Linear maps transforming the higher numerical ranges

Chi-Kwong Li, Yiu-Tung Poon, Nung-Sing Sze

Abstract

Let $k \in \{1, \ldots, n\}$. The k-numerical range of $A \in M_n$ is the set

$$W_k(A) = \{(\operatorname{tr} X^*AX)/k : X \text{ is } n \times k,\ X^*X = I_k\},$$

and the k-numerical radius of A is the quantity

$$w_k(A) = \max\{|z| : z \in W_k(A)\}.$$

Suppose $k > 1$, $k' \in \{1, \ldots, n'\}$ and $n' < C(n,k)\min\{k', n'-k'\}$. It is shown that there is a linear map $\phi : M_n \to M_{n'}$ satisfying $W_{k'}(\phi(A)) = W_k(A)$ for all $A \in M_n$ if and only if $n'/n = k'/k$ or $n'/n = k'/(n-k)$ is a positive integer. Moreover, if such a linear map ϕ exists, then there is a unitary matrix $U \in M_{n'}$ and nonnegative integers p, q with $p + q = n'/n$ such that ϕ has the form

$$A \mapsto U^*[\underbrace{A \oplus \cdots \oplus A}_{p} \oplus \underbrace{A^t \oplus \cdots \oplus A^t}_{q}]U$$

or

$$A \mapsto U^*[\underbrace{\psi(A) \oplus \cdots \oplus \psi(A)}_{p} \oplus \underbrace{\psi(A)^t \oplus \cdots \oplus \psi(A)^t}_{q}]U,$$

where $\psi : M_n \to M_n$ has the form $A \mapsto [(\operatorname{tr} A)I_n - (n-k)A]/k$. Linear maps $\widetilde{\phi} : M_n \to M_{n'}$ satisfying $w_{k'}(\widetilde{\phi}(A)) = w_k(A)$ for all $A \in M_n$ are also studied. Furthermore, results are extended to triangular matrices.

AMS Classifications: 15A04, 15A60, 47A12.

Keywords: Linear transformations, numerical range, numerical radius.

1 Introduction

There has been a great deal of interest in studying linear operator $\phi : \mathcal{M} \to \mathcal{M}$, where $\mathcal{M}$ is a matrix algebra or space, with a certain special property such as:

(a) $f(\phi(A)) = f(A)$ for all $A \in \mathcal{M}$, where f is a given function on $\mathcal{M}$;

(b) $\phi(\mathcal{S}) \subseteq \mathcal{S}$ or $\phi(\mathcal{S}) = \mathcal{S}$ for a certain subset $\mathcal{S} \subseteq \mathcal{M}$;

(c) $\phi(A) \sim \phi(B)$ in $\mathcal{M}$ whenever $A \sim B$ in $\mathcal{M}$ for a certain relation $\sim$ on $\mathcal{M}$.

Very often, ϕ has nice forms such as

$$A \mapsto MAN \qquad \text{or} \qquad A \mapsto MA^tN$$

for some suitable $M, N \in \mathcal{M}$. One may see [19] for a survey on the subject. Recently, there has been research on more general problems concerning linear transformations $\phi : \mathcal{M} \to \mathcal{M}'$ with some special properties such as

(a) $f'(\phi(A)) = f(A)$ for all $A \in \mathcal{M}$, where f and f' are appropriate functions on $\mathcal{M}$ and $\mathcal{M}'$;

(b) $\phi(\mathcal{S}) \subseteq \mathcal{S}'$ or $\phi(\mathcal{S}) = \mathcal{S}'$ for certain subsets $\mathcal{S} \subseteq \mathcal{M}$ and $\mathcal{S}' \subseteq \mathcal{M}'$;

(c) $\phi(A) \sim' \phi(B)$ in $\mathcal{M}'$ whenever $A \sim B$ in $\mathcal{M}$ for certain relations $\sim$ on $\mathcal{M}$ and $\sim'$ on $\mathcal{M}'$.

Such problems are more challenging and their study often lead to the discovery of unexpected results and hidden structures of the matrix algebras $\mathcal{M}$ and $\mathcal{M}'$; see [6, 10]. In this paper, we consider these types of problems. We solve a specific problem and develop some proof techniques that may be useful for future study in this area.

Let us first introduce some notations and definitions. Denote by M_n the algebra of $n \times n$ complex matrices. For $1 \le k \le n$, define (see Halmos [11]) the k-numerical range of $A \in M_n$ as

$$W_k(A) = \{(\mathrm{tr}\, X^*AX)/k : X \text{ is } n \times k,\ X^*X = I_k\}.$$

Since $W_n(A) = \{\mathrm{tr}\, A/n\}$, we always assume that $k < n$ to avoid trivial consideration. When $k = 1$, we have the classical numerical range $W_1(A)$, which is useful in studying matrices and operators; see [11]. Researchers have studied linear maps $\phi : M_n \to M_n$ such that

$$W_k(\phi(A)) = W_k(A) \quad \text{for all } A \in M_n. \tag{1.1}$$

By a result of Pellegrini [18], a linear map $\phi : M_n \to M_n$ satisfies (1.1) for $k = 1$ if and only if there is a unitary $U \in M_n$ such that ϕ has the form

(S1) $\quad A \mapsto U^*AU \quad$ or $\quad A \mapsto U^*A^tU.$

Pierce and Watkins [20] extended the result of Pellegrini to other values of k as long as $k \neq n/2$, and raised the open problem for the case $k = n/2$. In [12] (see also [17]), it was shown that for $k = n/2$, a linear map $\phi : M_n \to M_n$ satisfies (1.1) if and only if there is a unitary $U \in M_n$ such that ϕ has the form (S1), or

(S2) $\ n = 2k\ $ and

$$A \mapsto (\mathrm{tr}\, A/k)I_n - U^*AU \quad \text{or} \quad A \mapsto (\mathrm{tr}\, A/k)I_n - U^*A^tU. \tag{1.2}$$

In fact, for any $k \in \{1, \ldots, n-1\}$, a mapping ϕ of the form (1.2) satisfies

$$(n-k)W_{n-k}(A) = kW_k(\phi(A)) \quad \text{for all } A \in M_n.$$

In [6] the authors studied linear maps $\phi : M_n \to M_{n'}$ such that (1.1) holds with $k = 1$. It was shown that for $n' \le 2n-2$, a linear map $\phi : M_n \to M_{n'}$ satisfies (1.1) if and only if $n' \ge n$, there exist a unitary $U \in M_{n'}$ and a unital positive linear map $f : M_n \to M_{n'-n}$ such that ϕ has the form

$$A \mapsto U^*[A \oplus f(A)]U \quad \text{or} \quad A \mapsto U^*[A^t \oplus f(A)]U.$$

However, for $n' > 2n-2$, there are other linear maps $\phi : M_n \to M_{n'}$ satisfying (1.1) with complicated structure. The complete characterization of $\phi : M_n \to M_{n'}$ satisfying (1.1) is unknown.

The purpose of this paper is to study those linear operators $\phi : M_n \to M_{n'}$ satisfying

$$W_{k'}(\phi(A)) = W_k(A) \quad \text{for all } A \in M_n\,.$$

By modifying the map in [6], we can easily get a map $\phi : M_n \to M_{n'}$ with complicated structure satisfying

$$W_{k'}(\phi(A)) = W_1(A) \quad \text{for all } A \in M_n\,.$$

Therefore, we will only study the case when $k > 1$. It turns out that we also need to impose some conditions on n and n' to avoid the pathetic situation described below.

Let $k \in \{1, \dots, n\}$, and (α, β) be a pair of length k increasing subsequences of $\{1, \dots, n\}$. Denote by $A[\alpha, \beta]$ the submatrix of $A \in M_n$ lying in rows and columns indexed by α and β, respectively. Then the kth compound matrix of A is the $C(n,k) \times C(n,k)$ matrix $C_k(A)$ whose entries equal $\det A[\alpha,\beta]$ arranged in lexicographic order of α and β. The kth additive compound is defined by

$$\Delta_k(A) = \frac{\mathrm{d}}{\mathrm{dt}} C_k(I + tA)|_{t=0}.$$

It is known (e.g., see [16]) that the mapping

$$A \mapsto \Delta_k(A)$$

from M_n to $M_{C(n,k)}$ is linear and satisfies

$$W_k(A) = W_1(\Delta_k(A)) \qquad \text{for any } A \in M_n.$$

So, if $n' \ge 2C(n,k) - 1$ then there is a linear map $\psi : M_{C(n,k)} \to M_{n'}$ satisfying $W_1(\psi(X)) = W_1(X)$ for all $X \in M_n$ without nice structure. Thus, the linear map $\phi : M_n \to M_{n'}$ defined by

$$A \mapsto \psi(\Delta_k(A))$$

satisfies $W_k(A) = W_1(\phi(A))$ for all $A \in M_n$ and does not have nice structure. For larger n' one can extend the above idea to construct ϕ of the form

$$A \mapsto \psi_1(\Delta_k(A)) \oplus \cdots \oplus \psi_{k'}(\Delta_k(A))$$

satisfying $W_k(A) = W_{k'}(\phi(A))$ for all $A \in M_n$ without nice structure.

By the above discussion, we see that it is reasonable to impose appropriate assumption on n, n', k, k' to obtain nice characterizations of linear map $\phi : M_n \to M_{n'}$ satisfying $W_k(A) = W_{k'}(\phi(A))$ for all $A \in M_n$. This is done in Section 2. In fact, we show that the same result is valid for real linear map $\phi : H_n \to H_{n'}$, where H_m denotes the real linear space of all $m \times m$ complex Hermitian matrices. In Section 3, we extend the result to triangular matrices. Define the k-numerical radius of $A \in M_n$ by

$$w_k(A) = \max\{|z| : z \in W_k(A)\}.$$

In Section 4, we study those linear maps $\widetilde{\phi}$ satisfying

$$w_{k'}(\widetilde{\phi}(A)) = w_k(A) \qquad \text{for all } A \in M_n.$$

Some open problems are mentioned in Section 5.

Note that recently, researchers have also considered mappings preserving the classical numerical range and radius on more general operator algebras; see [1, 2, 4, 5, 7, 8, 14].

2 Results on Hermitian and Complex Matrices

The main theorem of this section is the following.

Theorem 2.1 *Let $(\mathcal{M}, \mathcal{M}') = (H_n, H_{n'})$ or $(M_n, M_{n'})$. Suppose $k \in \{2, \dots, n-1\}$, $k' \in \{1, \dots, n'\}$ and $n' < C(n,k)\min\{k', n'-k'\}$. There exists a linear map $\phi : \mathcal{M} \to \mathcal{M}'$ such that*

$$W_{k'}(\phi(A)) = W_k(A) \quad \text{for all } A \in \mathcal{M} \tag{2.1}$$

if and only if there is a unitary $U \in M_{n'}$ and nonnegative integers p, q with $p + q = n'/n$ such that one of the following holds:

(W1) *$n'/n = k'/k$ and ϕ has the form*

$$A \mapsto U^*[\underbrace{A \oplus \cdots \oplus A}_{p} \oplus \underbrace{A^t \oplus \cdots \oplus A^t}_{q}]U.$$

(W2) *$n'/n = k'/(n-k)$ and ϕ has the form*

$$A \mapsto U^*[\underbrace{\psi(A) \oplus \cdots \oplus \psi(A)}_{p} \oplus \underbrace{\psi(A)^t \oplus \cdots \oplus \psi(A)^t}_{q}]U,$$

where $\psi : \mathcal{M} \to \mathcal{M}$ is the mapping $A \mapsto [(\operatorname{tr} A)I_n - (n-k)A]/k$.

Proof of the sufficiency part. Suppose $n'/n = k'/k$. Then any mapping described in (W1) satisfies (2.1). If $n'/n = k'/(n-k)$, then the mapping $\phi : \mathcal{M} \to \mathcal{M}'$ described in (W2) satisfies

$$W_{k'}(\phi(A)) = W_{n-k}(\psi(A)) = W_k(A) \quad \text{for all } A \in \mathcal{M}. \qquad \blacksquare$$

In the following, we consider the converse. Suppose there exists a linear map $\phi : \mathcal{M} \to \mathcal{M}'$ such that $W_{k'}(\phi(A)) = W_k(A)$ for all $A \in \mathcal{M}$. We will show that n'/n is an integer and one of conditions (W1) or (W2) holds by establishing a sequence of lemmas.

Let $X \in H_m$. Denote the eigenvalues of X by

$$\lambda_1(X) \geq \cdots \geq \lambda_m(X).$$

Suppose $r \in \{1, \ldots, m-1\}$. Let

$$S_r(X) = \sum_{j=1}^{r} \lambda_j(X) \quad \text{and} \quad s_r(X) = \sum_{j=1}^{r} \lambda_{m-j+1}(X).$$

We have the following result concerning the r-numerical range; see [11, 12, 20] and their references.

Lemma 2.2 *Suppose $r \in \{1, \ldots, m-1\}$.*

(a) *$W_r(A) = W_r(U^*AU)$ for any unitary $U \in M_m$.*

(b) *$W_r(\alpha A + \beta I_m) = \alpha W_r(A) + \beta$ for any $\alpha, \beta \in \mathbb{C}$.*

(c) *If $A \in H_m$, then*

$$W_r(A) = [s_r(A)/r, S_r(A)/r] \,.$$

(d) *A matrix $B \in M_m$ satisfies $W_r(B) \subseteq \mathbb{R}$ if and only if $B = B^*$.*

(e) *A matrix $C \in M_m$ satisfies $W_r(C) = \{\lambda\}$ if and only if $C = \lambda I_m$.*

Lemma 2.3 *The mapping ϕ satisfies $\phi(H_n) \subseteq H_{n'}$ and $\phi(I_n) = I_{n'}$.*

Proof. If $A \in H_n$, then $W_{k'}(\phi(A)) = W_k(A) \subseteq \mathbb{R}$. By Lemma 2.2 (d), $\phi(A) \in H_{n'}$. Furthermore, since $W_{k'}(\phi(I_n)) = W_k(I_n) = \{1\}$, we have $\phi(I_n) = I_{n'}$ by Lemma 2.2 (e). ■

By the above lemma, we can focus on proving the result for the Hermitian case. Once it is done, the result on complex matrices will follow from the fact that $\phi(A) = \phi(H) + i\phi(G)$ for any complex matrix $A = H + iG \in M_n$ with $H, G \in H_n$.

A key step in our proof is to show that ϕ or $\phi \circ \psi^{-1}$ will map idempotents in H_n to idempotents in $H_{n'}$. Two idempotents $F, G \in H_m$ are said to be disjoint if $FG = GF = 0_m$.

Lemma 2.4 *Suppose $A, B \in H_m$ and $r \in \{1, \ldots, m-1\}$. The following conditions are equivalent.*

(a) *The sum of the first r diagonal entries of $A + B$ equals $S_r(A) + S_r(B)$.*

(b) *The sum of the last $m - r$ diagonal entries of $A + B$ equals $s_{m-r}(A) + s_{m-r}(B)$.*

(c) *$A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ such that A_1 has the r largest eigenvalues of A and B_1 has the r largest eigenvalues of B.*

Proof. Clearly, (a) and (b) are equivalent, and (c) implies (a). To prove (a) implies (c), let d_A be the sum of the first r diagonal entries of A, and d_B be the sum of the first r diagonal entries of B. Then $d_A \leq S_r(A)$ and $d_B \leq S_r(B)$. Now, the sum of the first r diagonal entries of $A+B$ equals $d_A + d_B = S_r(A) + S_r(B)$. So, $d_A = S_r(A)$ and $d_B = S_r(B)$. By [13, Lemma 4.1], $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ with $A_1, B_1 \in M_r$ such that A_1 has the r largest eigenvalues of A and B_1 has the r largest eigenvalues of B. ■

By Lemma 2.4, one readily deduces the following.

Lemma 2.5 *Suppose $A, B \in H_m$ and $1 < r < m$. The following conditions are equivalent.*

(a) $S_r(A+B) = S_r(A) + S_r(B)$.

(b) $s_{m-r}(A+B) = s_{m-r}(A) + s_{m-r}(B)$.

(c) *If $V \in M_m$ is unitary such that $V^*(A+B)V = \operatorname{diag}(c_1, \dots, c_m)$ with $c_1 \geq \cdots \geq c_m$, then $V^*AV = A_1 \oplus A_2$ and $V^*BV = B_1 \oplus B_2$ with $A_1, B_1 \in M_r$ such that A_1 (respectively, B_1) has the r largest eigenvalues of A (respectively, B).*

Lemma 2.6 *Suppose $k < n/2$ and $k' \leq n'/2$. If $E \in H_n$ is a rank one idempotent, then $\phi(E)$ is positive semidefinite.*

Proof. Suppose $Q \in H_n$ is a rank k idempotent such that $QE = EQ = 0$. Then $Q + E$ is a Hermitian idempotent with trace $k+1$. Since $k < n/2$,

$$W_{k'}(\phi(Q+E)) = W_k(Q+E) = [0,1],$$

$$W_{k'}(\phi(Q)) = W_k(Q) = [0,1], \qquad \text{and} \qquad W_{k'}(\phi(E)) = W_k(E) = [0, 1/k].$$

Then $S_{k'}(\phi(Q)) = k'$, $s_{k'}(\phi(Q)) = 0$, $S_{k'}(\phi(E)) = k'/k$, and $s_{k'}(\phi(E)) = 0$. So,

$$s_{k'}(\phi(Q) + \phi(E)) = s_{k'}(\phi(Q)) + s_{k'}(\phi(E)).$$

Let $V \in M_{n'}$ be unitary such that $V^*(\phi(Q+E))V = \operatorname{diag}(c_1, \dots, c_{n'})$ with $c_1 \geq \cdots \geq c_{n'}$. By Lemma 2.5, $V^*\phi(Q)V = Y_1 \oplus Y_2$ and $V^*\phi(E)V = Z_1 \oplus Z_2$ with $Y_2, Z_2 \in M_{k'}$ such that Z_2 has the k' smallest eigenvalues of $\phi(E)$. If Z_2 is the zero matrix, then Z_1 and hence, $\phi(E)$ is positive semidefinite as asserted. If Z_2 has a negative eigenvalue, then the largest eigenvalue of Z_2 is positive. Hence, Z_1 is positive definite. Now, suppose $V_1 \in M_{n'-k'}$ such that $V_1^* Y_1 V_1 = \operatorname{diag}(b_1, \dots, b_{n'-k'})$ with $b_1 \geq \ldots \geq b_{n'-k'}$. Then $\sum_{j=1}^{k'} b_j = S_{k'}(\phi(Q)) = k'$. Since Z_1 is positive definite and $k' \leq n'/2$, the sum of the first k' diagonal entries of $V_1^* Z_1 V_1 = a > 0$. Let X be the matrix consisting of the first k' columns of the unitary matrix $V(V_1 \oplus I_{k'})$. Then

$$1 < (k'+a)/k' = \operatorname{tr}(X^*\phi(Q+E)X)/k' \in W_{k'}(\phi(Q+E)),$$

which contradicts the fact that $W_{k'}(\phi(Q+E)) = [0,1]$. ■

Lemma 2.7 *Suppose $k \leq n/2$ and $k' \leq n'/2$. Let $E_1, \dots, E_n \in H_n$ be rank one idempotents such that $E_1 + \cdots + E_n = I_n$.*

(a) *Suppose $s \in \{1, \dots, k'-1\}$ such that*

$$\lambda_{n'-s}(\phi(E_1)) > \lambda_{n'-s+1}(\phi(E_1)).$$

Then there is an $n' \times s$ matrix S whose columns are orthonormal eigenvectors of the eigenvalues $\lambda_{n'-s+1}(\phi(E_1)), \ldots, \lambda_{n'}(\phi(E_1))$ such that $S^*\phi(E_j)S = \gamma_j I_s$ with

$$\gamma_1 = 1 - \sum_{j=2}^{n} \gamma_j = \lambda_{n'-s+1}(\phi(E_1)) = \cdots = \lambda_{n'}(\phi(E_1)),$$

and

$$\gamma_j = \lambda_{k'-s+1}(\phi(E_j)) = \cdots = \lambda_{n'-k'+s}(\phi(E_j)), \qquad j = 2, \ldots, n.$$

(b) *Suppose $r \in \{1, \ldots, k'-1\}$ such that*

$$\lambda_r(\phi(E_1)) > \lambda_{r+1}(\phi(E_1)).$$

Then there is an $n' \times r$ matrix R whose columns are orthonormal eigenvectors of the eigenvalues $\lambda_1(\phi(E_1)), \ldots, \lambda_r(\phi(E_1))$ such that $R^*\phi(E_j)R = \tilde{\gamma}_j I_r$ with

$$\tilde{\gamma}_1 = 1 - \sum_{j=2}^{n} \tilde{\gamma}_j = \lambda_1(\phi(E_1)) = \cdots = \lambda_r(\phi(E_1)),$$

and

$$\tilde{\gamma}_j = \lambda_{k'-r+1}(\phi(E_j)) = \cdots = \lambda_{n'-k'+r}(\phi(E_j)), \qquad j = 2, \ldots, n.$$

Proof. Assume that $\phi(E_1)$ has eigenvalues $a_1 \geq \cdots \geq a_{n'}$. Since

$$W_{k'}(\phi(E_1)) = W_k(E_1) = [0, 1/k],$$

$a_{n'} + \cdots + a_{n'-k'+1} = 0$. Let $B \in \{E_2, \ldots, E_n\}$. Then

$$W_{k'}(\phi(B)) = W_k(B) = [0, 1/k] \quad \text{and} \quad W_{k'}(\phi(E_1 + B)) = W_k(E_1 + B) = [0, 2/k].$$

Suppose $\phi(B)$ has eigenvalues $b_1 \geq \cdots \geq b_{n'}$.

(a) Note that $s_{k'}(\phi(E_1) + \phi(B)) = s_{k'}(\phi(E_1)) + s_{k'}(\phi(B))$. By Lemma 2.5, there is a unitary $V \in M_{n'}$ such that $V^*\phi(E_1)V = A_1 \oplus A_2$ and $V^*\phi(B)V = B_1 \oplus B_2$, where A_2 has eigenvalues $a_{n'}, \ldots, a_{n'-k'+1}$ and B_2 has eigenvalues $b_{n'}, \ldots, b_{n'-k'+1}$. Now,

$$W_{k'}(\phi(E_1) - \phi(B)) = W_k(E_1 - B) = [-1/k, 1/k].$$

Then $s_{k'}(\phi(E_1)) + s_{k'}(-\phi(B)) = s_{k'}(\phi(E_1 - B))$. By Lemma 2.5, there is a unitary $\tilde{V} \in M_{n'}$ such that $\tilde{V}^*\phi(E_1)\tilde{V} = \tilde{A}_1 \oplus \tilde{A}_2$ and $\tilde{V}^*\phi(B)\tilde{V} = \tilde{B}_1 \oplus \tilde{B}_2$, where $\tilde{A}_2$ has eigenvalues $a_{n'}, \ldots, a_{n'-k'+1}$ and $\tilde{B}_2$ has eigenvalues $b_1, \ldots, b_{k'}$. Since $a_{n'-s} > a_{n'-s+1}$, we may assume that the last s columns of V and $\tilde{V}$ are the eigenvectors of $\phi(E_1)$ corresponding to the eigenvalues $a_{n'-s+1}, \ldots, a_{n'}$. So, the lower $s \times s$ principal submatrices of B_2 and $\tilde{B}_2$ are the same, say, equal to $X \in H_s$. Suppose X has eigenvalues $d_1 \geq \cdots \geq d_s$. Because B_2 has eigenvalues $b_{n'-k'+1} \geq \cdots \geq b_{n'}$, it follows from the interlacing inequalities (see [9]) that

$$b_{n'-k'+j} \geq d_j, \quad j = 1, \ldots, s. \tag{2.2}$$

Because $\tilde{B}_2$ has eigenvalues $b_1 \geq \cdots \geq b_{k'}$, by the interlacing inequalities again, we have

$$d_j \geq b_{k'-s+j}, \quad j = 1, \ldots, s. \tag{2.3}$$

Since $k' \leq n'/2$, $b_{k'-s+j} \geq b_{n'-k'+j}$ for $1 \leq j \leq s$. By (2.2) and (2.3), we see that

$$b_{k'-s+j} = d_j = b_{n'-k'+j}, \quad j = 1, \ldots, s.$$

Thus,

$$d_1 = \cdots = d_s = b_{k'-s+1} = \cdots = b_{n'-k'+s}. \tag{2.4}$$

Use the last s columns of V to form the matrix S. Then $S^*BS = d_1 I_s$.

By the above arguments, $S^*\phi(E_j)S = \gamma_j I_s$ for $j = 2, \ldots, n$, where $\gamma_j = \lambda_{k'-s+1}(\phi(E_j)) = \lambda_{n'-k'+s}(\phi(E_j))$. By Lemma 2.3,

$$\phi(E_1 + E_2 + \cdots + E_n) = \phi(I_n) = I_{n'}.$$

It follows that

$$S^*\phi(E_1)S = I_s - \sum_{j=2}^{n} \gamma_j I_s \tag{2.5}$$

is a scalar matrix, where $\gamma_1 = 1 - \sum_{j=2}^{n} \gamma_j$. Clearly, $\gamma_1 = a_{n'-s+1} = a_{n'}$.

(b) Note that

$$W_{k'}(\phi(E_1 + B)) = W_k(E_1 + B) = [0, 2/k].$$

Thus, $S_{k'}(\phi(E_1) + \phi(B)) = S_{k'}(\phi(E_1)) + S_{k'}(\phi(B))$. Then there is a unitary $W \in M_{n'}$ such that $W^*(\phi(E_1))W = Y_1 \oplus Y_2$ and $W^*(\phi(B))W = Z_1 \oplus Z_2$ with $Y_1, Z_1 \in M_{k'}$, where Y_1 has eigenvalues $a_1, \ldots, a_{k'}$ and Z_1 has eigenvalues $b_1, \ldots, b_{k'}$. Now,

$$W_{k'}(\phi(E_1 - B)) = W_k(E_1 - B) = [-1/k, 1/k].$$

We see that $S_{k'}(\phi(E_1) + \phi(-B)) = S_{k'}(\phi(E_1)) + S_{k'}(\phi(-B))$. So there exists a unitary $\tilde{W} \in M_{n'}$ such that $\tilde{W}^*\phi(E_1)\tilde{W} = \tilde{Y}_1 \oplus \tilde{Y}_2$ and $\tilde{W}^*\phi(B)\tilde{W} = \tilde{Z}_1 \oplus \tilde{Z}_2$, where $\tilde{Y}_1$ has eigenvalues $a_1, \ldots, a_{k'}$ and $\tilde{Z}_1$ has eigenvalues $b_{n'}, \ldots, b_{n'-k'+1}$. Since $a_r > a_{r+1}$, we may assume that the first r columns of W and $\tilde{W}$ are the eigenvectors of $\phi(E_1)$ corresponding to the eigenvalues $a_1, \ldots, a_r$. So, the leading $r \times r$ submatrices of Z_1 and $\tilde{Z}_1$ are the same, say, equal to $T \in H_r$. Suppose T has eigenvalues $t_1 \geq \cdots \geq t_r$. Since Z_1 has eigenvalues $b_1 \geq \cdots \geq b_{k'}$, by the interlacing inequalities

$$t_j \geq b_{k'-r+j} \quad j = 1, \ldots, r. \tag{2.6}$$

Since $\tilde{Z}_1$ has eigenvalues $b_{n'-k'+1} \geq \cdots \geq b_{n'}$, by the interlacing inequalities

$$b_{n'-k'+j} \geq t_j, \quad j = 1, \ldots, r. \tag{2.7}$$

Since $k' \leq n'/2$, $b_{k'-r+j} \geq b_{n'-k'+j}$ for $1 \leq j \leq r$. By (2.6) and (2.7), we see that

$$b_{k'-r+j} = t_j = b_{n'-k+j}, \quad j = 1, \ldots, r.$$

Since $k' \le n'/2$,

$$t_1 = \cdots = t_r = b_{k'-r+1} = \cdots = b_{n'-k'+r}.$$

Use the first r columns of W to form the matrix R. Then $R^*TR = t_1 I_r$. Consequently, $R^*\phi(E_j)R = \tilde{\gamma}_j I_r$ for $j = 2, \ldots, n$, as $\tilde{\gamma}_j = \lambda_{k'-r+1}(\phi(E_j)) = \lambda_{n'-k'+r}(\phi(E_j))$. Moreover, $R^*\phi(E_1)R = I_r - \sum_{j=2}^n \tilde{\gamma}_j I_r$, where $\tilde{\gamma}_1 = 1 - \sum_{j=2}^n \tilde{\gamma}_j$. Clearly, $\tilde{\gamma}_1 = a_1 = a_r$. ∎

Lemma 2.8 *Suppose $k = n/2$ and $k' \le n'/2$. If there is a rank one idempotent $E \in H_n$ such that $\phi(E)$ has negative eigenvalues, then $k' = n'/2$ and $\phi \circ \psi^{-1}(F)$ is positive semidefinite for any rank one idempotent $F \in H_n$.*

Proof. Suppose there is a rank one idempotent E such that $\phi(E)$ has negative eigenvalues. Assume that $\phi(E)$ has eigenvalues $a_1 \ge \cdots \ge a_{n'-s} \ge 0 > a_{n'-s+1} \ge \cdots \ge a_{n'}$. Since

$$W_{k'}(\phi(E)) = W_k(E) = [0, 1/k],$$

$a_{n'} + \cdots + a_{n'-k'+1} = 0$. Thus, $s < k'$.

Let $E_1, \ldots, E_n \in H_n$ be rank one idempotents such that $E_1 = E$ and $\sum_{j=1}^n E_j = I_n$. By Lemma 2.7 (a), there is an $n' \times s$ matrix S whose columns are orthonormal eigenvectors of the negative eigenvalues of $\phi(E_1)$ such that $S^*\phi(E_j)S = \gamma_j I_s$ with

$$\gamma_1 = 1 - \sum_{j=2}^n \gamma_j = a_{n'} = \cdots = a_{n'-s+1} \tag{2.8}$$

and

$$\gamma_j = \lambda_{k'-s+1}(\phi(E_j)) = \lambda_{n'-k'+s}(\phi(E_j)), \qquad j = 2, \ldots, n. \tag{2.9}$$

We must have

$$a_1 = \cdots = a_{k'}.$$

Otherwise, there is $r < k'$ such that $a_r > a_{r+1}$. By Lemma 2.7 (b), there is an $n' \times r$ matrix R whose columns are orthonormal eigenvectors of the r largest eigenvalues of $\phi(E_1)$ such that $R^*\phi(E_j)R = \tilde{\gamma}_j I_r$ with

$$\tilde{\gamma}_j = \lambda_{k'-r+1}(\phi(E_j)) = \lambda_{n'-k'+r}(\phi(E_j)), \qquad j = 2, \ldots, n. \tag{2.10}$$

By (2.9) and (2.10), we have $\tilde{\gamma}_j = \lambda_{k'}(\phi(E_j)) = \gamma_j$ for $j = 2, \ldots, n$. But then, we have $R^*\phi(E_1)R = I_r - \sum_{j=2}^n \gamma_j I_r$, where $a_1 = 1 - \sum_{j=2}^n \gamma_j = a_{n'}$, which is a contradiction. Since $W_k(E_1) = W_{k'}(\phi(E_1))$, we see that $a_1 = \cdots = a_{k'} = 1/k$.

Observe that $\sum_{j=1}^n \gamma_j = 1$ and $\gamma_1 < 0$. Thus, there exists $j \ge 2$ such that $\gamma_j > 0$. We may assume that $\gamma_2 > 0$. Since $s_{k'}(\phi(E_2)) = 0$ and $\lambda_{n'-k'+s}(\phi(E_2)) = \gamma_2 > 0$, we see that $\lambda_{n'}(\phi(E_2)) < 0$. Suppose $\phi(E_2)$ has t negative eigenvalues. Applying the arguments on $\phi(E_1)$ to $\phi(E_2)$, we see that the last t eigenvalues of $\phi(E_2)$ all equal to $1 - \sum_{j \ne 2} \lambda_{k'-s+1}(\phi(E_j))$ and

$$1/k = \lambda_l(\phi(E_2)) \qquad \text{for } l = 1, \ldots, k'.$$

By (2.9), we have

$$1/k = \gamma_2 = \lambda_{k'-s+1}(\phi(E_2)) = \cdots = \lambda_{n'-k'+s}(\phi(E_2)).$$

Interchanging the roles of E_1 and E_2, we see that

$$1/k = \lambda_1(\phi(E_1)) = \cdots = \lambda_{n'-k'+t}(\phi(E_1)).$$

Moreover, since $\lambda_{k'}(\phi(E_1)) = \lambda_{k'}(\phi(E_2)) = 1/k$, we see that

$$\lambda_{n'}(\phi(E_1)) = 1 - \sum_{j=3}^{n} \lambda_{k'}(\phi(E_j)) - 1/k = \lambda_{n'}(\phi(E_2)).$$

Suppose $j \geq 3$ is such that $\phi(E_j)$ has negative eigenvalues. We can apply the above arguments on $\phi(E_2)$ to $\phi(E_j)$ to conclude that $\lambda_{n'}(\phi(E_j)) = \gamma_1$, and

$$1/k = \lambda_l(\phi(E_j)) \qquad \text{for } l = 1, \ldots, n' - k' + s. \tag{2.11}$$

Suppose $j \geq 3$ and $\phi(E_j)$ is positive semidefinite. Then $\lambda_{n'-k'+1}(\phi(E_j)) = \cdots = \lambda_{n'}(\phi(E_j)) = 0$. By (2.9), we have

$$\lambda_{k'-s+1}(\phi(E_j)) = \lambda_{n'-k'+s}(\phi(E_j)) = 0. \tag{2.12}$$

Relabeling $E_1, \ldots, E_n$ if necessary, we can assume that $\phi(E_2), \ldots, \phi(E_m)$ have negative eigenvalues, and $\phi(E_j)$ is positive semidefinite for $j > m$. Then each one of $\phi(E_1), \ldots, \phi(E_m)$ has smallest eigenvalue

$$\gamma_1 = 1 - \sum_{j=2}^{n} \lambda_{k'}(\phi(E_j)) = 1 - (m-1)/k < 0.$$

If $m < n$, then $\phi(E_n)$ is positive semidefinite with fewer than k' positive eigenvalues. Since $W_{k'}(\phi(E_n)) = W_k(E_n) = [0, 1/k]$, we see that $\lambda_1(E_n) > 1/k$. By Lemma 2.7 (b), we have

$$1/k < \lambda_1(\phi(E_n)) = 1 - \sum_{j=1}^{n} \lambda_{k'}(\phi(E_j)) = 1 - m/k < 1 - (m-1)/k < 0,$$

which is a contradiction. So, we have $m = n$. By (2.11), and the fact that $W_{k'}(\phi(E_j)) = W_k(E_j) = [0, 1/k]$ for $j = 1, \ldots, n$, we have

$$\begin{aligned} n' &= \operatorname{tr}(I_{n'}) = \sum_{j=1}^{n} \operatorname{tr} \phi(E_j) \\ &= \sum_{j=1}^{n} S_{k'}(\phi(E_j)) + \sum_{l=k'+1}^{n'-k'} \sum_{j=1}^{n} \lambda_l(\phi(E_j)) + \sum_{j=1}^{n} s_{k'}(\phi(E_j)) \\ &= n(k'/k) + n(n' - 2k')(1/k) = 2k' + 2(n' - 2k') = 2n' - 2k'. \end{aligned}$$

Hence, $n' = 2k'$.

Suppose $F \in H_n$ is a rank one idempotent. We claim that $\lambda_1(F) = 1/k$. Since $n = 2k \geq 4$, there is a rank one idempotent $G \in H_n$ such that $EG = GE = 0$ and $FG = GF = 0$. Moreover, there exist rank one idempotents $G_3, \ldots, G_n \in H_n$ such that $E+G+G_3+\cdots G_n = I_n$. Applying the previous argument with $(E, E_2, \ldots, E_n)$ replaced by $(E, G, G_3, \ldots, G_n)$, we see that G has negative eigenvalues. Now, there exist rank one idempotents $F_3, \ldots, F_n \in H_n$ such that $G+F+F_3+\cdots+F_n = I_n$. Applying the previous arguments with $(E, E_2, \ldots, E_n)$ replaced by $(G, F, F_3, \ldots, F_n)$, we see that $\phi(F)$ has largest eigenvalue $1/k$.

Now, observe that $\psi^{-1}(F) = I_n/k - F$. Since $\phi(F)$ has largest eigenvalue $1/k$, we conclude that $\phi(\psi^{-1}(F)) = I_{n'}/k - \phi(F)$, is positive semidefinite as asserted. ■

Lemma 2.9 *Suppose $k \leq n/2$, $2k' \leq n' < k' \cdot C(n, k)$, and $\phi(E)$ is positive semidefinite for any rank one idempotent $E \in H_n$. Then $\phi(F)$ and $\phi(G)$ are disjoint idempotents in $H_{n'}$ for any disjoint rank one idempotents $F, G \in H_n$.*

Proof. Let $E_1, \ldots, E_n \in H_n$ be rank one idempotents such that $F = E_1$, $G = E_2$, and $E_1 + \cdots + E_n = I_n$. Then $Y_j = \phi(E_j) \in H_{n'}$ is positive semidefinite and

$$W_{k'}(Y_j) = [0, 1/k] = [s_{k'}(Y_j)/k', S_{k'}(Y_j)/k']$$

for all $j = 1, \ldots, n$. We claim that there is $j \in \{1, \ldots, n\}$ such that the largest eigenvalue of Y_j has multiplicity $r < k'$. If it is not true, then for $j = 1, \ldots, n$,

$$\lambda_1(Y_j) = \cdots = \lambda_{k'}(Y_j) = 1/k,$$

as $S_{k'}(Y_j) = k'/k$. Now, for any $1 \leq j_1 < j_2 < \cdots < j_k \leq n$,

$$W_{k'}\left(\sum_{t=1}^{k} Y_{j_t}\right) = W_k\left(\sum_{t=1}^{k} E_{j_t}\right) = [0, 1].$$

Thus, there exists an $n' \times k'$ matrix U such that

$$k' = S_{k'}\left(\sum_{t=1}^{k} Y_{j_t}\right) = \operatorname{tr}\left(U^*\left(\sum_{t=1}^{k} Y_{j_t}\right)U\right) = \sum_{t=1}^{k} \operatorname{tr}\left(U^* Y_{j_t} U\right) \leq \sum_{t=1}^{k} S_{k'}(Y_{j_t}) = k'.$$

It follows that $\operatorname{tr}(U^* Y_{j_t} U) = k'/k$ and hence $U^* Y_{j_t} U = (1/k) I_{k'}$ for $t = 1, \ldots, k$. Since $Y_1 + \cdots + Y_n = I_{n'}$, we see that $U^* Y_t U = 0_{k'}$ for any $t \notin \{j_1, \ldots, j_k\}$.

Now, for any other choice of $1 \leq \tilde{j}_1 < \tilde{j}_2 < \cdots < \tilde{j}_k \leq n$, there is a corresponding $n' \times k'$ matrix $\tilde{U}$ such that $\tilde{U}^* \tilde{U} = I_{k'}$ such that $\tilde{U}^* Y_{\tilde{j}_t} \tilde{U} = (1/k) I_{k'}$ for $t = 1, \ldots, k$, and $\tilde{U}^* Y_t \tilde{U} = 0_{k'}$ for any $t \notin \{\tilde{j}_1, \ldots, \tilde{j}_k\}$. Suppose $\tilde{j}_p \notin \{j_1, \ldots, j_k\}$. Then $U^* Y_{\tilde{j}_p} U = 0_{k'}$ and $\tilde{U}^* Y_{\tilde{j}_p} \tilde{U} = (1/k) I_{k'}$. Thus, the columns of U belong to the kernel of $Y_{\tilde{j}_p}$ whereas the columns of $\tilde{U}$ belong to the kernel of $Y_{\tilde{j}_p} - (1/k) I_{k'}$. So, $U^* \tilde{U} = 0_{k'}$.

Combining the above arguments, we see that there are $C(n,k)$ matrices U's of size $n' \times k'$ such that $U^*U = I_{k'}$. Any two of such U have mutually orthogonal columns. So, there are $k' \cdot C(n,k)$ orthonormal columns. Hence $k' \cdot C(n,k) \le n'$, which contradicts our assumption.

By the above argument, we see that

$$\min\{r : \lambda_r(Y_t) > \lambda_{r+1}(Y_t) \text{ with } t \in \{1,\dots,n\}\} < k'.$$

Relabeling $Y_1,\dots,Y_n$ if necessary, we may assume that

$$\lambda_1(Y_1) = \cdots = \lambda_r(Y_1) > \lambda_{r+1}(Y_1)$$

and for $t = 2,\dots,n$,

$$\lambda_1(Y_t) = \cdots = \lambda_r(Y_t).$$

Note that the last k' eigenvalues of Y_j are all zeros. We claim that the first k' eigenvalues of Y_t can contain at most two distinct values. Otherwise, there are $1 \le s < s' < k'$ such that $\lambda_s(Y_t) > \lambda_{s+1}(Y_t)$ and $\lambda_{s'}(Y_t) > \lambda_{s'+1}(Y_t)$. But by Lemma 2.7 (b), $\lambda_1(Y_t) = \cdots = \lambda_{s'}(Y_t)$, which is impossible.

Note that the last k' eigenvalues of Y_t are all zeros. Applying Lemma 2.7(b) to Y_1, we have

$$\lambda_{k'-r+1}(Y_t) = \lambda_{n'-k'+r}(Y_t) = 0$$

for $t = 2,\dots,n$. Then there is $r_t < k' - r + 1 \le k'$ such that

$$\lambda_1(Y_t) = \lambda_{r_t}(Y_t) > \lambda_{r_t+1}(Y_t) = \lambda_{n'}(Y_t) = 0,$$

i.e., Y_t is unitarily similar to $\gamma_t I_{r_t} \oplus 0_{n'-r_t}$ for $t = 2,\dots,n$. Interchanging the role of Y_1 and Y_t, we conclude that Y_1 is unitarily similar to $\gamma_1 I_{r_1} \oplus 0_{n'-r_1}$. Since $W_{k'}(Y_t) = W_k(E_t) = [0, 1/k]$, $r_t\gamma_t = k'/k$.

Furthermore, we can see from Lemma 2.7 (b) that for $s \ne t$, all eigenvectors of Y_s corresponding to the eigenvalue γ_s are eigenvectors of Y_t corresponding to the eigenvalue 0. Hence, $Y_sY_t = 0$ for any $s \ne t$. Since $Y_1 + \cdots + Y_n = I_{n'}$, $\gamma_t = 1$ and $r_1 + \cdots + r_n = n'$. Hence, $r_t = k'/k = r$ for all $t = 1,\dots,n$ and $k'/k = n'/n$. This shows that every Y_t is unitarily similar to $I_r \oplus 0_{n'-r}$. Hence, $A \mapsto \phi(A)$ maps disjoint idempotents to disjoint idempotents. ∎

Proof of the necessity part of Theorem 1. Suppose $k < n/2$ and $k' \le n'/2$. By Lemmas 2.6 and 2.9, ϕ will map idempotents to idempotents. So, (see Corollary 4.3 in [10] and also [3, Theorem 2.1]), ϕ has the asserted form.

Suppose $k = n/2$ and $k' \le n'/2$. Apply Lemma 2.8; then apply Lemmas 2.6 and 2.9 to $\phi \circ \psi^{-1}$ to get the conclusion.

Suppose $k > n/2$ and $k' \le n'/2$. Then $\phi \circ \psi^{-1}$ satisfies $W_{n-k}(A) = W_{k'}(\phi \circ \psi^{-1}(A))$ for all $A \in M_n$. So, $\phi \circ \psi^{-1}$ has the desired form.

Suppose $k > n/2$ and $k' > n'/2$. Replace ϕ by $\Psi \circ \phi \circ \psi^{-1}$ with $\Psi : M_{n'} \to M_{n'}$ defined by $\Psi(X) = [(\operatorname{tr} X)I_{n'} - k'X]/(n' - k')$ for all $X \in M_{n'}$. Then $W_{n-k}(A) = W_{n'-k'}(\phi(A))$ for all $A \in M_n$. So, $\Psi \circ \phi \circ \psi^{-1}$ has the asserted form. It follows that ϕ has the same form as well. ∎

3 Results on Triangular Matrices

Let T_n be the set of $n \times n$ upper triangular matrices. In this section, we study those linear maps $\phi : T_n \to T_{n'}$ satisfying

$$W_{k'}(\phi(A)) = W_k(A) \quad \text{for all } A \in T_n. \tag{3.1}$$

Clearly, if a map ϕ has the form (W1) or (W2) in Theorem 2.1 for some unitary U such that $\phi(T_n) \subseteq T_{n'}$, then condition (3.1) holds. The following theorem shows that the converse of the above statement is also valid, and gives a condition on U to ensure that $\phi(T_n) \subseteq T_{n'}$.

Theorem 3.1 *Suppose $k \in \{2, \ldots, n-1\}$, $k' \in \{1, \ldots, n'\}$ and $n' < C(n,k)\min\{k', n'-k'\}$. There exists a linear map $\phi : T_n \to T_{n'}$ such that*

$$W_{k'}(\phi(A)) = W_k(A) \quad \textit{for all } A \in T_n$$

if and only if there is a unitary $U = (u_{ij}) \in M_{n'}$ and nonnegative integers p, q with $p + q = n'/n$ such that

$$\sum_{j=0}^{p-1} \overline{u}_{(jn+a),d}\; u_{(jn+b),c} + \sum_{j=p}^{p+q-1} \overline{u}_{(jn+b),d}\; u_{(jn+a),c} = 0 \tag{3.2}$$

for all $1 \le a \le b \le n$ and $1 \le c < d \le n'$, and one of the following holds:

(T1) *$n'/n = k'/k$ and ϕ has the form*

$$A \mapsto U^*[\underbrace{A \oplus \cdots \oplus A}_{p} \oplus \underbrace{A^t \oplus \cdots \oplus A^t}_{q}]U.$$

(T2) *$n'/n = k'/(n-k)$ and ϕ has the form*

$$A \mapsto U^*[\underbrace{\psi(A) \oplus \cdots \oplus \psi(A)}_{p} \oplus \underbrace{\psi(A)^t \oplus \cdots \oplus \psi(A)^t}_{q}]U,$$

where $\psi : T_n \to T_n$ is the mapping $A \mapsto [(\operatorname{tr} A)I_n - (n-k)A]/k$.

Let us further analyze condition (3.2) in the following. For any $1 \le a \le b \le n$ and $1 \le c \le n'$, define

$$u_c^{ba} = \begin{pmatrix} v_c^b \\ w_c^a \end{pmatrix} \quad \text{with} \quad v_c^b = \begin{pmatrix} u_{bc} \\ \vdots \\ u_{((p-1)n+b)c} \end{pmatrix} \quad \text{and} \quad w_c^a = \begin{pmatrix} u_{(pn+a)c} \\ \vdots \\ u_{((p+q-1)n+a)c} \end{pmatrix}.$$

Then (3.2) reduces to

$$(v_d^a, v_c^b) + (w_d^b, w_c^a) = (u_d^{ab}, u_c^{ba}) = 0 \quad \text{for all } 1 \le a \le b \le n \text{ and } 1 \le c < d \le n',$$

where $(\cdot,\cdot)$ denotes the usual inner product, i.e., $(x,y)=y^*x$. Suppose $U=(u_{ij})$ satisfies (3.2). Clearly, the set $\{u_1^{aa},\ldots,u_{n'}^{aa}\}$ forms an orthogonal set on $\mathbb{C}^{n'/n}$. Then at most n'/n vectors of the set can be nonzero. Hence, at most $n(n'/n)=n'$ vectors of the set

$$\{u_1^{11},\cdots u_{n'}^{11}\}\cup\cdots\cup\{u_1^{nn},\cdots u_{n'}^{nn}\}$$

can be nonzero. As U is an $n'\times n'$ unitary matrix, exactly one vector in $\{u_c^{11},\ldots u_c^{nn}\}$ can be nonzero. Otherwise, U has a zero column. Furthermore, if $a\neq b$, then at most one of v_c^b and w_c^a can be nonzero, as only one of u_c^{bb} and u_c^{aa} can be nonzero. Thus, we deduce from (3.2) that

$$(v_d^a,v_c^b)=0=(w_d^b,w_c^a)\quad\text{for all } 1\le a<b\le n \text{ and } 1\le c<d\le n'.$$

In conclusion, we have the following

Proposition 3.2 *A unitary matrix $U=(u_{ij})\in M_{n'}$ satisfies (3.2) if and only if*

(i) *for each $1\le c\le n'$, there is a $a\in\{1,\ldots,n\}$ such that*

$$(v_d^a,v_c^a)+(w_d^a,w_c^a)=1$$

and v_c^b and w_c^b are zero vectors for all $b\neq a$; and

(ii) *for any $1\le a<b\le n$ and $1\le c<d\le n'$,*

$$(v_d^a,v_c^a)+(w_d^a,w_c^a)=(v_d^a,v_c^b)=(w_d^b,w_c^a)=0.$$

Example If $n'=6$, $n=p=2$ and $q=1$, then

$$U=\begin{pmatrix}1/\sqrt{2}&0&0&0&1/\sqrt{2}&0\\0&0&0&1/\sqrt{2}&0&1/\sqrt{2}\\1/\sqrt{2}&0&0&0&-1/\sqrt{2}&0\\0&0&0&1/\sqrt{2}&0&-1/\sqrt{2}\\0&0&1&0&0&0\\0&1&0&0&0&0\end{pmatrix}$$

satisfies (3.2). But if U_1 is the matrix obtained from U by interchanging its $(4,4)$-th and $(4,6)$-th entries, then U_1 does not satisfy (3.2) as $(v_4^2,v_5^1)=1$.

Proof of Theorem 3.1. Note that the left side of (3.2) equals the (d,c)-th entry of

$$U^*[\underbrace{E_{ab}\oplus\cdots\oplus E_{ab}}_{p}\oplus\underbrace{E_{ba}\oplus\cdots\oplus E_{ba}}_{q}]U.$$

Hence, (3.2) holds if and only if

$$U^*[\underbrace{A\oplus\cdots\oplus A}_{p}\oplus\underbrace{A^t\oplus\cdots\oplus A^t}_{q}]U\in T_{n'}\quad\text{for all } A\in T_n.$$

Therefore, if ϕ has the form (T1), $\phi(T_n) \subseteq T_{n'}$. If ϕ has the form (T2), since $\psi(T_n) = T_n$, we have the same conclusion too. Therefore the sufficiency part holds.

For the converse, take any diagonal matrix R with real diagonal entries. Then

$$W_{k'}(\phi(R)) = W_k(R) \subseteq \mathbb{R}.$$

By Lemma 2.2(d), $\phi(R)^* = \phi(R)$. As $\phi(R) \in T_{n'}$, $\phi(R)$ must be a diagonal matrix with real diagonal entries.

Now for any diagonal matrix $D \in D_n$, write $D_1 = (D + D^*)/2$ and $D_2 = (D - D^*)/(2i)$. Then D_1 and D_2 are diagonal matrices with real diagonal entries. It follows that $\phi(D_1)^* = \phi(D_1)$ and $\phi(D_2)^* = \phi(D_2)$. As $D = D_1 + iD_2$,

$$\phi(D^*)^* = (\phi(D_1) - i\phi(D_2))^* = \phi(D_1)^* + i\phi(D_2)^* = \phi(D_1) + i\phi(D_2) = \phi(D).$$

Every $A \in M_n$ can be expressed as $T_1 + T_2^*$ for some upper triangular matrices T_1 and T_2. Define $\Phi : M_n \to M_{n'}$ by

$$\Phi(A) = \phi(T_1) + \phi(T_2)^*.$$

Clearly, Φ is linear. Suppose A can be written as $U_1 + U_2^*$ for some $U_1, U_2 \in T_n$ distinct from T_1 and T_2. Let $D = T_1 - U_1 = U_2^* - T_2^*$. Then D is a diagonal matrix. Observe that

$$\begin{aligned} 0 &= \phi(D) - \phi(D^*)^* \\ &= \phi(T_1 - U_1) - \phi(U_2 - T_2)^* \\ &= \phi(T_1) + \phi(T_2)^* - \phi(U_1) - \phi(U_2)^* \\ &= \Phi(T_1 + T_2^*) - \Phi(U_1 + U_2^*). \end{aligned}$$

Hence, Φ is well defined. On the other hand, we see that for any $A \in M_m$ and $1 \le r \le m$,

$$\begin{aligned} W_r(A + A^*) &= \{\operatorname{tr}(X^*AX)/r + \operatorname{tr}(X^*A^*X)/r : X \text{ is } m \times r,\ X^*X = I_r\} \\ &= \{\operatorname{tr}(X^*AX)/r + \overline{\operatorname{tr}(X^*AX)/r} : X \text{ is } m \times r,\ X^*X = I_r\} \\ &= \{z + \overline{z} : z \in W_r(A)\}. \end{aligned}$$

Since every matrix $H \in H_n$ can be expressed as $H = T + T^*$ with $T \in T_n$,

$$\begin{aligned} W_{k'}(\Phi(H)) &= W_{k'}(\phi(T) + \phi(T)^*) = \{z + \overline{z} : z \in W_{k'}(\phi(T))\} \\ &= \{z + \overline{z} : z \in W_k(T)\} = W_k(T + T^*) = W_k(H). \end{aligned}$$

Hence, $\Phi : M_n \to M_{n'}$ is a linear map such that

$$W_{k'}(\Phi(H)) = W_k(H) \quad \text{for all } H \in H_n.$$

By Theorem 2.1, there exist a unitary $U \in M_{n'}$ and nonnegative integers p, q with $p+q = n'/n$ such that Φ satisfies (W1) or (W2) in Theorem 2.1. Since $\phi(A) = \Phi(A)$ for all $A \in T_n$, ϕ has the form (T1) or (T2). Finally, we check that U satisfies (3.2) as $\phi(E_{ab}) \in T_{n'}$ for all $a \le b$. ∎

Remark 3.3 Denote by $T(n_1, \ldots, n_r)$ the algebra of upper block triangular matrices $A = (A_{ij})$ such that $A_{ii} \in M_{n_i}$ for $i = 1, \ldots, r$. One can extend Theorem 3.1 to linear map $\phi : T(n_1, \ldots, n_r) \to T(m_1, \ldots, m_s)$ for $n_1 + \cdots + n_r = n$, $m_1 + \cdots + m_s = n'$, and $n' < C(n,k)\min\{k', n'-k'\}$. The result and proofs are basically the same provided that U satisfies (3.2) for all $1 \le a, b \le n$, $1 \le c, d \le n'$ such that $E_{ab} \in T(n_1, \ldots, n_r)$ and $E_{cd} \notin T(m_1, \ldots, m_s)$. Since the corresponding statements are rather tedious, we omit the details.

Note also that if a linear map $\phi : T(n_1, \ldots, n_r) \to T(m_1, \ldots, m_s)$ satisfies $W_{k'}(\phi(A)) = W_k(A)$ for all $A \in T(n_1, \ldots, n_k)$, then one can replace $T(m_1, \ldots, m_s)$ by other block triangular matrix algebras such as $T(m_1 + m_2, m_3 \ldots, m_s)$ or $T(m_1 + m_2, m_3 + m_4, \ldots, m_s)$, etc.

4 k-Numerical Radius

Theorem 4.1 *Let $(\mathcal{M}, \mathcal{M}') = (H_n, H_{n'})$, $(M_n, M_{n'})$ or $(T_n, T_{n'})$, $k \in \{2, \ldots, n-1\}$, $k' \in \{1, \ldots, n'\}$ and $n' < C(n,k)\min\{k', n'-k'\}$. Then a linear operator $\widetilde{\phi} : \mathcal{M} \to \mathcal{M}'$ satisfies*

$$w_{k'}(\widetilde{\phi}(A)) = w_k(A) \quad \text{for all } A \in \mathcal{M}, \tag{4.1}$$

and $\widetilde{\phi}(X) = I_{n'}$ for some $X \in \mathcal{M}$ if and only if there is a complex unit μ such that $\phi = \mu\widetilde{\phi}$ satisfies

$$W_{k'}(\phi(A)) = W_k(A) \quad \text{for all } A \in \mathcal{M},$$

equivalently, ϕ has the form in Theorem 2.1 or Theorem 3.1.

Lemma 4.2 *For any $T = (t_{ij}) \in T_n$, if*

$$\frac{1}{k}\left|\sum_{i=1}^{k} t_{n_i n_i}\right| = w_k(T) \quad \text{for all } 1 \le n_1 < \cdots < n_k \le n, \tag{4.2}$$

then T is a diagonal matrix.

Proof. Suppose $t_{ij} \neq 0$ for some $i < j$. Denote by $X[i,j] \in M_2$ the submatrix of $X \in M_n$ lying in the rows and columns indexed by i and j. Then $W_1(T[i,j])$ is an elliptical disk with the length of minor axis equal to $|t_{ij}|$, and foci t_{ii} and t_{jj}; see [11]. Thus, there is a unitary $U \in M_2$ such that the $(1,1)$ entry of $U^*T[i,j]U$ equals $t_{ii} + z$ and

$$\left|z + t_{ii} + \sum_{i=2}^{k} t_{n_i n_i}\right| > \left|t_{ii} + \sum_{i=2}^{k} t_{n_i n_i}\right| = k w_k(T),$$

where $1 \le n_2 < \cdots < n_k \le n$ are chosen from $\{1, \ldots, n\} \setminus \{i,j\}$. Let $V \in M_n$ be obtained from I_n by replacing $I_n[i,j]$ with U, and $V^*TV = (\tilde{t}_{rs})$. Then

$$k w_k(T) = k w_k(V^*TV) \ge \left|\tilde{t}_{ii} + \sum_{i=2}^{k} \tilde{t}_{n_i n_i}\right| = \left|z + t_{ii} + \sum_{i=2}^{k} t_{n_i n_i}\right| > k w_k(T),$$

which is a contradiction. ∎

Lemma 4.3 *Let $\mathcal{M} = H_n, M_n$ or T_n. Suppose $k \in \{1, \dots, n-1\}$. Given any matrix $A \in \mathcal{M}$, $A = \mu I_n$ with $|\mu| = 1$ if and only if for any $B \in M$, there is θ (depending on B) with $|\theta| = 1$ such that*

$$w_k(A + \theta B) = w_k(A) + w_k(B) = 1 + w_k(B),$$

i.e., there is an $n \times k$ matrix U (depending on B) with $U^*U = I_k$ such that

$$w_k(A) = |\mathrm{tr}\,(U^*AU)|/k = 1 \quad \textit{and} \quad w_k(B) = |\mathrm{tr}\,(U^*BU)|/k. \tag{4.3}$$

Proof. Suppose $A = \mu I_n$ for some $|\mu| = 1$. For any $B \in \mathcal{M}$, if $w_k(B) = |\mathrm{tr}\,(U^*BU)|/k$ for some $n \times k$ matrix U with $U^*U = I_k$, then

$$|\mathrm{tr}\,(U^*AU)|/k = |\mathrm{tr}\,(U^*(\mu I_n)U)|/k = 1 = w_k(A).$$

For the converse, suppose for any $B \in \mathcal{M}$, there is an $n \times k$ matrix U with $U^*U = I_k$ such that (4.3) holds.

Let $K = I_k \oplus 0_{n-k}$. For any $n \times k$ matrix X with $X^*X = I_k$, we write $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ with $X_1 \in M_k$. Clearly, $\mathrm{tr}\,(X_1^*X_1 + X_2^*X_2) = \mathrm{tr}\,(X^*X) = k$. Then

$$(k - \mathrm{tr}\,(X_2^*X_2))/k = \mathrm{tr}\,(X_1^*X_1)/k = \mathrm{tr}\,(X^*KX)/k \in W_k(K) = [0, 1].$$

It follows that $\mathrm{tr}\,(X^*KX)/k = 1$ if and only if $\mathrm{tr}\,(X_2^*X_2) = 0$. Since $X_2^*X_2$ is positive semidefinite, $\mathrm{tr}\,(X_2^*X_2) = 0$ if and only if $X_2^*X_2 = 0_k$. Thus, X_1 must be unitary.

Suppose $\mathcal{M} = H_n$ or M_n. Take any $n \times k$ matrix V with $V^*V = I_k$, we extend V to an $n \times n$ unitary matrix $W = \begin{pmatrix} V & V' \end{pmatrix}$ with some suitable $n \times (n-k)$ matrix V'. Choose $B = WKW^*$. Then there is an $n \times k$ matrix U with $U^*U = I_k$ such that

$$1 = w_k(A) = |\mathrm{tr}\,(U^*AU)|/k \quad \text{and} \quad w_k(K) = w_k(WKW^*) = |\mathrm{tr}\,(U^*WKW^*U)|/k.$$

By the above argument, $W^*U = X = \begin{pmatrix} X_1 \\ 0 \end{pmatrix}$ for some unitary matrix $X_1 \in M_k$. Thus, $U = VX_1$ and

$$|\mathrm{tr}\,(V^*AV)|/k = |\mathrm{tr}\,(X_1^*V^*AVX_1)|/k = |\mathrm{tr}\,(U^*AU)| = w_k(A) = 1.$$

It follows that all elements of $W_k(A)$ lie on the unit circle. Since $W_k(A)$ is convex, $W_k(A)$ must be a singleton set. By Lemma 2.2(e), $A = \mu I_n$ for some $|\mu| = 1$.

It remains to show the case for $\mathcal{M} = T_n$. For any $1 \le n_1 < \cdots < n_k \le n$, let $P = (p_{ij})$ be the $n \times n$ permutation matrix with $p_{n_i i} = 1$ for $i = 1, \dots, k$ and $B = PKP^* \in T_n$. Then there is an $n \times k$ matrix U with $U^*U = I_k$ such that

$$1 = w_k(T) = |\mathrm{tr}\,(U^*TU)|/k \quad \text{and} \quad w_k(K) = w_k(PKP^*) = |\mathrm{tr}\,(U^*PKP^*U)|/k.$$

By the above argument, $P^*U = X = \begin{pmatrix} X_1 \\ 0 \end{pmatrix}$ for some unitary matrix $X_1 \in M_k$. Thus,

$$kw_k(T) = |\mathrm{tr}\,(U^*TU)| = |\mathrm{tr}\,(X^*P^*TPX)| = |\mathrm{tr}\,(X_1^*T_1X_1)| = |\mathrm{tr}\,T_1| = \left|\sum_{i=1}^{k} t_{n_in_i}\right|,$$

where T_1 is the $k \times k$ principal submatrix of P^*TP. As $n_1, \ldots, n_k$ are arbitrary, T satisfies (4.2). By Lemma 4.2, we conclude that T is a diagonal matrix.

Finally we show that the diagonal entries of T are the same. Suppose $t_{ii} \neq t_{jj}$ for some $i \neq j$. For simplicity, we assume that $t_{11} \neq t_{22}$. Take $B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \oplus I_{k-1} \oplus 0_{n-k-1}$. Then $w_k(B) = 1$ and $|\mathrm{tr}\,(X^*BX)|/k = w_k(B)$ if and only if the $n \times k$ matrix X has the form $\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \\ 0 & 0 \end{pmatrix}$ with $X_1 = \begin{pmatrix} \alpha/\sqrt{2} \\ \alpha/\sqrt{2} \end{pmatrix}$ for some $|\alpha| = 1$ and unitary $X_2 \in M_{k-1}$. In this case,

$$\left|\frac{1}{2}(t_{11} + t_{22}) + \sum_{i=3}^{k+1} t_{ii}\right| = |\mathrm{tr}\,(X^*TX)| = k.$$

Let $\alpha = t_{11} + \sum_{i=3}^{k+1} t_{ii}$ and $\beta = t_{22} + \sum_{i=3}^{k+1} t_{ii}$. Since T satisfies (4.2), we see that

$$|(\alpha + \beta)/2| = k = |\alpha| = |\beta|,$$

and hence $t_{11} = t_{22}$, which is the desired contradiction. ■

The following lemma is a modification of [15, Lemma 2], we give the proof here for the sake of completeness.

Lemma 4.4 *Let* $(\mathcal{M}, \mathcal{M}') = (H_n, H_{n'})$, $(M_n, M_{n'})$ *or* $(T_n, T_{n'})$, $k \in \{1, \ldots, n-1\}$ *and* $k' \in \{1, \ldots, n'\}$. *If* $\widetilde{\phi} : \mathcal{M} \to \mathcal{M}'$ *is a linear map satisfying (4.1) and* $\widetilde{\phi}(I_n) = I_{n'}$, *then*

$$W_{k'}(\widetilde{\phi}(A)) = W_k(A) \quad \text{for all } A \in \mathcal{M}.$$

Proof. Suppose $W_k(A) \not\subseteq W_{k'}(\widetilde{\phi}(A))$. Let $z \in W_k(A) \backslash W_{k'}(\widetilde{\phi}(A))$. Since $W_{k'}(\widetilde{\phi}(A))$ is compact, there exists some $\lambda \in \mathbb{C}$ such that

$$|z + \lambda| > |z' + \lambda|$$

for all $z' \in W_{k'}(\widetilde{\phi}(A))$. Here,

$$w_k(A + \lambda I_n) > w_{k'}(\widetilde{\phi}(A) + \lambda I_{n'}) = w_{k'}(\widetilde{\phi}(A + \lambda I_n)) = w_k(A + \lambda I_n)$$

which is impossible. Therefore, $W_k(A) \subseteq W_{k'}(\widetilde{\phi}(A))$. Similarly, we have $W_{k'}(\widetilde{\phi}(A)) \subseteq W_k(A)$. The result follows. ■

Proof of Theorem 4.1. The sufficiency part is clear. For the necessity part, suppose there is $X \in \mathcal{M}$ such that $\widetilde{\phi}(X) = I_{n'}$. For any $B \in \mathcal{M}$, there exists some $\theta \in \mathbb{C}$ with $|\theta| = 1$ such that

$$w_k(X + \theta B) = w_{k'}(I_{n'} + \theta\widetilde{\phi}(B)) = w_{k'}(I_{n'}) + w_{k'}(\widetilde{\phi}(B)) = w_k(X) + w_k(B).$$

By Lemma 4.3, $X = \mu I_n$ for some $\mu \in \mathbb{C}$ with $|\mu| = 1$. We see that the map $A \mapsto \mu\widetilde{\phi}(A)$ maps I_n to $I_{n'}$ and satisfies (4.3). Then the result follows by Lemma 4.4. ∎

5 Open problems

There are many open problems deserved further study. We mention a few of them in the following.

1. If $n' = C(n,k)\min\{k', n'-k'\}$, there are exceptional maps for the range preservers have the form
$$A \mapsto U^*\Delta_k(A)U \text{ or } A \mapsto U^*\Delta_k(A)^t U$$
with $k' = 1$. Are there other exceptional maps?

2. If $n' \le 2C(n,k)\min\{k', n'-k'\} - 2$, there are exceptional maps for the range preservers have the form
$$A \mapsto U^*[\Delta_k(A) \oplus f(\Delta_k(A))]U \text{ or } A \mapsto U^*[\Delta_k(A)^t \oplus f(\Delta_k(A))]U$$
for some unital positive linear map $f : M_{C(n,k)} \to M_{n'-C(n,k)}$, here $k' = 1$. Are there other exceptional maps?

3. In Theorem 4.1, an assumption that $\widetilde{\phi}(X) = I_{n'}$ for some $X \in \mathcal{M}$ is needed. For $k' = 1$, since $w_1(A) = w_1(A \oplus 0)$, the condition is clearly necessary. Can this assumption be removed when $k' > 1$?

4. How about extending the results to infinite dimensional operators, nest algebras, etc.?

5. What about other types of generalized numerical ranges and radii?

Acknowledgement

Li is an honorary professor of the Heilongjiang University, and also an honorary professor of the University of Hong Kong. His research was partially supported by a USA NSF grant and a HK RCG grant. Sze would like to thank Dr. Jor-Ting Chan for his guidance and encouragement.

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Department of Mathematics, College of William and Mary,
Williamsburg, VA 23187-8795, USA
ckli@math.wm.edu

Department of Mathematics, Iowa State University, Ames, Iowa 50011, USA
ytpoon@iastate.edu

Department of Mathematics, The University of Hong Kong, Hong Kong.
nungsingsze@graduate.hku.hk