcal{H})$ as a real space and noting that

$$w(A_1 + iA_2, \ldots, A_{2m-1} + iA_{2m}) = w(A_1, A_2, \ldots, A_{2m})$$

for any self-adjoint operators $A_1, \ldots, A_{2m}$. □

Here are some consequences of our results.

Corollary 2.5. *Let $\mathcal{V} = \mathcal{S}(\mathcal{H})$ or $\mathcal{B}(\mathcal{H})$. The following are equivalent for a map $f : \mathcal{V}^m \to \mathcal{V}^m$:*

(a) *$w(\mathbf{A} + \mathbf{B}) = w(f(\mathbf{A}) + f(\mathbf{B}))$ for all $\mathbf{A}, \mathbf{B} \in \mathcal{V}^m$.*

(b) *f is additive and satisfies $w(\mathbf{A}) = w(f(\mathbf{A}))$ for all $\mathbf{A} \in \mathcal{V}^m$.*

(c) *f is real linear and satisfies $w(\mathbf{A}) = w(f(\mathbf{A}))$ for all $\mathbf{A} \in \mathcal{V}^m$.*

(d) *There is a unitary operator $U \in \mathcal{B}(\mathcal{H})$, a linear isometry $\Gamma = (\gamma_{ij}) \in M_k(\mathbf{R})$ with $\ell_2(\Gamma u) = \ell_2(u)$ for all $u \in \mathbf{R}^k$ such that f has the form*

$$(A_1, \ldots, A_m) \mapsto \left(\sum_{j=1}^{m} \gamma_{1j} U^* \psi(A_j) U, \ldots, \sum_{j=1}^{m} \gamma_{mj} U^* \psi(A_j) U \right),$$

with $k = m$ if $\mathcal{V} = \mathcal{S}(\mathcal{H})$, or the form

$$(A_1 + iA_2, \ldots, A_{2m-1} + iA_{2m}) \mapsto \left(\sum_{j=1}^{2m} U^*(\gamma_{1j}\psi(A_j) + i\gamma_{2j}\psi(A_j))U, \ldots, U^*(\gamma_{2m-1,j}\psi(A_j) + i\gamma_{2m,j}\psi(A_j))U \right),$$

with $k = 2m$ if $\mathcal{V} = \mathcal{B}(\mathcal{H})$. In both cases, ψ has either the form $X \mapsto X$ or $X \mapsto X^t$.

Proof. Clearly $(d) \implies (c) \implies (b) \implies$ (a). On the other hand, if (a) holds then

$$0 = w(\mathbf{0}) = w(\mathbf{A} - \mathbf{A}) = w(f(\mathbf{A}) + f(-\mathbf{A})).$$

It follows that $f(-\mathbf{A}) = -f(\mathbf{A})$, whence (d) follows from Theorem 2.4. □

Corollary 2.6. *Let $(\mathcal{V}, \mathbf{F}) = (\mathcal{S}(\mathcal{H}), \mathbf{R})$ or $(\mathcal{B}(\mathcal{H}), \mathbf{C})$. The following are equivalent for a map $f : \mathcal{V}^m \to \mathcal{V}^m$:*

(a) *$f(\mathbf{0}) = \mathbf{0}$ and $W(\mathbf{A} - \mathbf{B}) = W(f(\mathbf{A}) - f(\mathbf{B}))$ for all $\mathbf{A}, \mathbf{B} \in \mathcal{V}^m$.*

(b) *$W(\mathbf{A} + \mathbf{B}) = W(f(\mathbf{A}) + f(\mathbf{B}))$ for all $\mathbf{A}, \mathbf{B} \in \mathcal{V}^m$.*

(c) *f is additive and satisfies $W(\mathbf{A}) = W(f(\mathbf{A}))$ for all $\mathbf{A} \in \mathcal{V}^m$.*

(d) *f is ($\mathbf{F}$-)linear and satisfies $W(\mathbf{A}) = W(f(\mathbf{A}))$ for all $\mathbf{A} \in \mathcal{V}^m$.*

(e) *There is a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that f has the form*

$$(A_1, \ldots, A_m) \mapsto (U^* A_1 U, \ldots, U^* A_m U) \quad or \quad (A_1, \ldots, A_m) \mapsto \left(U^* A_1^t U, \ldots, U^* A_m^t U\right),$$

where X^t is the transpose of X with respect to a fixed orthonormal basis.

Proof. The implications $(e) \implies (d) \implies (c) \implies (b) \implies (a)$ are clear. If (a) holds, then f has the form in Theorem 2.4 with $R = 0$. Since $\{(1, 0, \ldots, 0)\} = W((I, 0, \ldots, 0)) = W(f(I, 0, \ldots, 0))$, it follows that the only nonzero γ_{j1} is $\gamma_{11} = 1$. If we let $\mathbf{X_j} \in \mathcal{V}^m$ have zero entries except for an I in the jth position, applying this same argument to $W(f(\mathbf{X_j}))$ (and to $W(f(i\mathbf{X_j}))$ if $\mathcal{V} = \mathcal{B}(\mathcal{H})$) shows that $\Gamma = I$, whence we have (e). □

3 Joint numerical radius defined by smooth norms

For any function ν on $\mathbf{F}^m$, one can define the ν-joint numerical radius of $\mathbf{A} \in \mathcal{V}^m$ by

$$w_\nu(\mathbf{A}) = \sup\{\nu(a_1, \ldots, a_m) : (a_1, \ldots, a_m) \in W(A_1, \ldots, A_m)\}.$$

If ν is a norm on $\mathbf{F}^m$, then w_ν will be a norm on $\mathcal{V}^m$. Roughly speaking, if the unit norm ball for the dual norm ν^* has 'enough' extreme points, then we can characterize the linear isometries of w_ν completely. We shall henceforth denote the unit norm ball for ν by $\mathcal{B}_\nu = \{x \in \mathbf{F}^m : \nu(x) \leq 1\}$. Recall that, given any $X \in \mathcal{V}$ and $c = (c_1, \ldots, c_m) \in \mathbf{F}^m$, we let $c \otimes X = (c_1 X, \ldots, c_m X) \in \mathcal{V}^m$.

Theorem 3.1. *Let $(\mathcal{V}, \mathbf{F}) = (\mathcal{S}(\mathcal{H}), \mathbf{R})$ or $(\mathcal{B}(\mathcal{H}), \mathbf{C})$. Let ν be a norm on $\mathbf{F}^m$ such that the set E of extreme points of $\mathcal{B}_{\nu^*}$ has the following property: There exist linearly independent $v_1, \ldots, v_m \in E$ such that, for any $j > 1$, there is a $u_j \in E$ so that dim span $(v_j, u_j) = 2$ and span $(v_1, \ldots, v_j) =$ span $(v_1, \ldots, v_{j-1}, u_j)$. Then an $\mathbf{F}$-linear map $f : \mathcal{V}^m \to \mathcal{V}^m$ satisfies*

$$w_\nu(\mathbf{A}) = w_\nu(f(\mathbf{A})) \qquad \text{for all } \mathbf{A} \in \mathcal{V}^m$$

if and only if there is a unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a linear ν-isometry $\Gamma = (\gamma_{ij}) \in M_m(\mathbf{F})$ with $\nu(\Gamma u) = \nu(u)$ for all $u \in \mathbf{F}^m$ such that f has the form

$$(A_1, \ldots, A_m) \mapsto \left(\sum_{j=1}^m \gamma_{1j} U^* \psi(A_j) U, \ldots, \sum_{j=1}^m \gamma_{mj} U^* \psi(A_j) U \right), \tag{3}$$

with ψ taking the form $X \mapsto X$ or $X \mapsto X^t$, where X^t is the transpose of X with respect to a fixed orthonormal basis.

Proof. We shall mimic and closely follow the proof of Theorem 2.1. As before, sufficiency is easy to check and we define an inner product $(\mathbf{A}, \mathbf{B})$ on $\mathcal{V}^m$ the same way. Since $\mathbf{F}^m$ is reflexive, $\nu = (\nu^*)^*$. Let E denote the extreme points of $\mathcal{B}_{\nu^*}$. Then

$$\begin{aligned} w_\nu(\mathbf{A}) &= \sup\{\nu(a_1, \ldots, a_m) : (a_1, \ldots, a_m) \in W(A_1, \ldots, A_m)\} \\ &= \sup\{\nu(\operatorname{tr} A_1 xx^*, \ldots, \operatorname{tr} A_m xx^*) : x \in \mathcal{H}, (x, x) = 1\} \\ &= \sup\left\{ \left| \sum_{j=1}^m \operatorname{tr} A_j xx^* \bar{r}_j \right| : x \in \mathcal{H}, (x, x) = 1, r = (r_1, \ldots, r_m) \in \mathcal{B}_{\nu^*} \right\} \\ &= \sup\{|(\mathbf{A}, \mathbf{B})| : \mathbf{B} \in \mathcal{E}\} \end{aligned}$$

where

$$\mathcal{E} = \{(r_1 xx^*, \ldots, r_m xx^*) : x \in \mathcal{H}, (x, x) = 1, r = (r_1, \ldots, r_m) \in E\}.$$

Thus w_ν is the dual of a norm w_ν^* on $\mathcal{V}^m$ whose unit norm ball is the closed convex hull of its extreme points $\mathcal{E}$. Now let $f : \mathcal{V}^m \to \mathcal{V}^m$ be a $\mathbf{F}$-linear map preserving w_ν; it must be the dual of a bijective linear map g preserving the induced norm w_ν^* on $\mathcal{V}^m$, in which case $g(\mathcal{E}) = \mathcal{E}$. We shall show that g has the form in (3) except with Γ being a ν^*-isometry instead. But note that if Q is a ν-isometry on $\mathbf{F}^m$ (i.e. $\nu(Qx) = \nu(x)$ for all $x \in \mathbf{F}^m$), then Q^* is a ν^*-isometry on $\mathbf{F}^m$. It follows that f has the desired form (3).

Fix a unit vector $x \in \mathcal{H}$. By the hypotheses on E, there exist $u_2, \ldots, u_m \in E$ and linearly independent $v_1, \ldots, v_m \in E$ such that, for $j = 2, \ldots, m$, $u_j = \alpha_j v_j + w_{j-1}$ for some nonzero scalar α_j and some nonzero vector $w_{j-1} \in V_{j-1} = \text{span}(v_1, \ldots, v_{j-1})$. Since $v_1 \otimes xx^* \in \mathcal{E}$, we may write $g(v_1 \otimes xx^*) = a \otimes yy^*$ for some unit vector $y \in \mathcal{H}$ and some $a \in E$. By linearity, $g(c \otimes xx^*)$ has the form $b_c \otimes yy^*$ for any vector $c \in V_1$. We shall use induction to show that, for any vector $c \in \mathbf{F}^m$, $g(c \otimes xx^*)$ has the form $b_c \otimes yy^*$ for some vector $b_c \in \mathbf{F}^m$.

Suppose this statement is true for vectors $c \in V_{j-1}$. Let $\mathbf{Z} = w_{j-1} \otimes xx^*$ so we may write $g(\mathbf{Z}) = r \otimes R$ where $R = yy^*$ and $r \in \mathbf{F}^m$ is nonzero since g is bijective. Since $u_j, v_j \in E$, $\mathbf{X} = u_j \otimes xx^* \in \mathcal{E}$ and $\mathbf{Y} = v_j \otimes xx^* \in \mathcal{E}$, so we may write $g(\mathbf{X}) = p \otimes P$ and $g(\mathbf{Y}) = q \otimes Q$ for $p, q \in E$ and for some rank 1 (hermitian) projections P, Q. But $\mathbf{X} = \alpha_j \mathbf{Y} + \mathbf{Z}$, so $g(\mathbf{X}) = \alpha_j g(\mathbf{Y}) + g(\mathbf{Z})$, whence $p_k P = \alpha_j q_k Q + r_k R$ for all $k = 1, \ldots, m$. Since p, q, r are nonzero vectors, we must have $P = Q = R$. Since g is linear, we see that $g(c \otimes xx^*)$ must have the asserted form for any $c \in V_j$, and hence, by induction, for all $c \in \mathbf{F}^m$.

Thus we can define a map $h : \mathbf{F}^m \to \mathbf{F}^m$ by $h(a_1, \ldots, a_m) = (b_1, \ldots, b_m)$ where

$$g(a_1 xx^*, \ldots, a_m xx^*) = (b_1 yy^*, \ldots, b_m yy^*).$$

Since g is a bijective linear preserver of $\mathcal{E}$, h is a linear ν^*-isometry. Let $T = h^{-1}$ and let $U \in \mathcal{B}(\mathcal{H})$ be a unitary (real orthogonal if $\mathbf{F} = \mathbf{R}$) matrix satisfying $Uy = x$; then $\hat{g} = L \circ g$ fixes $(r_1 xx^*, \ldots, r_m xx^*)$ for all $(r_1, \ldots, r_m) \in \mathbf{F}^m$, where

$$L(A_1, \ldots, A_m) = \left(\sum_{j=1}^{m} t_{1j} U A_j U^*, \ldots, \sum_{j=1}^{m} t_{mj} U A_j U^* \right).$$

Note that g has the desired form if and only if $\hat{g}$ does, and that $\hat{g}(\mathcal{E}) = \mathcal{E}$. Moreover, since x was arbitrary, we see that $\hat{g}(\mathbf{F}^m \otimes \mathcal{P}) \subseteq \mathbf{F}^m \otimes \mathcal{P}$, and can apply Lemma 2.3 to conclude that $\hat{g}$ has the desired form. □

Recall that a norm ν is smooth if every point x with $\nu(x) = 1$ has precisely one supporting functional f of norm one (that is, $\nu^*(f) = f(x) = 1$). Some common examples of smooth norms are the ℓ_p norms for $1 < p < \infty$.

Corollary 3.2. *Let $(\mathcal{V}, \mathbf{F}) = (\mathcal{S}(\mathcal{H}), \mathbf{R})$ or $(\mathcal{B}(\mathcal{H}), \mathbf{C})$. Let ν be a smooth norm on $\mathbf{F}^m$. A $\mathbf{F}$-linear map $f : \mathcal{V}^m \to \mathcal{V}^m$ satisfies*

$$w_\nu(\mathbf{A}) = w_\nu(f(\mathbf{A})) \qquad \textit{for all } \mathbf{A} \in \mathcal{V}^m$$

if and only if there is a unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a linear isometry $\Gamma = (\gamma_{ij}) \in M_m(\mathbf{F})$ with $\nu(\Gamma u) = \nu(u)$ for all $u \in \mathbf{F}^m$ such that f has the form

$$(A_1, \dots, A_m) \mapsto \left(\sum_{j=1}^{m} \gamma_{1j} U^* \psi(A_j) U, \dots, \sum_{j=1}^{m} \gamma_{mj} U^* \psi(A_j) U \right), \tag{4}$$

with ψ taking the form $X \mapsto X$ or $X \mapsto X^t$, where X^t is the transpose of X with respect to a fixed orthonormal basis.

Proof. This follows immediately from Theorem 3.1 by noting that:

1. If the dual norm ν^* on X^* is strictly convex (respectively smooth) then the norm ν on the original Banach space X is smooth (respectively strictly convex).

2. The converse of the preceding statement obviously holds for reflexive spaces like $\mathbf{F}^m$ (but not in general).

3. A norm $\|\cdot\|$ is *strictly convex* if and only if the unit norm ball for $\|\cdot\|$ has an extreme point in every direction.

Hence a smooth norm satisfies the hypotheses of Theorem 3.1 and the conclusion follows. □

The results on distance-preserving maps and additive maps from Section 2 (Theorem 2.4 and Corollary 2.5) generalize to the norms in this section, using the same arguments as before.

4 Joint numerical radius defined by symmetric norms

Recall that ν on $\mathbf{F}^m$ is a *symmetric norm* if it is a norm such that $\nu(x) = \nu(Px)$ for any *generalized permutation* matrix P, i.e., $P = DQ$ for a permutation matrix Q and $D = \operatorname{diag}(d_1, \dots, d_n)$ with $|d_1| = \cdots = |d_n| = 1$. Commonly used symmetric norms on $\mathbf{F}^m$ include the ℓ_p norms defined by

$$\ell_p(x_1, \dots, x_m) = \left(\sum_{j=1}^{m} |x_j|^p \right)^{1/p} \qquad p \in [1, \infty),$$

and the k-norm defined by

$$\|x\|_k = \max \{ |x_{j_1}| + \cdots + |x_{j_k}| : 1 \le j_1 < j_2 < \cdots < j_k \le m \}.$$

It is known that (see [10] and also [2]) if ν is a symmetric norm not equal to a multiple of the ℓ_2-norm, then the isometry group for ν must be one of the following:

(1) the group of generalized permutation matrices, or

(2) $\mathbf{F}^m = \mathbf{R}^4$, and the isometry group is generated by generalized permutation matrices and the matrix A or B, where

$$A = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \text{ or}$$

(3) $\mathbf{F}^m = \mathbf{R}^2$, and the isometry group is a dihedral group with $8k$ elements for some positive integer k.

We can extend the results in Section 2 to w_ν for some symmetric norms ν on $\mathbf{F}^m$.

Theorem 4.1. *Let $(\mathcal{V}, \mathbf{F}) = (\mathcal{S}(\mathcal{H}), \mathbf{R})$ or $(\mathcal{B}(\mathcal{H}), \mathbf{C})$. Let ν be a symmetric norm on $\mathbf{F}^m$ and suppose $\mathcal{B}_{\nu^*}$ has an extreme point of the form $(\gamma, 0, \ldots, 0)$. Then a linear map $f : \mathcal{V}^m \to \mathcal{V}^m$ satisfies*

$$w_\nu(\mathbf{A}) = w_\nu(f(\mathbf{A})) \qquad \text{for all } \mathbf{A} \in \mathcal{V}^m$$

if and only if one of the following holds:

(a) ν is a multiple of the sup norm ℓ_∞, and there is a permutation $(i_1, \ldots, i_m)$ of $(1, \ldots, m)$ such that f has the form

$$(A_1, \ldots, A_m) \mapsto (\psi_1(A_{i_1}), \ldots, \psi_m(A_{i_m})),$$

where for each $j = 1, \ldots, m$, there is a unitary matrix $U_j \in \mathcal{B}(\mathcal{H})$ and $\xi_j \in \mathbf{F}$ with $|\xi_j| = 1$ such that ψ_j has the form

$$X \mapsto \xi_j U_j^* X U_j \quad \text{or} \quad X \mapsto \xi_j U_j^* X^t U_j.$$

(b) There is a unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a linear isometry $\Gamma = (\gamma_{ij}) \in M_m(\mathbf{F})$ for the norm ν such that f has the form

$$(A_1, \ldots, A_m) \mapsto \left(\sum_{j=1}^m \gamma_{1j} \psi(A_j), \ldots, \sum_{j=1}^m \gamma_{mj} \psi(A_j) \right),$$

where ψ has either the form $X \mapsto U^* X U$ or $X \mapsto U^* X^t U$.

Remark 4.2. *In the case where $\mathbf{F}^m \neq \mathbf{R}^4$ or $\mathbf{R}^2$, and ν is a symmetric norm that is not a multiple of the ℓ_2 or ℓ_∞ norms, we see that linear isometries of w_ν must have the form*

$$(A_1, \ldots, A_m) \mapsto (\xi_1 \psi(A_{i_1}), \ldots, \xi_m \psi(A_{i_m}))$$

where ψ has either the form $X \mapsto U^* X U$ or $X \mapsto U^* X^t U$, for some unitary operator $U \in \mathcal{B}(\mathcal{H})$, a permutation $(i_1, \ldots, i_m)$ of $(1, \ldots, m)$, and $\xi_1, \ldots, \xi_m \in \mathbf{F}$ with $|\xi_j| = 1$.

Proof. The sufficiency is clear. To prove the converse, we consider the dual norm w_ν^* of w_ν. The set of extreme points of $\mathcal{B}_{w_\nu^*}$ is

$$\mathcal{E} = \{(r_1xx^*, \ldots, r_mxx^*) : x \in \mathcal{H}, x^*x = 1, (r_1, \ldots, r_m) \in E\},$$

where E is the set of the extreme points of $\mathcal{B}_{\nu^*}$. Assume that $(\gamma, 0, \ldots, 0) \in E$ for some $\gamma > 0$. We may assume that $\gamma = 1$; otherwise, replace ν^* by $\gamma\nu^*$.

Let $\mathcal{P} = \{zz^* : z \in \mathcal{H}, z^*z = 1\}$ and $\tilde{\mathcal{P}} = \{\mu Q : \mu \in \mathbf{F}, |\mu| = 1, Q \in \mathcal{P}\}$. Let e_j denote the jth row of the identity matrix I_m, and recall that $c \otimes X = (c_1X, \ldots, c_mX)$ for $X \in M_n$ and $c = (c_1, \ldots, c_m) \in \mathbf{F}^m$. In particular, let $x \in \mathcal{H}$ be a unit vector; since f^* is a w_ν^*-isometry preserving $\mathcal{E}$, $f^*(e_j \otimes xx^*) = v_j \otimes y_jy_j^*$ for some $v_j \in E$ and unit vector $y_j \in \mathcal{H}$ for $j = 1, \ldots, m$. We consider two cases.

Case 1. Suppose ν is the sup norm. Then $E = \{\xi e_i : |\xi| = 1\}$, and for $j \in \{1, \ldots, m\}$ we have $f^*(e_j \otimes xx^*) = \mu_j e_{\tau(j)} \otimes y_jy_j^*$, where τ is a permutation of $(1, \ldots, m)$ and $\mu_j \in \mathbf{F}$ with $|\mu_j| = 1$. We may compose f^* with the map $(X_1, \ldots, X_m) \mapsto (\bar{\mu}_1X_{\tau(1)}, \ldots, \bar{\mu}_mX_{\tau(m)})$ and assume that $f^*(e_j \otimes xx^*) = (e_j \otimes y_jy_j^*)$ for $j \in \{1, \ldots, m\}$. Applying Lemma 2.2 with $g = f^*$ and $e = e_1$, we see that either $f^*(e_1 \otimes \mathcal{V}) = e_1 \otimes \mathcal{V}$ or $f^*(e_1 \otimes \mathcal{V}) \subseteq \mathbf{F}^m \otimes y_1y_1^*$.

Suppose, by way of contradiction, that the latter case holds. Then $f^*(e_1 \otimes A) = \psi(A) \otimes y_1y_1^*$ for some injective linear map $\psi : \mathcal{V} \to \mathbf{F}^m$. Since $f^*(e_1 \otimes \mathcal{P}) \subseteq E \otimes \mathcal{P}$, $\psi(\mathcal{P}) \subseteq E$. Since $\mathcal{P}$ is connected, $\psi(\mathcal{P})$ is a connected subset of E, so $\psi(\mathcal{P}) \subseteq \{\mu e_1 : |\mu| = 1\}$. But then $\psi(\mathcal{V}) \subseteq \mathbf{F}e_1$, so ψ is a rank one injective map, which is impossible since dim $\mathcal{V} > 1$. Hence $f^*(e_1 \otimes \mathcal{V}) = e_1 \otimes \mathcal{V}$.

We may write $f^*(e_1 \otimes X) = e_1 \otimes \psi_1(X)$ for some $\psi_1 : \mathcal{V} \to \mathcal{V}$. Since f^* is bijective and $f^*(E \otimes \mathcal{P}) = E \otimes \mathcal{P}$, ψ_1 is bijective and $\psi_1(\tilde{\mathcal{P}}) = \tilde{\mathcal{P}}$. As $\tilde{\mathcal{P}}$ is the set of extreme points of $\mathcal{B}_{w^*}$, ψ_1^* preserves the numerical radius and has the form (see [8]) $X \mapsto \xi U_1^*XU_1$ or $X \mapsto \xi U_1^*X^tU_1$ for some unitary U_1 and $\xi \in \mathbf{F}$ with $|\xi| = 1$. It follows that ψ_1 has the same form; since $\psi_1(xx^*) = y_1y_1^*$, $\xi = 1$. Similarly, we can show that $f^*(e_j \otimes X) = e_j \otimes \psi_j(X)$, where both sides have the nonzero component at the jth position, and that ψ_j has the form $X \mapsto U_j^*XU_j$ or $X \mapsto U_j^*X^tU_j$ for some unitary U_j. Thus, case (a) of the Theorem holds.

Case 2. Suppose ν is not the sup norm. Let $a = (\alpha_1, \ldots, \alpha_m)$ be a vector in E with as few zero entries as possible. Without loss of generality we may assume $\alpha_1 \geq \cdots \geq \alpha_k > 0$ and $\alpha_j = 0$ for $j > k \geq 2$. We shall show that the hypotheses of Theorem 3.1 apply.

Let $v_1 = a$. For $2 \leq j \leq k$, let v_j be the vector in $\mathbf{F}^m$ having the same entries as a with the exception of having $-\alpha_j$ in the jth coordinate instead. For $j > k$, let v_j be the vector in $\mathbf{F}^m$ whose first $k-1$ entries are $\alpha_1, \ldots, \alpha_{k-1}$, jth entry is α_k, and all other entries are zero. Thus $v_1, \ldots, v_m$ are linearly independent extreme points of $\mathcal{B}_{\nu^*}$. Let V_j denote the span of $v_1, \ldots, v_j$.

For $j \geq 2$, let $u_j = e_j \in E$. If $2 \leq j \leq k$, $2\alpha_ju_j = v_1 - v_j$, so span $(V_{j-1}, v_j) =$ span (V_{j-1}, u_j). If $j > k$, then $v_j - \alpha_ku_j \in V_k \subseteq V_{j-1}$, so again span $(V_{j-1}, v_j) =$ span (V_{j-1}, u_j). Thus the hypotheses for Theorem 3.1 are satisfied and the conclusion follows. □

Note that this result fails if ν is the ℓ_1 norm on $\mathbf{R}^2$. For example, the map

$$f(A,B) = \frac{1}{2}(A+B+U^*(A-B)U, A+B-U^*(A-B)U)$$

is a w_{ℓ_1} isometry on $\mathcal{S}(\mathcal{H})^2$ for any unitary U, but does not have the form asserted by the theorem. Thus, the assumption that $(\gamma, 0, \dots, 0) \in E$ is needed, at least when $\mathbf{F} = \mathbf{R}$. It turns out that this assumption is not necessary when $\mathbf{F} = \mathbf{C}$ however.

Theorem 4.3. *Let ν be a symmetric norm on $\mathbf{C}^m$ not equal to a multiple of the sup norm ℓ_∞. Then a linear map $f : \mathcal{B}(\mathcal{H})^m \to \mathcal{B}(\mathcal{H})^m$ satisfies*

$$w_\nu(\mathbf{A}) = w_\nu(f(\mathbf{A})) \qquad \textit{for all } \mathbf{A} \in \mathcal{V}^m$$

if and only if there is a unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a linear isometry $\Gamma = (\gamma_{ij}) \in M_m(\mathbf{F})$ for the norm ν such that f has the form

$$(A_1, \dots, A_m) \mapsto \left(\sum_{j=1}^m \gamma_{1j}\psi(A_j), \dots, \sum_{j=1}^m \gamma_{mj}\psi(A_j) \right),$$

where ψ has either the form $X \mapsto U^*XU$ or $X \mapsto U^*X^tU$.

Proof. Since ν is not a multiple of the sup norm, the norm ball $\mathcal{B}_{\nu^*}$ has an extreme point of the form $(x_1, \dots, x_m)$ with $x_1 \geq x_2 \geq \cdots \geq x_k > 0$, $2 \leq k \leq m$, and $x_j = 0$ for $j > k$. We shall show that the hypotheses of Theorem 3.1 apply.

Since

$$\det \begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_k \\ x_1 & wx_2 & x_3 & & \vdots \\ x_1 & x_2 & wx_3 & & \vdots \\ \vdots & & & \ddots & \vdots \\ x_1 & \dots & \dots & \dots & wx_k \end{bmatrix}$$

is a polynomial of degree $k-1$ in w, we may choose a nonreal w so that $|w| = 1$ and the determinant is nonzero. For $1 \leq j \leq k$, let v_j denote the vector in $\mathbf{C}^m$ whose first k entries are given by the jth row of the above matrix, and whose other entries are zero. For $j > k$, let v_j be the vector in $\mathbf{C}^m$ whose first $k-1$ entries are $x_1, \dots, x_{k-1}$, jth entry is x_k, and all other entries are zero. Thus $v_1, \dots, v_m$ are linearly independent extreme points of $\mathcal{B}_{\nu^*}$. Let V_j denote the span of $v_1, \dots, v_j$.

If $2 \leq j \leq k$, let u_j be the vector whose jth entry is $\overline{w}x_j$ and whose other entries match those of v_j. Then u_j is an extreme point and $(\overline{w}-w)v_1 + (1-\overline{w})v_j + (w-1)u_j = 0$, so span $(V_{j-1}, v_j) =$ span (V_{j-1}, u_j). If $j > k$, let u_j be the vector whose jth entry is $\overline{w}x_k$ and whose other entries match those of v_j. Then u_j is an extreme point and $u_j - \overline{w}v_j$ is a nonzero vector in $V_k \subset V_{j-1}$, so span $(V_{j-1}, v_j) =$ span (V_{j-1}, u_j). Thus the hypotheses for Theorem 3.1 are satisfied and the conclusion follows. □

As in Section 2, we may obtain results on

- distance-preserving maps f satisfying $w_\nu(\mathbf{A}-\mathbf{B}) = w_\nu(f(\mathbf{A})-f(\mathbf{B}))$ for all $\mathbf{A},\mathbf{B} \in \mathcal{V}^m$, and
- additive maps f satisfying $w_\nu(\mathbf{A}) = w_\nu(f(\mathbf{A}))$ for all $\mathbf{A} \in \mathcal{V}^m$

generalizing Theorem 2.4 and Corollary 2.5 by using the same arguments as before. We omit their discussion.

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