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# GENERALIZED NUMERICAL RANGES AND QUANTUM ERROR CORRECTION

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ABSTRACT. For a noisy quantum channel, a quantum error correcting code of dimension $k$ exists if and only if the joint rank-$k$ numerical range associated with the error operators of the channel is non-empty. In this paper, geometric properties of the joint rank $k$-numerical range are obtained and their implications to quantum computing are discussed. It is shown that for a given $k$ if the dimension of the underlying Hilbert space of the quantum states is sufficiently large, then the joint rank $k$-numerical range of operators is always star-shaped and contains the convex hull of the rank $\hat{k}$-numerical range of the operators for sufficiently large $\hat{k}$. In case the operators are infinite dimensional, the joint rank $\infty$-numerical range of the operators is a convex set closely related to the joint essential numerical ranges of the operators.

KEYWORDS: *Quantum error correction, joint higher rank numerical range, joint essential numerical range, self-adjoint operator, Hilbert space.*

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## 1. INTRODUCTION

In quantum computing, information is stored in *quantum bits*, abbreviated as *qubits*. Mathematically, a qubit is represented by a $2 \times 2$ rank one Hermitian matrix $Q = vv^*$, where $v \in \mathbf{C}^2$ is a unit vector. A state of $N$-qubits $Q_1, \dots, Q_N$ is represented by their tensor products in $M_n$ with $n = 2^N$. A *quantum channel* for states of $N$-qubits corresponds to a *trace preserving completely positive linear map* $\Phi : M_n \to M_n$. By the structure theory of completely positive linear map [2], there are $T_1, \dots, T_r \in M_n$ with $\sum_{j=1}^r T_j^* T_j = I_n$ such that

$$\Phi(X) = \sum_{j=1}^{r} T_j X T_j^*. \tag{1.1}$$

In the context of quantum error correction, $T_1, \dots, T_r$ are known as the *error operators*.

Let $\mathbf{V}$ be a $k$-dimensional subspace of $\mathbf{C}^n$ and $P$ the orthogonal projection of $\mathbf{C}^n$ onto $\mathbf{V}$. Then $\mathbf{V}$ is a *quantum error correcting code* for the quantum channel $\Phi$ if there exists another trace preserving completely positive linear map $\Psi : M_n \to M_n$ such that $\Psi \circ \Phi(A) = A$ for all $A \in PM_nP$. By the results in [9], this happens if and only if there are scalars $\gamma_{ij}$ with $1 \le i, j \le r$ such that

$$PT_i^*T_jP = \gamma_{ij}P.$$

Let $\mathcal{P}_k$ be the set of rank $k$ orthogonal projections in $M_n$. Define the *joint rank $k$-numerical range* of an $m$-tuple of matrices $\mathbf{A} = (A_1, \ldots, A_m) \in M_n^m$ by

$$\Lambda_k(\mathbf{A}) = \{(a_1, \ldots, a_m) \in \mathbf{C}^m : \text{ there is } P \in \mathcal{P}_k \text{ such that } PA_jP = a_jP \text{ for } j = 1, \ldots, m\}.$$

Then the quantum channel $\Phi$ defined in (1.1) has an error correcting code of $k$-dimension if and only if

$$\Lambda_k(T_1^*T_1, T_1^*T_2, \ldots, T_r^*T_r) \ne \emptyset.$$

Evidently, $(a_1, \ldots, a_m) \in \Lambda_k(\mathbf{A})$ if and only if there exists an $n \times k$ matrix $U$ such that

$$U^*U = I_k, \quad \text{and} \quad U^*A_jU = a_jI_k \qquad \text{for } j = 1, \ldots, m.$$

Let $\mathbf{x}, \mathbf{y} \in \mathbf{C}^n$. Denote by $\langle \mathbf{A}\mathbf{x}, \mathbf{y}\rangle$ the vector $(\langle A_1\mathbf{x}, \mathbf{y}\rangle, \ldots, \langle A_m\mathbf{x}, \mathbf{y}\rangle) \in \mathbf{C}^m$. Then $\mathbf{a} \in \Lambda_k(\mathbf{A})$ if and only if there exists an orthonormal set $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ in $\mathbf{C}^n$ such that $\langle \mathbf{A}\mathbf{x}_i, \mathbf{x}_j\rangle = \delta_{ij}\mathbf{a}$, where $\delta_{ij}$ is the Kronecker delta. When $k = 1$, $\Lambda_1(\mathbf{A})$ reduces to the (classical) *joint numerical range*

$$W(\mathbf{A}) = \{(\mathbf{x}^*A_1\mathbf{x}, \ldots, \mathbf{x}^*A_m\mathbf{x}) : \mathbf{x} \in \mathbf{C}^n,\ \mathbf{x}^*\mathbf{x} = 1\}$$

of $\mathbf{A}$, which is quite well studied; see [11] and the references therein. It turns out that even for a single matrix $A \in M_n$, the study of $\Lambda_k(A)$ is highly non-trivial, and the results are useful in quantum computing, say, in constructing binary unitary channels; see [3, 4, 5, 6, 7, 13, 15, 18].

More generally, let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators acting on a Hilbert space $\mathcal{H}$, which may be infinite dimensional. One can extend the definition of $\Lambda_k(A)$ to $A \in \mathcal{B}(\mathcal{H})$. If $\mathcal{H}$ is infinite dimensional, one may allow $k = \infty$ by letting $\mathcal{P}_k$ be the set of infinite rank orthogonal projections in $\mathcal{B}(\mathcal{H})$ in the definition; see [14, 16]. There are a number of reasons to consider rank $k$-numerical range of infinite dimensional operators. First, many quantum mechanical phenomena are better described using infinite dimensional Hilbert spaces. Also, a practical quantum computer must be able to handle a large number of qubits so that the underlying Hilbert space must have a very large dimension. We will also consider the joint rank $k$-numerical range of an $m$-tuple $\mathbf{A} = (A_1, \ldots, A_m)$ of infinite dimensional operators $A_1, \ldots, A_m$ for positive integers $k$ and $k = \infty$. It is interesting to note that $\Lambda_\infty(\mathbf{A})$ has intimate connection with the *joint essential numerical range* of $\mathbf{A}$ defined as

$$W_e(\mathbf{A}) = \cap\{\mathbf{cl}\,(W(\mathbf{A} + \mathbf{F})) : \mathbf{F} \in \mathcal{F}(\mathcal{H})^m\},$$

where $\mathcal{F}(\mathcal{H})$ denotes the set of finite rank operators in $\mathcal{B}(\mathcal{H})$ and $\mathbf{cl}\,(\mathcal{S})$ denotes the closure of the set $\mathcal{S}$. Clearly, the joint essential numerical range is useful for the study of the joint behaviors of operators under perturbations of finite rank (or compact) operators.

The purpose of this paper is to study the joint rank $k$-numerical range of $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{B}(\mathcal{H})^m$. Understanding the properties of $\Lambda_k(\mathbf{A})$ is useful for constructing quantum error correcting codes and studying their properties such as their stability under perturbation.

Our paper is organized as follows. In Section 2, we present some basic properties of $\Lambda_k(\mathbf{A})$. Section 3 concerns the geometric properties of $\Lambda_k(\mathbf{A})$. We show that if $\dim \mathcal{H}$ is sufficiently large, then $\Lambda_k(\mathbf{A})$ is always star-shaped and contains the convex hull of $\Lambda_{\hat{k}}(\mathbf{A})$ for sufficiently large $\hat{k}$. In Section 4, we study the connection between $W_e(\mathbf{A})$, $\Lambda_k(\mathbf{A})$ and its closure $\mathbf{cl}\,(\Lambda_k(\mathbf{A}))$. We show that $\Lambda_\infty(\mathbf{A})$ is always convex, and is a subset of the set of star centers of $\Lambda_k(\mathbf{A})$ for each positive integer $k$. We also show that

$$W_e(\mathbf{A}) = \cap_{k\geq 1} \mathbf{cl}\,(\Lambda_k(\mathbf{A}))\,.$$

Moreover, we obtain several equivalent formulations of $\Lambda_\infty(\mathbf{A})$, including

$$\Lambda_\infty(\mathbf{A}) = \cap\{\Lambda_k(\mathbf{A}+\mathbf{F}) : \mathbf{F} \in \mathcal{F}(\mathcal{H})^m\}.$$

The results extend those in [1, 12].

Let $\mathcal{S}(\mathcal{H})$ be the real linear space of self-adjoint operators in $\mathcal{B}(\mathcal{H})$. Suppose

$$A_j = H_{2j-1} + iH_{2j} \quad \text{with } H_{2j-1}, H_{2j} \in \mathcal{S}(\mathcal{H}) \quad for\ j = 1, \ldots, m.$$

Then $\Lambda_k(\mathbf{A}) \subseteq \mathbf{C}^m$ can be identified with $\Lambda_k(H_1, \ldots, H_{2m}) \subseteq \mathbf{R}^{2m}$. Thus, we will focus on the joint rank $k$-numerical ranges of self-adjoint operators in our discussion.

## 2. BASIC PROPERTIES OF $\Lambda_k(\mathbf{A})$

PROPOSITION 2.1. *Suppose $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$, and $T = (t_{ij})$ is an $m \times n$ real matrix. If $B_j = \sum_{i=1}^m t_{ij} A_i$ for $j = 1, \ldots, n$, then*

$$\{\mathbf{a}T : \mathbf{a} \in \Lambda_k(\mathbf{A})\} \subseteq \Lambda_k(\mathbf{B}).$$

*Equality holds if $\{A_1, \ldots, A_m\}$ is linearly independent and*

$$\operatorname{span}\{A_1, \ldots, A_m\} = \operatorname{span}\{B_1, \ldots, B_n\}.$$

*Proof.* The set inclusion follows readily from definitions. Evidently, the equality holds if $n = m$ and $T$ is invertible.

Suppose $\{A_1, \ldots, A_m\}$ is linearly independent and

$$\operatorname{span}\{A_1, \ldots, A_m\} = \operatorname{span}\{B_1, \ldots, B_n\}.$$

First consider the special case when $A_i = B_i$ for $1 \le i \le m$. Then $T = [I_m|T_1]$ for some $m \times (n-m)$ matrix $T_1$. Let $(b_1, \dots, b_n) \in \Lambda_k(\mathbf{B})$. Then there exists a rank $k$ orthogonal projection $P$ such that $PB_iP = b_iP$ for $i = 1, \dots, n$. Therefore, we have $(b_1, \dots, b_m) \in \Lambda_k(\mathbf{A})$ and for $1 \le j \le n$,

$$b_jP = PB_jP = P\left(\sum_{i=1}^m t_{ij}A_i\right)P = P\left(\sum_{i=1}^m t_{ij}B_i\right)P = \left(\sum_{i=1}^m t_{ij}b_i\right)P$$

implying that $b_j = \left(\sum_{i=1}^m t_{ij}b_i\right)$. Therefore, $(b_1, \dots, b_n) = (b_1, \dots, b_m)\,T$.

For the general case, by applying a permutation, if necessary, we may assume that $\{B_1, \dots, B_m\}$ is a basis of span $\{B_1, \dots, B_n\}$. Then there exists an $m \times m$ invertible matrix $S = \left(s_{ij}\right)$ such that $A_j = \sum_{i=1}^m s_{ij}B_i$ for $j = 1, \dots, m$. For $1 \le j \le m$, we have

$$B_j = \sum_{i=1}^m t_{ij}A_i = \sum_{i=1}^m t_{ij}\left(\sum_{k=1}^m s_{ki}B_k\right) = \sum_{k=1}^m \left(\sum_{i=1}^m s_{ki}t_{ij}\right)B_k\,.$$

Therefore, $\sum_{i=1}^m s_{ki}t_{ij} = \delta_{kj}$ and $ST = [I_m|T_1]$ for some $m \times (n-m)$ matrix $T_1$. Hence, we have

$$\Lambda_k(\mathbf{B}) = \Lambda_k\left(B_1, \dots, B_m\right)[I|T_1] = \Lambda_k\left(B_1, \dots, B_m\right)ST = \Lambda_k\left(A_1, \dots, A_m\right)T. \quad \square$$

In view of the above proposition, in the study of the geometric properties of $\Lambda_k(\mathbf{A})$, we may always assume that $A_1, \dots, A_m$ are linearly independent.

PROPOSITION 2.2. *Let* $\mathbf{A} = (A_1, \dots, A_m) \in \mathcal{S}(\mathcal{H})^m$, *and let* $k < \dim \mathcal{H}$.
(a) *For any real vector* $\boldsymbol{\mu} = (\mu_1, \dots, \mu_m)$,

$$\Lambda_k(A_1 - \mu_1 I, \dots, A_m - \mu_m I) = \Lambda_k(\mathbf{A}) - \boldsymbol{\mu}.$$

(b) *If* $(a_1, \dots, a_m) \in \Lambda_k(\mathbf{A})$ *then* $(a_1, \dots, a_{m-1}) \in \Lambda_k(A_1, \dots, A_{m-1})$.
(c) $\Lambda_{k+1}(\mathbf{A}) \subseteq \Lambda_k(\mathbf{A})$.

REMARK 2.3. By Proposition 2.2 (a), we can replace $A_j$ by $A_j - \mu_j I$ for $j = 1, \dots, m$, without affecting the geometric properties of $\Lambda_k(A_1, \dots, A_m)$.

Suppose $\dim \mathcal{H} = n < 2k - 1$ and $A_1 = \operatorname{diag}(1, 2, \dots, n)$. Then $\Lambda_k(A_1) = \emptyset$. By Proposition 2.2 (c), we see that $\Lambda_k(\mathbf{A}) = \emptyset$ for any $A_2, \dots, A_m$. Thus, $\Lambda_k(\mathbf{A})$ can be empty if $\dim \mathcal{H}$ is small. However, a result of Knill, Laflamme and Viola [10] shows that $\Lambda_k(\mathbf{A})$ is non-empty if $\dim \mathcal{H}$ is sufficiently large. By modifying the proof of Theorem 3 in [10], we can get a slightly better bound in the following proposition. The proof given here is essentially the same as that of Theorem 3 and 4 in [10], except for the choice of $\mathbf{x}_1$. We include the details here for completeness.

PROPOSITION 2.4. *Let* $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$. *For* $m \ge 1$ *and* $k > 1$. *If*

$$\dim \mathcal{H} = n \ge (k-1)(m+1)^2,$$

*then* $\Lambda_k(\mathbf{A}) \ne \emptyset$.

*Proof.* We may assume that $\dim \mathcal{H} = n = (k-1)(m+1)^2$. Otherwise, replace each $A_j$ by $U^*A_jU$ for some $U$ such that $U^*U = I_n$. Let

$$q = (m+1)(k-1)+1.$$

Choose an eigenvector $\mathbf{x}_1$ of $A_1$ with $\|\mathbf{x}_1\| = 1$. Then choose a unit vector $\mathbf{x}_2$ orthogonal to $\mathbf{x}_1, A_2\mathbf{x}_1, \ldots, A_m\mathbf{x}_1$. By the assumption on $n$, we can choose an orthonormal set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_q\}$ of $q$ vectors in $\mathbf{C}^n$ such that for $1 < r \le q$, $\mathbf{x}_r$ is orthogonal to $A_j\mathbf{x}_i$ for all $1 \le i < r$ and $1 \le j \le m$. Let $X$ be the $n \times q$ matrix with $\mathbf{x}_i$ as the $i$-th column. Then $X^*A_jX$ is a diagonal matrix for $1 \le j \le m$. By Tverberg's Theorem [17], we can partition the set $\{i : 1 \le i \le q\}$ into $k$ disjoint subset $R_j$, $1 \le j \le k$ such that $R = \cap_{j=1}^k \mathbf{conv}\,\{\langle \mathbf{A}\mathbf{x}_i, \mathbf{x}_i\rangle : i \in R_j\} \ne \emptyset$. Suppose $\mathbf{a} \in R$. Then there exist non-negative numbers $t_{ij}$, $1 \le j \le k$, $i \in R_j$ such that for all $1 \le j \le k$, $\sum_{i\in R_j} t_{ij} = 1$ and $\sum_{i\in R_j} t_{ij}\langle \mathbf{A}\mathbf{x}_i, \mathbf{x}_i\rangle = \mathbf{a}$. Let $\mathbf{y}_j = \sum_{i\in R_j} \sqrt{t_{ij}}\mathbf{x}_i$ for $1 \le j \le k$. Then $\{\mathbf{y}_1, \ldots, \mathbf{y}_k\}$ is orthonormal and $\langle \mathbf{A}\mathbf{y}_j, \mathbf{y}_j\rangle = \mathbf{a}$ for all $1 \le j \le k$. $\square$

PROPOSITION 2.5. *Suppose $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$ and $1 \le r < k \le \dim \mathcal{H}$. Let $\mathcal{V}_r$ be the set of operator $X : \mathcal{H}_1^\perp \to \mathcal{H}$ such that $X^*X = I_{\mathcal{H}_1^\perp}$ for an $r$-dimensional subspace $\mathcal{H}_1$ of $\mathcal{H}$. Then*

$$\Lambda_k(\mathbf{A}) \subseteq \cap\{\Lambda_{k-r}(X^*A_1X, \ldots, X^*A_mX) : X \in \mathcal{V}_r\} \tag{2.1}$$

*and*

$$\mathbf{conv}\,\Lambda_k(\mathbf{A}) \subseteq \mathbf{conv}\,\left(\cap\{\Lambda_{k-r}(X^*A_1X, \ldots, X^*A_mX) : X \in \mathcal{V}_r\}\right). \tag{2.2}$$

*Proof.* Suppose $(a_1, \ldots, a_m) \in \Lambda_k(\mathbf{A})$. Let $\mathcal{H}_2$ be a $k$-dimensional subspace and $V : \mathcal{H}_2 \to \mathcal{H}$ such that $V^*V = I_{\mathcal{H}_2}$ and $V^*A_jV = a_jI_k$ for $j = 1, \ldots, m$. Let $X \in \mathcal{V}_r$ and $X^*X = I_{\mathcal{H}_1^\perp}$ for an $r$-dimensional subspace $\mathcal{H}_1$ of $\mathcal{H}$. Then

$$\mathcal{H}_0 = X^*\left(V\left(\mathcal{H}_2\right) \cap X\left(\mathcal{H}_1^\perp\right)\right)$$

has dimension at least $s = k - r$. Let $U : \mathcal{H}_0 \hookrightarrow \mathcal{H}$ be given by $Ux = x$ for all $x \in \mathcal{H}_0$. Then we have $U^*U = I_{\mathcal{H}_0}$ and $U^*\left(X^*A_jX\right)U = a_jI_{\mathcal{H}_0}$ for $j = 1, \ldots, m$. Thus, (2.1) holds, and the inclusion (2.2) follows. $\square$

Proposition 2.5 extends [14, Proposition 4.8] corresponding to the case when $m = 2$. In such a case, the set inclusion (2.1) becomes a set equality if $\dim \mathcal{H} < \infty$ or if $(A_1, A_2)$ is a commuting pair, i.e., $A_1 + iA_2$ is normal; see [14, Corollary 4.9]. The following example shows that the set equality in (2.1) may not hold even in the finite dimensional case if $m \ge 3$.

EXAMPLE 2.6. Let $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $B_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. For $k > 1$, let $A_j = B_j \otimes I_k$ for $j = 1, 2, 3$.

(a) We have $\Lambda_k(A_1, \ldots, A_m) = \Lambda_1(B_1, B_2, B_3) = \{\mathbf{a} \in \mathbf{R}^3 : \|\mathbf{a}\| = 1\}$, which is not convex.

(b) If $r = k-1$ and $X \in \mathcal{V}_r$ and $X^*X = I_{\mathcal{H}_1^\perp}$ for an $r$-dimensional subspace $\mathcal{H}_1$, then $\dim \mathcal{H}_1^\perp = 2k - r = k+1 \geq 3$ so that

$$\Lambda_{k-r}\left(X^*A_1X, X^*A_2X, X^*A_3X\right) = \Lambda_1\left(X^*A_1X, X^*A_2X, X^*A_3X\right)$$

is convex [11].

Consequently, $\cap\{\Lambda_{k-r}\left(X^*A_1X, X^*A_2X, X^*A_3X\right) : X \in \mathcal{V}_r\}$ is convex and cannot be equal to $\Lambda_k(A_1, A_2, A_3)$.

For $m > 3$, we can take $A_1$, $A_2$, $A_3$ as above and $A_j = 0_{2k}$ for $3 < j \leq m$. Then we have

$$\Lambda_k(A_1, A_2, A_3) \neq \cap\{\Lambda_{k-r}(X^*A_1X, \ldots, X^*A_mX) : X \in \mathcal{V}_r\}.$$

To verify (a), suppose $U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$ is such that

$$U_1, U_2 \in M_k, \quad U^*U = U_1^*U_1 + U_2^*U_2 = I_k \quad \text{and} \quad U^*A_jU = a_jI_k.$$

Then $U_1^*U_1 - U_2^*U_2 = a_1I_k$. It follows that

$$U_1^*U_1 = (1+a_1)I_k \quad \text{and} \quad U_2^*U_2 = (1-a_1)I_k.$$

Thus, $U_1U_1^* = (1+a_1)I_k$ and $U_2U_2^* = (1-a_1)I_k$. As a result,

$$U_i^*U_jU_j^*U_i = (1+a_1)(1-a_1)I = U_i^*U_iU_j^*U_j \quad \text{for } (i,j) \in \{(1,2),(2,1)\},$$

and

$$\begin{aligned}
(a_1^2 + a_2^2 + a_3^2)I_k &= \sum_{j=1}^{3}(U^*A_jU)^2 \\
&= (U_1^*U_1 - U_2^*U_2)^2 + (U_1^*U_2 + U_2^*U_1)^2 + (iU_1^*U_2 - iU_2^*U_1)^2 \\
&= (U_1^*U_1 + U_2^*U_2)^2 \\
&= I_k.
\end{aligned}$$

Thus, $\Lambda_k(A_1, A_2, A_3) \subseteq \{(a_1, a_2, a_3) \in \mathbf{R}^3 : a_1^2 + a_2^2 + a_3^2 = 1\}$.

Conversely, suppose $(a_1, a_2, a_3) \in \mathbf{R}^3$ such that $a_1^2 + a_2^2 + a_3^2 = 1$. Let

$$(\alpha, \beta) = \begin{cases} (0,1) & if\ a_1 = -1, \\ \\ \dfrac{(1+a_1, a_2 - ia_3)}{\sqrt{2(1+a_1)}} & otherwise. \end{cases}$$

Let $U = \begin{pmatrix} \alpha I_k \\ \beta I_k \end{pmatrix}$. Direct computation shows that $U^*A_jU = a_jI_k$ for $1 \leq j \leq 3$.

By a similar argument or putting $k = 1$, we see that $\Lambda_1(B_1, B_2, B_3)$ has the same form. $\square$

It is natural to ask if the set equality in (2.2) can hold for for $m > 2$. Also,

$$\begin{aligned}&\mathbf{conv}\ (\cap\{\Lambda_{k-r}(X^*A_1X,\dots,X^*A_mX) : X\in\mathcal{V}_r\})\\ &\subseteq \cap\{\mathbf{conv}\,\Lambda_{k-r}(X^*A_1X,\dots,X^*A_mX) : X\in\mathcal{V}_r\}.\end{aligned}$$

It is interesting to determine whether the two sets are equal.

## 3. GEOMETRIC PROPERTIES OF $\Lambda_k(\mathbf{A})$

Let $\mathbf{A}=(A_1,\dots,A_m)\in\mathcal{S}(\mathcal{H})^m$. Much is known about $\Lambda_1(\mathbf{A})$; for example see [11] and its references. For instance, $\Lambda_1(\mathbf{A})$ is always convex if $m\le 3$ unless $(\dim\mathcal{H},m)=(2,3)$; If $(\dim\mathcal{H},m)=(2,3)$ or $n>1$, $m\ge 4$, there are examples $\mathbf{A}\in\mathcal{S}(\mathcal{H})^m$ such that $\Lambda_1(\mathbf{A})$ is not convex. Furthermore, for any $A_1,A_2,A_3\in\mathcal{S}(\mathcal{H})$ such that span $\{I,A_1,A_2,A_3\}$ has dimension 4, there is always an $A_4\in\mathcal{S}(\mathcal{H})$ for which $\Lambda_1(A_1,\dots,A_4)$ is not convex.

In the following, we show that $\Lambda_k(\mathbf{A})$ is always star-shaped if $\dim\mathcal{H}$ is sufficiently large. Moreover, it always contains the convex hull of $\Lambda_{\hat{k}}(\mathbf{A})$ for $\hat{k}=(m+2)k$. If $k=1$ and $m>2$, we can lower the bound of the dimension of the Hilbert space to get the star-shapedness result. We begin with the following.

THEOREM 3.1. *Let $\mathbf{A}=(A_1,\dots,A_m)\in\mathcal{S}(\mathcal{H})^m$ and $k$ be a positive integer. If $\Lambda_{\hat{k}}(\mathbf{A})\neq\emptyset$ for some $\hat{k}\ge(m+2)k$, then $\Lambda_k(\mathbf{A})$ is star-shaped and contains the convex subset $\mathbf{conv}\,\Lambda_{\hat{k}}(\mathbf{A})$ so that every element in $\mathbf{conv}\,\Lambda_{\hat{k}}(\mathbf{A})$ is a star center of $\Lambda_k(\mathbf{A})$.*

Note that $\Lambda_{\hat{k}}(\mathbf{A})$ may be empty if $\dim\mathcal{H}$ is small relative to $\hat{k}$. Even if $\Lambda_{\hat{k}}(\mathbf{A})$ is non-empty, it may be much smaller than its convex hull; for example, see Example 2.6. So, the conclusion in Theorem 3.1 is rather remarkable.

*Proof.* We may assume $\mathbf{a}=\mathbf{0}\in\Lambda_{\hat{k}}(\mathbf{A})$. Then there exists $Y$ such that $Y^*Y=I_{(m+2)k}$ and $Y^*A_jY=\mathbf{0}$ for all $1\le j\le m$.

Let $\mathbf{b}\in\Lambda_k(\mathbf{A})$. Then there exists $X$ such that $X^*X=I_k$ and

$$(X^*A_1X,\dots,X^*A_mX)=(b_1I_k,\dots,b_mI_k).$$

Suppose $\mathcal{X}$ and $\mathcal{Y}$ are the range spaces of $X$ and $Y$, respectively. Then we have

$$\begin{aligned}&\dim\left(\mathcal{Y}\cap(\mathcal{X}+A_1(\mathcal{X})+\cdots+A_m(\mathcal{X}))^\perp\right)\\ &\ge\dim\mathcal{Y}-\dim\,(\mathcal{X}+A_1(\mathcal{X})+\cdots+A_m(\mathcal{X}))\ge k.\end{aligned}$$

Let $\mathcal{Y}_1$ be an $k$-dimensional subspace of $\mathcal{Y}\cap(\mathcal{X}+A_1(\mathcal{X})+\cdots+A_m(\mathcal{X}))^\perp$ and $\mathcal{Y}_2=\mathcal{Y}\cap(\mathcal{X}+\mathcal{Y}_1)^\perp$. Set $Z=[X|Y_1|Y_2]$, where $Y_i$ has columns forming an orthonormal basis of $\mathcal{Y}_i$ for $i=1,2$. Then we have $Z^*Z=I_{(m+2)k}$ and for $1\le j\le m$, $Z^*A_jZ$ has the form

$$\begin{pmatrix} b_jI_k & \mathbf{0}_k & * \\ \mathbf{0}_k & \mathbf{0}_k & * \\ * & * & * \end{pmatrix}.$$

Let $C_j = b_j I_k \oplus \mathbf{0}_k$. For $t \in [0,1]$, we have

$$t(b_1,\dots,b_m) \in \Lambda_k(C_1,\dots,C_m) \subseteq \Lambda_k(Z^*A_1Z,\dots,Z^*A_mZ) \subseteq \Lambda_k(\mathbf{A}).$$

Clearly, $\Lambda_{\hat{k}}(\mathbf{A}) \subseteq \Lambda_k(\mathbf{A})$. Since every element of $\Lambda_{\hat{k}}(\mathbf{A})$ is a star center of $\Lambda_k(\mathbf{A})$ and the set of star centers of a star-shaped set is convex, we see that every element in $\mathbf{conv}\,\Lambda_{\hat{k}}(\mathbf{A})$ is a star center of $\Lambda_k(\mathbf{A})$. Hence, $\mathbf{conv}\,\Lambda_{\hat{k}}(\mathbf{A}) \subseteq \Lambda_k(\mathbf{A})$. □

If $\dim \mathcal{H}$ is finite, then $\Lambda_k(\mathbf{A})$ is always closed. But this may not be the case if $\dim \mathcal{H}$ is infinite. Using Theorem 3.1, we can prove the star-shapedness of $\mathbf{cl}\,(\Lambda_k(\mathbf{A}))$.

COROLLARY 3.2. *Let $\mathbf{A} = (A_1,\dots,A_m) \in \mathcal{S}(\mathcal{H})^m$ and $k$ be a positive integer. If $\mathbf{cl}\,(\Lambda_{\hat{k}}(\mathbf{A})) \neq \emptyset$ for some $\hat{k} \geq (m+2)k$, then $\mathbf{cl}\,(\Lambda_k(\mathbf{A}))$ is star-shaped and contains the convex subset $\mathbf{conv}\,\mathbf{cl}\,(\Lambda_{\hat{k}}(\mathbf{A}))$ so that every element in $\mathbf{conv}\,\mathbf{cl}\,(\Lambda_{\hat{k}}(\mathbf{A}))$ is a star center of $\mathbf{cl}\,(\Lambda_k(\mathbf{A}))$.*

*Proof.* Suppose $\mathbf{a} \in \mathbf{cl}\,(\Lambda_{\hat{k}}(\mathbf{A}))$ and $\mathbf{b} \in \Lambda_k(\mathbf{A})$. Then for every $\varepsilon$ there is $\tilde{\mathbf{a}} = (\tilde{a}_1,\dots,\tilde{a}_m) \in \Lambda_{\hat{k}}(\mathbf{A})$ such that $\ell_1(\tilde{\mathbf{a}} - \mathbf{a}) < \varepsilon$. By Theorem 3.1, we see that the line segment joining $\tilde{\mathbf{a}}$ and $\mathbf{b}$ lies in $\Lambda_k(\mathbf{A})$. Consequently, the line segment joining $\mathbf{a}$ and $\mathbf{b}$ lies in $\mathbf{cl}\,(\Lambda_k(\mathbf{A}))$. The proof of the last assertion is similar to that of Theorem 3.1. □

It is easy to see that a star center of $\Lambda_k(\mathbf{A})$ is also a star center of $\mathbf{cl}\,(\Lambda_k(\mathbf{A}))$. However, the converse may not hold. The following example from [12, Example 3.2] illustrates this.

EXAMPLE 3.3. Consider $\mathcal{H} = \ell^2$ with canonical basis $\{e_n : n \geq 1\}$. Let $\mathbf{A} = (A_1,\dots,A_4)$ with $A_1 = \mathrm{diag}\,(1,0,1/3,1/4,\dots.)$, $A_2 = \mathrm{diag}\,(1,0) \oplus 0$,

$$A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 0 \quad \text{and} \quad A_4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \oplus 0.$$

Then $(1,1,0,0) \in W(\mathbf{A})$ and $(0,0,0,0) \in W(\mathbf{A}) \cap W_e(\mathbf{A})$ is a star-center of the closure of $W(\mathbf{A})$. However, $(1/2,1/2,0,0) \notin W(\mathbf{A})$ so that $(0,0,0,0)$ is not a star-center of $W(\mathbf{A})$. In fact, $W(\mathbf{A})$ is not convex even though $\mathbf{cl}\,(W(\mathbf{A}))$ is convex.

By Proposition 2.4, we see that $\Lambda_{\hat{k}}(\mathbf{A})$ is non-empty if $\dim \mathcal{H}$ is sufficiently large. So, $\Lambda_k(\mathbf{A})$ is star-shaped and contains a convex set. The same comment also holds for $\mathbf{cl}\,(\Lambda_k(\mathbf{A}))$. More specifically, we have the following.

THEOREM 3.4. *Let $\mathbf{A} = (A_1,\dots,A_m) \in \mathcal{S}(\mathcal{H})^m$. If*

$$\dim \mathcal{H} \geq ((m+2)k-1)(m+1)^2,$$

*then both $\Lambda_k(\mathbf{A})$ and $\mathbf{cl}\,(\Lambda_k(\mathbf{A}))$ are star-shaped.*

In Theorem 3.7, we will show that the classical joint numerical range is star-shaped with a much milder restriction on $\dim \mathcal{H}$ comparing with that in Theorem 3.4. To demonstrate this, we need two related results.

PROPOSITION 3.5. *Suppose* $\dim \mathcal{H} = n$ *and* $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$ *is such that* $\{A_1, \ldots, A_m\}$ *is linearly independent. Assume that* $\mathbf{0} \in \Lambda_{n-1}(\mathbf{A})$*, i.e., there is a basis such that* $A_j$ *has operator matrix* $\begin{pmatrix} * & * \\ * & 0_{n-1} \end{pmatrix}$ *for* $j = 1, \ldots, m$.

(a) *If* $m = 2n-1$*, then there is an invertible* $S \in M_m(\mathbf{R})$ *such that*

$$\Lambda_1(\mathbf{A}) = \left\{ (1+u_1, u_2, \ldots, u_m)S : u_1, \ldots, u_m \in \mathbf{R}, \sum_{j=1}^m u_j^2 = 1 \right\}$$

*so that* $\Lambda_1(\mathbf{A})$ *is not star-shaped.*

(b) *If* $m < 2n-1$*, then* $\Lambda_1(\mathbf{A})$ *is star-shaped with* $\mathbf{0}$ *as a star center.*

*Proof.* (a) If $m = 2n-1$, there is an invertible $m \times m$ real matrix $T = (t_{ij})$ such that for $B_j = \sum_{i=1} t_{ij} A_i$ for $j = 1, \ldots, m$ with

$$\mathbf{B} = (B_1, \ldots, B_m) = (E_{11}, E_{12} + E_{21}, -iE_{12} + iE_{21}, \ldots, -iE_{1n} + iE_{n1}).$$

Let $\mathbf{x} = \mu(\cos t, \sin t(v_2 + iv_3), \ldots, \sin t(v_{m-1} + iv_m))^t$ be a unit vector in $\mathbf{C}^n$ such that $|\mu| = 1$, $t \in [0, \pi/2]$, and $v_2, \ldots, v_m \in \mathbf{R}$ with $\sum_{j=2}^m v_j^2 = 1$. Then

$$\begin{aligned} \langle \mathbf{Bx}, \mathbf{x} \rangle &= ((1+\cos(2t))/2, v_2 \sin(2t), v_2 \sin(2t), \ldots, v_m \sin(2t)) \\ &= (1+u_1, u_2, \ldots, u_m)D, \end{aligned}$$

where $D = [1/2] \oplus I_{n-1}$, $u_1 = \cos(2t)$ and $u_j = v_j \sin(2t)$ for $j = 2, \ldots, m$. It follows that

$$\Lambda_1(\mathbf{B}) = \left\{ (1+u_1, u_2, \ldots, u_m)D : u_1, \ldots, u_m \in \mathbf{R}, \sum_{j=1}^m u_j^2 = 1 \right\}.$$

By Proposition 2.1, $\Lambda_1(\mathbf{A}) = \{\mathbf{b}T^{-1} : \mathbf{b} \in \Lambda_1(\mathbf{B})\}$. The result follows.

(b) Now, suppose $m < 2n-1$. By adding more $A_j$, if necessary, we only need to consider the case when $m = 2n-2$. Let $\mathbf{v}_j$ be the row vector obtained by removing the first entry of the first row of $A_j$ for $j = 1, \ldots, m$.

**Case 1** Suppose span $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ has real dimension $m-1 = 2n-3$. Then there is a unitary matrix of the form $U = [1] \oplus U_0 \in M_n$ such that the $(1,n)$ entry of $U^* A_j U$ is real for $j = 1, \ldots, m$. Hence, there is an invertible $m \times m$ real matrix $T = (t_{ij})$ such that for $B_j = \sum_{i=1} t_{ij} A_i$ for $j = 1, \ldots, m$ with

$$\begin{aligned} \mathbf{B} &= (B_1, \ldots, B_m) \\ &= (E_{11}, E_{12} + E_{21}, -iE_{12} + iE_{21}, \ldots, -iE_{1,n-1} + iE_{n-1,1}, E_{1n} + E_{n1}). \end{aligned}$$

Suppose $\mathbf{b} \in \Lambda_1(\mathbf{B})$, $\mathbf{b} \neq \mathbf{0}$. Then there exists a unit vector $\mathbf{x} = \mu(u_0, u_1 + iu_2, \ldots, u_{m-1} + iu_m)^t$ such that $|\mu| = 1$ and $u_0 > 0$, with

$$\mathbf{b} = \langle \mathbf{Bx}, \mathbf{x} \rangle = u_0(u_0, 2u_1, \ldots, 2u_{m-1}).$$

For any $t \in (0,1)$, we can choose a unit vector of the form $\mathbf{x}_t = \sqrt{t}(u_0, u_1 + iu_2, \ldots, u_{m-1} + i\tilde{u}_m)^t$ with $t\tilde{u}_m^2 = 1 - \sum_{j=0}^{m-1} tu_j^2$ so that

$$\langle \mathbf{B}\mathbf{x}_t, \mathbf{x}_t \rangle = t\langle \mathbf{B}\mathbf{x}, \mathbf{x} \rangle.$$

**Case 2.** Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ has real dimension $m = 2n-2$. Then there is an invertible $m \times m$ real matrix $T = (t_{ij})$ such that

$$\begin{aligned} \mathbf{B} &= (B_1, \ldots, B_m) = \left( \sum_{i=1}^m t_{i1} A_i, \ldots, \sum_{i=1}^m t_{im} A_i \right) \\ &= (a_1 E_{11}, \ldots, a_m E_{11}) + (E_{12} + E_{21}, iE_{12} + iE_{21}, \ldots, E_{1n} + E_{n1}, -iE_{1n} + iE_{n1}) \end{aligned}$$

with $a_1, \ldots, a_m \in \mathbf{R}$. Suppose $\mathbf{b} \in \Lambda_1(\mathbf{B})$, $\mathbf{b} \neq \mathbf{0}$. Then there exists a unit vector $\mathbf{x} = \mu(u_0, u_1 + iu_2, \ldots, u_{m-1} + iu_m)^t$ such that $|\mu| = 1$ and $u_0 > 0$, with

$$\mathbf{b} = \langle \mathbf{B}\mathbf{x}, \mathbf{x} \rangle = u_0(a_1 u_0 + 2u_1, a_2 u_0 + 2u_2, \ldots, a_m u_0 + 2u_m).$$

For any $t \in (0,1)$, consider a vector of the form

$$\mathbf{x}_\xi = (\xi u_0, w_1 + iw_2, w_3 + iw_4, \ldots, w_{m-1} + iw_m)^t,$$

where $\xi \geq t$ and $w_j = a_j u_0 (t - \xi^2)/(2\xi) + tu_j/\xi$ for $j = 1, \ldots, m$. Then

$$\xi u_0 (a_j \xi u_0 + 2w_j) = tu_0(a_j u_0 + 2u_j), \qquad j = 1, \ldots, m,$$

so that

$$\langle \mathbf{B}\mathbf{x}_\xi, \mathbf{x}_\xi \rangle = t\langle \mathbf{B}\mathbf{x}, \mathbf{x} \rangle.$$

If $\xi = \sqrt{t}$, then $\mathbf{x}_\xi = \sqrt{t}(u_0, u_1 + iu_2, \ldots, u_{m-1} + iu_m)^t$ has norm less than 1; if $\xi \to \infty$, then $\|\mathbf{x}_\xi\| \geq |\xi u_0| \to \infty$. Thus, there is $\xi > t$ such that $\mathbf{x}_\xi$ is a unit vector satisfying $\langle \mathbf{B}\mathbf{x}_\xi, \mathbf{x}_\xi \rangle = t\langle \mathbf{B}\mathbf{x}, \mathbf{x} \rangle$. So, $\Lambda_1(\mathbf{B})$ is star-shaped with $\mathbf{0}$ as a star center. By Proposition 2.1, $\Lambda_1(\mathbf{A}) = \{\mathbf{b}T^{-1} : \mathbf{b} \in \Lambda_1(\mathbf{B})\}$. The result follows. □

THEOREM 3.6. *Let $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$. If $\Lambda_{\hat{k}}(\mathbf{A}) \neq \emptyset$ for some $\hat{k} > (m+1)/2$, then $\Lambda_1(\mathbf{A})$ is star-shaped and contains $\mathbf{conv}\, \Lambda_{\hat{k}}(\mathbf{A})$ such that every element in $\mathbf{conv}\, \Lambda_{\hat{k}}(\mathbf{A})$ is a star center of $\Lambda_1(\mathbf{A})$.*

*Proof.* We may assume $\mathbf{a} = \mathbf{0} \in \Lambda_{\hat{k}}(\mathbf{A})$ with $\hat{k} > (m+1)/2$. Suppose $\mathbf{x} \in \mathcal{H}$ is a unit vector and $\mathbf{b} = \langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle \in \Lambda_1(\mathbf{A})$. Suppose $X$ is such that $X^*X = I_{\hat{k}}$ and $X^*A_jX = 0_{\hat{k}}$ for $j = 1, \ldots, m$. Let $Y$ be such that $Y^*Y = I_{\hat{k}+1}$, and the range space of $Y$ contains the range space of $X$ and $\mathbf{x}$. Suppose $\mathbf{B} = (B_1, \ldots, B_m) = (Y^*A_1Y, \ldots, Y^*A_mY)$. Then we may assume that $B_j$ has the form $\begin{pmatrix} * & * \\ * & 0_{\hat{k}} \end{pmatrix}$ for $j = 1, \ldots, m$. Clearly, span $\{B_1, \ldots, B_m\}$ has dimension at most $m < 2\hat{k} - 1$. By Proposition 3.5, the line segment joining $\mathbf{0}$ and $\mathbf{b}$ lies entirely in $\Lambda_1(\mathbf{B}) \subseteq \Lambda_1(\mathbf{A})$. Thus, $\mathbf{0}$ is a star center of $\Lambda_1(\mathbf{A})$. Since the set of star centers of $\Lambda_1(\mathbf{A})$ is convex, we see that every element in $\mathbf{conv}\, \Lambda_{\hat{k}}(\mathbf{A})$ is a star center of $\Lambda_1(\mathbf{A})$. □

THEOREM 3.7. *Let* $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$. *If*

$$\dim \mathcal{H} \geq \left[\frac{m+1}{2}\right](m+1)^2,$$

*then* $\Lambda_1(\mathbf{A})$ *is star-shaped.*

*Proof.* Let $\hat{k} = \left[\frac{m+1}{2}\right] + 1 > \frac{m+1}{2}$. Then $\dim \mathcal{H} \geq (\hat{k}-1)(m+1)^2$ and $\Lambda_{\hat{k}}(A) \neq \emptyset$ by Proposition 2.4. The result then follows from Theorem 3.6. $\square$

## 4. RESULTS ON $\Lambda_\infty(\mathbf{A})$

In this section, we always assume that $\mathcal{H}$ has infinite dimension. Denote by $\mathcal{P}_\infty$ the set of infinite rank orthogonal projections in $\mathcal{S}(\mathcal{H})$. For $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$ let

$$\Lambda_\infty(\mathbf{A}) = \{(\gamma_1, \ldots, \gamma_m) \in \mathbf{R}^m : \text{ there is } P \in \mathcal{P}_\infty \\ \text{such that } PA_iP = \gamma_i P \text{ for all } 1 \leq i \leq m\}.$$

By the result in Section 3, we have the following.

PROPOSITION 4.1. *Suppose* $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$, *where* $\mathcal{H}$ *is infinite-dimensional. Then* $\Lambda_k(\mathbf{A})$ *is star-shaped for each positive integer* $k$. *Moreover, if* $\mathbf{a} \in \Lambda_\infty(\mathbf{A})$, *then* $\mathbf{a}$ *is a star center for* $\Lambda_k(\mathbf{A})$ *for every positive integer* $k$.

When $m = 2$, it was conjectured in [16] and confirmed in [14] that

$$\Lambda_\infty(A_1, A_2) = \bigcap_{k \geq 1} \Lambda_k(A_1, A_2);$$

in [1, Theorem 4], it was proven that

$$\Lambda_\infty(A_1, A_2) = \cap\{W(A_1 + F_1, A_2 + F_2) : F_1, F_2 \in \mathcal{S}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})\}.$$

In the following, we extend the above results to $\Lambda_\infty(A_1, \ldots, A_m)$ for $m > 2$. Moreover, we show that $\Lambda_\infty(\mathbf{A}) = \bigcap_{k \geq 1} S_k(\mathbf{A})$, where $S_k(\mathbf{A})$ is the set of star centers of $\Lambda_k(\mathbf{A})$. Hence, $\Lambda_\infty(\mathbf{A})$ is always convex.

THEOREM 4.2. *Suppose* $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$, *where* $\mathcal{H}$ *is infinite-dimensional. For each* $k \geq 1$, *let* $S_k(\mathbf{A})$ *be the set of star-center of* $\Lambda_k(\mathbf{A})$. *Then*

$$\Lambda_\infty(\mathbf{A}) = \cap_k S_k(\mathbf{A}) = \cap_k \Lambda_k(\mathbf{A}) = \cap\{W(\mathbf{A} + \mathbf{F}) : \mathbf{F} \in \mathcal{S}(\mathcal{H})^m \cap \mathcal{F}(\mathcal{H})^m\}. \tag{4.1}$$

*Consequently,* $\Lambda_\infty(\mathbf{A})$ *is convex.*

*Proof.* It follows from definitions and Theorem 3.1 that

$$\Lambda_\infty(\mathbf{A}) \subseteq \cap_k S_k(\mathbf{A}) \subseteq \cap_k \Lambda_k(\mathbf{A}).$$

We are going to prove that $\cap_k \Lambda_k(\mathbf{A}) \subseteq \cap\{W(\mathbf{A} + \mathbf{F}) : \mathbf{F} \in \mathcal{S}(\mathcal{H})^m \cap \mathcal{F}(\mathcal{H})^m\}$. Suppose $\mathbf{F} = (F_1, \ldots, F_m) \in \mathcal{S}(\mathcal{H})^m \cap \mathcal{F}(\mathcal{H})^m$ and $K = \sum_{i=1}^m \text{rank } (F_i) + 1$. Let

$\boldsymbol{\mu} = (\mu_1, \dots, \mu_m) \in \cap_k \Lambda_k(\mathbf{A})$. Then there exists a rank $K$ orthogonal projection $P$ such that $PA_jP = \mu_j P$ for $1 \le j \le m$. Let

$$\begin{aligned} \mathcal{H}_0 &= \text{range}\, P \cap \ker F_1 \cap \ker F_2 \cap \cdots \cap \ker F_m \\ &= \text{range}\, P \cap (\text{range}\, F_1 + \text{range}\, F_2 + \cdots + \text{range}\, F_m)^{\perp} . \end{aligned}$$

Then $\dim \mathcal{H}_0 \ge 1$. Let $\mathbf{x}$ be a unit vector in $\mathcal{H}_0$. Then we have $\langle (\mathbf{A} + \mathbf{F})\, \mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle = \boldsymbol{\mu}$. Therefore, $\boldsymbol{\mu} \in W(\mathbf{A} + \mathbf{F})$. Hence, we have $\cap_k \Lambda_k(\mathbf{A}) \subseteq \cap \{ W(\mathbf{A} + \mathbf{F}) : \mathbf{F} \in \mathcal{S}(\mathcal{H})^m \cap \mathcal{F}(\mathcal{H})^m \}$.

Next, we prove that $\cap \{ W(\mathbf{A} + \mathbf{F}) : \mathbf{F} \in \mathcal{S}(\mathcal{H})^m \cap \mathcal{F}(\mathcal{H})^m \} \subseteq \Lambda_\infty(\mathbf{A})$. Suppose

$$\boldsymbol{\mu} \in \cap \{ W(\mathbf{A} + \mathbf{F}) : \mathbf{F} \in \mathcal{S}(\mathcal{H})^m \cap \mathcal{F}(\mathcal{H})^m \}.$$

By Remark 2.3, we may assume that $\boldsymbol{\mu} = \mathbf{0}$. Then $\mathbf{0} \in W(\mathbf{A})$ and there exists a unit vector $\mathbf{x}_1$ such that $\langle \mathbf{A}\mathbf{x}_1, \mathbf{x}_1 \rangle = \mathbf{0}$. Suppose we have chosen an orthonormal set of vectors $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ such that $\langle \mathbf{A}\mathbf{x}_i, \mathbf{x}_j \rangle = \mathbf{0}$ for all $1 \le i,\ j \le n$. Let $\mathcal{H}_0$ be the subspace spanned by

$$\{\mathbf{x}_i : 1 \le i \le n\} \cup \{A_j \mathbf{x}_i : 1 \le i \le n,\ 1 \le j \le m\}$$

and $P$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_0$. Suppose

$$\mathbf{B} = \left( (I-P)A_1(I-P)|_{\mathcal{H}_0^\perp}, \dots, (I-P)A_m(I-P)|_{\mathcal{H}_0^\perp} \right).$$

Let $\mathbf{b} = (b_1, \dots, b_m)$ be a star-center of $W(\mathbf{B})$. Then

$$\mathbf{b} I_{\mathcal{H}_0} \oplus \mathbf{B} = (b_1 P + (I-P)A_1(I-P), \dots, b_m P + (I-P)A_m(I-P)) = \mathbf{A} + \mathbf{F}$$

for some $\mathbf{F} \in \mathcal{S}(\mathcal{H})^m \cap \mathcal{F}(\mathcal{H})^m$. Therefore, $\mathbf{0} \in W(\mathbf{b} I_{\mathcal{H}_0} \oplus \mathbf{B})$. Hence, there exists a unit vector $\mathbf{x} \in \mathcal{H}$ such that $\mathbf{0} = \langle (\mathbf{A} + \mathbf{F})\mathbf{x}, \mathbf{x} \rangle$. Let $\mathbf{x} = \mathbf{y} + \mathbf{z}$, where $\mathbf{y} \in \mathcal{H}_0$ and $\mathbf{z} \in \mathcal{H}_0^\perp$. Then $\|\mathbf{y}\|^2 + \|\mathbf{z}\|^2 = \|\mathbf{x}\|^2 = 1$. If $\mathbf{z} = \mathbf{0}$, then $\mathbf{0} = \mathbf{b} \in W(\mathbf{B})$. If $\mathbf{z} \ne \mathbf{0}$, then by Proposition 4.1, we have

$$\mathbf{0} = \langle (\mathbf{A} + \mathbf{F})\mathbf{x}, \mathbf{x} \rangle = \|\mathbf{y}\|^2 \mathbf{b} + \|\mathbf{z}\|^2 \langle \mathbf{B} \left( \frac{\mathbf{z}}{\|\mathbf{z}\|} \right), \left( \frac{\mathbf{z}}{\|\mathbf{z}\|} \right) \rangle \in W(\mathbf{B})$$

So there exists a unit vector $\mathbf{x}_{n+1} \in \mathcal{H}_0^\perp$ such that

$$\mathbf{0} = \langle (\mathbf{A} + \mathbf{F})\mathbf{x}_{n+1}, \mathbf{x}_{n+1} \rangle = \langle \mathbf{B}\mathbf{x}_{n+1}, \mathbf{x}_{n+1} \rangle = \langle \mathbf{A}\mathbf{x}_{n+1}, \mathbf{x}_{n+1} \rangle$$

Hence, inductively, we can choose an orthonormal sequence of vectors $\{\mathbf{x}_n\}_{n=1}^\infty$ such that

$$\langle \mathbf{A}\mathbf{x}_i, \mathbf{x}_j \rangle = \mathbf{0} \ \text{ for all } i,\ j .$$

Thus, we have

$$\begin{aligned} \Lambda_\infty(\mathbf{A}) &\subseteq \cap_k S_k(\mathbf{A}) \subseteq \cap_k \Lambda_k(\mathbf{A}) \\ &\subseteq \cap \{ W(\mathbf{A} + \mathbf{F}) : \mathbf{F} \in \mathcal{S}(\mathcal{H})^m \cap \mathcal{F}(\mathcal{H})^m \} \subseteq \Lambda_\infty(\mathbf{A}). \end{aligned}$$

Since $S_k(\mathbf{A})$ is convex for all $k \ge 1$, the last statement follows. $\square$

The last equality in (4.1) establishes a relationship between $\Lambda_\infty(\mathbf{A})$ and the joint numerical ranges of finite rank perturbation of $\mathbf{A}$. The following result gives an extension.

THEOREM 4.3. *Suppose* $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$. *Let* $n \geq 1$ *and* $\mathbf{F}_0 \in \mathcal{S}(\mathcal{H})^m \cap \mathcal{F}(\mathcal{H})^m$. *Then the following sets are equal.*

(a) $\Lambda_\infty(\mathbf{A})$.
(b) $\Lambda_\infty(\mathbf{A} + \mathbf{F}_0)$.
(c) $\cap\{\Lambda_n(\mathbf{A} + \mathbf{F}) : \mathbf{F} \in \mathcal{F}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m\}$.
(d) $\cap_{k\geq 1}\left(\cap\{\Lambda_k(\mathbf{A} + \mathbf{F}) : \mathbf{F} \in \mathcal{F}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m\}\right)$.

*Proof.* By Theorem 4.2, we have

$$\begin{aligned} & \Lambda_\infty(\mathbf{A} + \mathbf{F}_0) \\ = \; & \cap\{W(\mathbf{A} + \mathbf{F}_0 + \mathbf{F}) : \mathbf{F} \in \mathcal{S}(\mathcal{H})^m \cap \mathcal{F}(\mathcal{H})^m\} \\ = \; & \cap\{W(\mathbf{A} + \mathbf{F}) : \mathbf{F} \in \mathcal{S}(\mathcal{H})^m \cap \mathcal{F}(\mathcal{H})^m\} \\ = \; & \Lambda_\infty(\mathbf{A}). \end{aligned}$$

This proves the equality of the sets in (a) and (b). For the equality of the sets of (a) and (c), let $\mathbf{F} \in \mathcal{F}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m$. Then we have

$$\Lambda_\infty(\mathbf{A}) = \Lambda_\infty(\mathbf{A} + \mathbf{F}) \subseteq \Lambda_n(\mathbf{A} + \mathbf{F}) \subseteq W(\mathbf{A} + \mathbf{F}).$$

It follows that

$$\begin{aligned} \Lambda_\infty(\mathbf{A}) & \subseteq \cap\{\Lambda_n(\mathbf{A} + \mathbf{F}) : \mathbf{F} \in \mathcal{S}(\mathcal{H})^m \cap \mathcal{F}(\mathcal{H})^m\} \\ & \subseteq \cap\{W(\mathbf{A} + \mathbf{F}) : \mathbf{F} \in \mathcal{S}(\mathcal{H})^m \cap \mathcal{F}(\mathcal{H})^m\} = \Lambda_\infty(\mathbf{A}). \end{aligned}$$

The equivalence of (a) and (d) follows immediately. $\square$

Recall that $\mathcal{V}_r$ is the set of $X : \mathcal{H}_1^\perp \to \mathcal{H}$ such that $\dim \mathcal{H}_1 = r$ and $X^*X = I_{\mathcal{H}_1^\perp}$. Then $X^*\mathbf{A}X$ is a compression of $\mathbf{A}$ to $\mathcal{H}_1^\perp$. The next result is an analog to Theorem 4.3 for $\Lambda_k(X^*\mathbf{A}X)$.

THEOREM 4.4. *Suppose* $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{S}(\mathcal{H})^m$. *Let* $n$, $r_0 \geq 1$ *and* $X_0 \in \mathcal{V}_{r_0}$. *Then for* $X^*\mathbf{A}X = (X^*A_1X, \ldots, X^*A_mX)$, *the following sets are equal.*

(a) $\Lambda_\infty(\mathbf{A})$.
(b) $\Lambda_\infty(X_0^*\mathbf{A}X_0)$.
(c) $\cap\{\Lambda_n(X^*\mathbf{A}X) : X \in \cup_{r\geq 1}\mathcal{V}_r\}$.
(d) $\cap_{k\geq 1}(\cap\{\Lambda_k(X^*\mathbf{A}X) : X \in \cup_{r\geq 1}\mathcal{V}_r\})$.

*Proof.* By (4.1) and (2.1), we have

$$\Lambda_\infty(\mathbf{A}) = \cap_k \Lambda_k(\mathbf{A}) = \cap_k \Lambda_{k+r}(\mathbf{A}) \subseteq \cap_k \Lambda_k(X_0^*\mathbf{A}X_0) = \Lambda_\infty(X_0^*\mathbf{A}X_0) \subseteq \Lambda_\infty(\mathbf{A}).$$

This proves the equality of the sets in (a) and (b). For the equality of the sets of (a) and (c), we will first show that

$$\cap\{\Lambda_1(X^*\mathbf{A}X) : X \in \cup_{r\geq 1}\mathcal{V}_r\} \subseteq \Lambda_\infty(\mathbf{A}). \tag{4.2}$$

Let $\boldsymbol{\mu} \in \cap\{\Lambda_1(X^*\mathbf{A}X) : X \in \cup_{r\geq 1}\mathcal{V}_r\}$. By Remark 2.3, we may assume that $\boldsymbol{\mu} = \mathbf{0}$. Then there exists a unit vector $\mathbf{x}_1$ such that $\langle \mathbf{A}\mathbf{x}_1, \mathbf{x}_1\rangle = \mathbf{0}$. Suppose we have chosen an orthonormal set of vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ such that $\langle \mathbf{A}\mathbf{x}_i, \mathbf{x}_j\rangle = \mathbf{0}$ for all $1 \leq i,\ j \leq N$. Let $\mathcal{H}_1$ be the subspace spanned by

$$\{\mathbf{x}_i : 1 \leq i \leq N\} \cup \{A_j\mathbf{x}_i : 1 \leq i \leq N,\ 1 \leq j \leq m\}$$

and $X : \mathcal{H}_1^{\perp} \to \mathcal{H}$ be given by $X(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in \mathcal{H}_1^{\perp}$. Then $\mathbf{0} \in \Lambda_1(X^*\mathbf{A}X)$. So there exists a unit vector $\mathbf{x}_{N+1} \in \mathcal{H}_1^{\perp}$ such that

$$\mathbf{0} = \langle (X^*\mathbf{A}X)\mathbf{x}_{N+1}, \mathbf{x}_{N+1}\rangle = \langle \mathbf{A}\mathbf{x}_{N+1}, \mathbf{x}_{N+1}\rangle .$$

Inductively, we can find an orthonormal sequence $\{\mathbf{x}_i\}$ in $\mathcal{H}$ such that $\langle \mathbf{A}\mathbf{x}_i, \mathbf{x}_j\rangle = \mathbf{0}$ for all $i,\ j$. Hence, $\mathbf{0} \in \Lambda_\infty(\mathbf{A})$.

To continue the proof of the equality of the sets of (a) and (c). Let $X \in \cup_{r\geq 1}\mathcal{V}_r$. Then we have

$$\Lambda_\infty(\mathbf{A}) = \Lambda_\infty(X^*\mathbf{A}X) \subseteq \Lambda_n(X^*\mathbf{A}X) \subseteq \Lambda_1(X^*\mathbf{A}X).$$

It follows that

$$\begin{aligned} \Lambda_\infty(\mathbf{A}) &\subseteq \cap\{\Lambda_\infty(X^*\mathbf{A}X) : X \in \cup_{r\geq 1}\mathcal{V}_r\} \\ &\subseteq \cap\{\Lambda_1(X^*\mathbf{A}X) : X \in \cup_{r\geq 1}\mathcal{V}_r\} \subseteq \Lambda_\infty(\mathbf{A}). \end{aligned}$$

The equality of the sets of (a) and (d) follows immediately. $\square$

Recall that the joint essential numerical range of $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$ is defined by

$$W_e(\mathbf{A}) = \cap\{\mathbf{cl}\,(W(\mathbf{A}+\mathbf{F})) : \mathbf{F} \in \mathcal{S}(\mathcal{H})^m \cap \mathcal{F}(\mathcal{H})^m\}.$$

Using the last two theorems, we have the following.

COROLLARY 4.5. *Let $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$, where $\mathcal{H}$ is infinite dimensional. Denote by $\tilde{S}_k(\mathbf{A})$ the set of star center of $\mathbf{cl}\,(\Lambda_k(\mathbf{A}))$. Then*

$$W_e(\mathbf{A}) = \bigcap_{k\geq 1} \mathbf{cl}\ (\Lambda_k(\mathbf{A})) = \bigcap_{k\geq 1} \tilde{S}_k(\mathbf{A}).$$

*In addition, let $n \geq 1$ and $\mathbf{F}_0 \in \mathcal{S}(\mathcal{H})^m \cap \mathcal{F}(\mathcal{H})^m$. Moreover, let $\mathbf{V}$ be a finite dimensional subspace of $\mathcal{H}$ and $X_0 : \mathbf{V}^{\perp} \to \mathcal{H}$ such that $X_0^*X_0 = I_{\mathbf{V}^{\perp}}$. Then the following sets are equal.*

(a) $W_e(\mathbf{A})$.
(b) $W_e(\mathbf{A}+\mathbf{F}_0)$.
(c) $W_e(X_0^*\mathbf{A}X_0)$.
(d) $\cap\{\mathbf{cl}\ (\Lambda_n(\mathbf{A}+\mathbf{F})) : \mathbf{F} \in \mathcal{F}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m\}$.
(e) $\cap\{\mathbf{cl}\ (\Lambda_n(X^*\mathbf{A}X)) : X \in \cup_{r\geq 1}\mathcal{V}_r\}$.
(f) $\cap_{k\geq 1}\left(\cap\{\mathbf{cl}\ (\Lambda_k(\mathbf{A}+\mathbf{F})) : \mathbf{F} \in \mathcal{F}(\mathcal{H})^m \cap \mathcal{S}(\mathcal{H})^m\}\right)$.
(g) $\cap_{k\geq 1}\left(\cap\{\mathbf{cl}\ ((\Lambda_k(X^*\mathbf{A}X))) : X \in \cup_{r\geq 1}\mathcal{V}_r\}\right)$.

Let $A \in \mathcal{S}(\mathcal{H})$ and $k$ be a positive integer. Denote by $\mathcal{U}_k$ the set of $X : \mathbf{C}^k \to \mathcal{H}$ such that $X^*X = I_k$ and

$$\lambda_k(A) = \sup\{\min \sigma(X^*AX) : X \in \mathcal{U}_k\},$$

where $\sigma(B)$ is the spectrum of the operator $B$. For $\mathbf{c} = (c_1, \ldots, c_m) \in \mathbf{R}^m$ and $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$, let $\mathbf{c} \cdot \mathbf{A} = \sum_{i=1}^m c_i A_i$. Define

$$\Omega_\infty(\mathbf{A}) = \bigcap_{\mathbf{c} \in \mathbf{R}^m} \{\mathbf{a} \in \mathbf{R}^m : \mathbf{c} \cdot \mathbf{a} \le \lambda_k(\mathbf{c} \cdot \mathbf{A}) \text{ for all } k \ge 1\}.$$

The following extends [14, Theorem 2.1].

THEOREM 4.6. *Let $\mathbf{A} \in \mathcal{S}(\mathcal{H})^m$, where $\mathcal{H}$ is infinite dimensional. Then*

$$\Lambda_\infty(\mathbf{A}) \subseteq \Omega_\infty(\mathbf{A}) = W_e(\mathbf{A}).$$

*Proof.* For $k \ge 1$, let

$$\Omega_k(\mathbf{A}) = \bigcap_{\mathbf{c} \in \mathbf{R}^m} \{\mathbf{a} \in \mathbf{R}^m : \mathbf{c} \cdot \mathbf{a} \le \lambda_k(\mathbf{c} \cdot \mathbf{A})\}.$$

Clearly, $\Omega_\infty(\mathbf{A}) = \bigcap_{k\ge 1} \Omega_k(\mathbf{A})$. Suppose $k \ge 1$ and $\mathbf{a} = (a_1, \ldots, a_m) \in \Lambda_k(\mathbf{A})$. Then there exists $P \in \mathcal{P}_k$ such that $PA_jP = a_jP$ for all $1 \le j \le m$. For every $\mathbf{c} \in \mathbf{R}^m$, we have $P(\mathbf{c} \cdot \mathbf{A})P = (\mathbf{c} \cdot \mathbf{a})P$. Hence,

$$\mathbf{c} \cdot \mathbf{a} = \lambda_k(\mathbf{c} \cdot P\mathbf{A}P) = \lambda_k(P(\mathbf{c} \cdot \mathbf{A})P) \le \lambda_k(\mathbf{c} \cdot \mathbf{A}).$$

Therefore, $\Lambda_k(\mathbf{A}) \subseteq \Omega_k(\mathbf{A})$. Since $\Omega_k(\mathbf{A})$ is closed, we have $\mathbf{cl}\,(\Lambda_k(\mathbf{A})) \subseteq \Omega_k(\mathbf{A})$. Hence,

$$\Lambda_\infty(\mathbf{A}) = \cap_{k\ge1}\Lambda_k(\mathbf{A}) \subseteq \cap_{k\ge1}\mathbf{cl}\,(\Lambda_k(\mathbf{A})) = W_e(\mathbf{A}) \subseteq \cap_{k\ge1}\Omega_k(\mathbf{A}) = \Omega_\infty(\mathbf{A}).$$

To show that $\Omega_\infty(\mathbf{A}) \subseteq W_e(\mathbf{A})$. Suppose $\mathbf{a} \in \Omega_k(\mathbf{A})$. By Remark 2.3, we may assume that $\mathbf{a} = \mathbf{0}$. So, $\lambda_k(\mathbf{c} \cdot \mathbf{A}) \ge 0$ for all $k \ge 1$ and $\mathbf{c} \in \mathbf{R}^m$. For $\mathbf{F} = (F_1, \ldots, F_m) \in \mathcal{S}(\mathcal{H})^m \cap \mathcal{F}(\mathcal{H})^m$, let $K = \sum_{i=1}^m \text{rank }(F_i) + 1$. Then

$$\lambda_1(\mathbf{c} \cdot (\mathbf{A} + \mathbf{F})) \ge \lambda_{K+1}(\mathbf{c} \cdot \mathbf{A}) \ge 0 \text{ and } \lambda_1(-(\mathbf{c} \cdot (\mathbf{A} + \mathbf{F}))) \ge \lambda_{K+1}(-\mathbf{c} \cdot \mathbf{A}) \ge 0.$$

Therefore, $\mathbf{c} \cdot \mathbf{0} = 0 \in \mathbf{cl}\,(W(\mathbf{c} \cdot (\mathbf{A} + \mathbf{F}))) = \mathbf{c} \cdot \mathbf{cl}\,(W(\mathbf{A} + \mathbf{F}))$. Hence, $\mathbf{c} \cdot \mathbf{0} \in \mathbf{c} \cdot W_e(\mathbf{A})$ for all $\mathbf{c} \in \mathbf{R}^m$. By the convexity of $W_e(\mathbf{A})$, we have $\mathbf{0} \in W_e(\mathbf{A})$. $\square$

EXAMPLE 4.7. For $n \ge 1$, let $B_n = \begin{bmatrix} \frac{1}{n} & 0 \\ 0 & -\frac{1}{n} \end{bmatrix}$, $C_n = \begin{bmatrix} \frac{1}{n} & 0 \\ 0 & 0 \end{bmatrix}$, $A_1 = \oplus_{n=1}^\infty B_n$, $A_2 = \oplus_{n=1}^\infty C_n$ and $\mathbf{A} = (A_1, A_2)$. Then $(0, 0) \in \Omega_\infty(\mathbf{A})$ but $\Lambda_\infty(\mathbf{A}) = \emptyset$.

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