H-UNITARY AND LORENTZ MATRICES: A REVIEW

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Dedicated to Professor Yung-Chow Wong, Emeritus Professor of The University of Hong Kong, on his 90th birthday

Abstract. Many properties of H-unitary and Lorentz matrices are derived using elementary methods. Complex matrices which are unitary with respect to the indefinite inner product induced by an invertible Hermitian matrix H, are called H-unitary, and real matrices that are orthogonal with respect to the indefinite inner product induced by an invertible real symmetric matrix, are called Lorentz. The focus is on the analogues of singular value and CS decompositions for general H-unitary and Lorentz matrices, and on the analogues of Jordan form, in a suitable basis with certain orthonormality properties, for diagonalizable H-unitary and Lorentz matrices. Several applications are given, including connected components of Lorentz similarity orbits, products of matrices that are simultaneously positive definite and H-unitary, products of reflections, stability and robust stability.

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1. Introduction.

Let $M_n = M_n(\mathbb{F})$ be the algebra of $n \times n$ matrices with entries in the field $\mathbb{F} = \mathbb{C}$, the complex numbers, or $\mathbb{F} = \mathbb{R}$, the real numbers. If $H \in M_n$ is an invertible Hermitian (symmetric in the real case) matrix, a matrix $A \in M_n$ is called H-*unitary* if $A^*HA = H$.

The literature on the subject is voluminous, starting with the invention of non-Euclidean geometry in the 19-th century; in the 20-th century studies of H-unitary matrices were motivated, besides considerable theoretical mathematical interest, by applications in physics, in particular, in relativity theory, and later by applications in electrical engineering, where functions with values in the group of H-unitary matrices play a significant role. Without attempting to give a literature guide on the subject, which would take us too far afield, we indicate an early influential source [1], and books [19], [9], [2], also [17, Chapter 2], where H-unitary matrices are treated in considerable depth from the point of view of the theory of matrices, in contrast with the point of view of Lie group theory in many other sources. For applications of H-unitary valued functions in engineering and interpolation, see, e.g., the books [16],[3], and for an exposition from the point of view of numerical methods see the recent review [14].

In this paper we present several canonical forms of H-unitary matrices, and demonstrate some of their applications. The exposition is kept purposely on an elementary level, but at the same time is self-contained (with few exceptions), to make the article accessible to a large audience. Thus, occasionally results are stated and proved not in the most general known form. Many results in this paper are known, in which cases we provide short transparent proofs. Hopefully, this will give a gentle introduction on the subject to beginners and researchers in fields other than matrix theory.

To avoid the well-known cases of unitary or real orthogonal matrices, we assume throughout that H is indefinite. In our discussion, we often assume that $H = J :=$

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$I_p \oplus -I_q$ for some positive integers p and q with $p+q=n$. There is no harm to do so because of the following observation.

OBSERVATION 1.1. *If $S \in M_n$ is invertible, then $A \in M_n$ is H-unitary if and only if $S^{-1}AS$ is S^*HS-unitary.*

In the real case, a matrix A is often called *Lorentz* if it is J-unitary. We will use the terminology "J-unitary" instead of "Lorentz" for convenience.

The following notation will be used in the paper.
$M_{p\times q} = M_{p\times q}(\mathbb{F})$: the $\mathbb{F}$-vector space of $p\times q$ matrices with entries in $\mathbb{F}$;
A^*: the conjugate transpose of $A \in M_{p\times q}$; it reduces to the transpose A^t of A in the real case;
$\mathrm{Spec}\,(A)$: the spectrum of a matrix A;
$\mathrm{diag}\,(X_1,\ldots,X_r) = X_1 \oplus \cdots \oplus X_r$: the block diagonal matrix with diagonal blocks $X_1,\ldots,X_r$ (in the given order);
$\sqrt{A}$: the unique positive definite square root of a positive definite matrix A;
I_p: the $p\times p$ identity matrix;
$[x,y] = y^*Hx$: the indefinite inner product induced by H;
$\mathcal{U}_{\mathbb{F}}^H$: the group of all H-unitary matrices with entries in $\mathbb{F}$.

On several occasions we will use the identification of the complex field as a subalgebra of real 2×2 matrices:

$$x+iy \in \mathbb{C},\ x,y\in\mathbb{R} \quad \longleftrightarrow \quad \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \in M_2(\mathbb{R}). \tag{1.1}$$

2. CS Decomposition. In this section we let $\mathbb{F}=\mathbb{R}$ or $\mathbb{F}=\mathbb{C}$, and $J = I_p \oplus -I_q$. Let $\mathcal{U}_n$ be the unitary group in M_n, and let $\mathcal{U}(p,q)$ be group of matrices $U_1 \oplus U_2$ such that $U_1 \in \mathcal{U}_p$ and $U_2 \in \mathcal{U}_q$.

OBSERVATION 2.1. *A matrix $A \in M_n$ is J-unitary if and only if UAV is J-unitary for any/all $U,V \in \mathcal{U}(p,q)$.*

The following lemma is useful (its verification is straightforward):

LEMMA 2.2. *A matrix $\begin{pmatrix} \sqrt{I_p+MM^*} & M \\ M^* & \sqrt{I_q+M^*M} \end{pmatrix}$ is J-unitary, as well as positive definite, for every $p\times q$ matrix M.*

For any (usual) unitary matrix $A \in M_n$, there are matrices $X,Y \in \mathcal{U}(p,q)$ such that

$$XAY = I_r \oplus \begin{pmatrix} C & S \\ -S^t & C \end{pmatrix} \oplus I_s,$$

where $C,S \in M_{p-r}$ are diagonal matrices with positive diagonal entries satisfying $C^2+S^2 = I_{p-r}$. This is known as the CS $(\cos-\sin)$ decomposition of A, see, e.g., [13, p. 78]. We have the following analogous CS $(\cosh-\sinh)$ decomposition theorem for a J-unitary matrix.

THEOREM 2.3. *A matrix $A \in M_n$ is J-unitary if and only if there exist $X,Y \in \mathcal{U}(p,q)$ and a $p\times q$ matrix $D=[d_{ij}]$, where $d_{11} \ge \cdots \ge d_{mm} > 0$ for some $m \le \min\{p,q\}$ and all other entries of D are zero, such that*

$$XAY = \begin{pmatrix} \sqrt{I_p+DD^t} & D \\ D^t & \sqrt{I_q+D^tD} \end{pmatrix}. \tag{2.1}$$

Moreover, the matrix D is uniquely determined by A.

Proof. Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ such that $A_{11} \in M_p$ and $A_{22} \in M_q$. Suppose $U_1, V_1 \in \mathcal{U}_p$ and $U_2, V_2 \in \mathcal{U}_q$ are such that

$$U_1 A_{12} V_2 = D_1 = \begin{pmatrix} \tilde{D}_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad U_2 A_{21} V_1 = D_2 = \begin{pmatrix} \tilde{D}_2 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\tilde{D}_1$ and $\tilde{D}_2$ are diagonal matrices with positive diagonal entries arranged in descending order. Let $U = U_1 \oplus U_2, V = V_1 \oplus V_2 \in \mathcal{U}(p,q)$. Then

$$B = UAV = \begin{pmatrix} R & D_1 \\ D_2 & S \end{pmatrix}.$$

Since $B^*JB = J$, we see that

$$R^*R - D_2^*D_2 = I_p, \quad D_1^*D_1 - S^*S = -I_q, \quad R^*D_1 - D_2^*S = 0.$$

Since $B^*JBJ = I_n$, we have $BJB^* = J$, and hence

$$RR^* - D_1D_1^* = I_p, \quad D_2D_2^* - SS^* = -I_q, \quad RD_2^* - D_1S^* = 0.$$

Note that $R^*R = I_p + D_2^*D_2$ and $RR^* = I_p + D_1D_1^*$ have the same eigenvalues, and thus,

$$\tilde{D}_1 = \tilde{D}_2 = d_1 I_{m_1} \oplus \cdots \oplus d_r I_{m_r}$$

for some $d_1 > d_2 > \cdots > d_r > 0$ and positive integers $m_1, \dots, m_r$. Furthermore, $RR^* = I_p + D_1D_1^*$ implies that R has orthogonal rows with lengths, which equal the singular values or R, arranged in descending order; $R^*R = I_p + D_2^*D_2$ implies that R has orthogonal columns with lengths, which are equal the singular values of R, arranged in descending order. As a result,

$$R = \sqrt{1+d_1^2}X_1 \oplus \cdots \oplus \sqrt{1+d_r^2}X_r \oplus X_{r+1},$$

where $X_j \in \mathcal{U}_{m_j}$ for $j = 1, \dots, r$ and $X_{r+1} \in \mathcal{U}_{p-m}$ with $m = m_1 + \cdots + m_r$. Similarly, one can show that

$$S = \sqrt{1+d_1^2}Y_1 \oplus \cdots \oplus \sqrt{1+d_r^2}Y_r \oplus Y_{r+1},$$

where $Y_j \in \mathcal{U}_{m_j}$ for $j = 1, \dots, r$ and $Y_{r+1} \in \mathcal{U}_{q-m}$. Suppose

$$Z = X_1 \oplus \cdots \oplus X_r \oplus X_{r+1} \oplus Y_1 \oplus \cdots \oplus Y_r \oplus Y_{r+1} \in \mathcal{U}(p,q),$$

$X = Z^*U$ and $Y = V$. Then XAY has the asserted form.

The uniqueness of D follows from (2.1), because $\sqrt{1+d_{jj}^2} \pm d_{jj}$, $j = 1, \dots, m$, are the singular values of A different from 1. ■

A different proof (using the exchange operator, in the terminology of [14]) of Theorem 2.3 is given in [14]. The proof of the above theorem only uses the elementary facts: (a) every rectangular matrix has a singular value decomposition, (b) XY and YX have the same eigenvalues for any $X, Y \in M_n$, (c) $Z \in M_n$ has orthogonal rows and columns with lengths arranged in decreasing size if and only if Z is a direct sum of multiples of unitary (if $\mathbb{F} = \mathbb{C}$) or real orthogonal (if $\mathbb{F} = \mathbb{R}$) matrices. Yet, we can deduce other known canonical forms, which have been obtained by more sophisticated techniques involving Lie theory, functions and power series of matrices, etc.

THEOREM 2.4. *Let* $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_n$ *such that* $A_{11} \in M_p$ *and* $A_{22} \in M_q$. *Then* A *is* J*-unitary if and only if any one of the following four conditions holds true.*

(a) *There is* $U \in \mathcal{U}(p,q)$ *and a* $p \times q$ *matrix* M *such that*

$$UA = \begin{pmatrix} \sqrt{I_p + MM^*} & M \\ M^* & \sqrt{I_q + M^*M} \end{pmatrix}.$$

(b) *There is* $U \in \mathcal{U}(p,q)$ *and a* $p \times q$ *matrix* L *such that*

$$UA = (I_p \oplus iI_q) \exp\left(i \begin{pmatrix} 0_p & L \\ -L^* & 0_q \end{pmatrix} \right) (I_p \oplus iI_q)^* = \exp \begin{pmatrix} 0_p & L \\ L^* & 0_q \end{pmatrix}.$$

(c) *There is* $U \in \mathcal{U}(p,q)$ *and a* $p \times q$ *matrix* K *with all singular values less than one such that*

$$UA = \begin{pmatrix} I_p & K \\ K^* & I_q \end{pmatrix} \begin{pmatrix} I_p & -K \\ -K^* & I_q \end{pmatrix}^{-1}.$$

(d) *Setting* $X = A_{11}(I_p + A_{21}^* A_{21})^{-1/2}$ *and* $Y = A_{22}(I_q + A_{12}^* A_{12})^{-1/2}$, *we have*

$$X \in \mathcal{U}_p, \quad Y \in \mathcal{U}_q, \quad \text{and} \quad X^* A_{12} = A_{21}^* Y.$$

Clearly, one can write analogous conditions with U on the right, with the same right hand sides as in (a), (b), (c), and get special forms for AU. We omit the statements.

Note that the formula in (a) is a *polar decomposition* of A. It follows in particular, that both factors in the polar decomposition of an J-unitary matrix are also J-unitary, a well-known fact in Lie theory. Thus, the matrices U and M in (a) are determined uniquely. Similarly, the matrices on the right sides in conditions (b) and (c) are different representations of the positive definite part of A, and are also uniquely determined.

Proof. By Theorem 2.3, $A \in M_n$ is J-unitary if and only if there are $X, Y \in \mathcal{U}(p,q)$ satisfying (2.1). Setting $U = YX$, we get the equivalent condition (a) in view of Lemma 2.2.

To prove the equivalent condition (b), suppose A is J-unitary, and XAY has the form (2.1). Note that

$$\begin{pmatrix} \sqrt{1+d_j^2} & d_j \\ d_j & \sqrt{1+d_j^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \exp\left(i \begin{pmatrix} 0 & \ell_j \\ -\ell_j & 0 \end{pmatrix} \right) \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix},$$

where $\sinh \ell_j = d_j$. Hence,

$$XAY = (I_p \oplus iI_q) \exp\left(i \begin{pmatrix} 0 & \tilde{L} \\ -\tilde{L}^t & 0 \end{pmatrix} \right) (I_p \oplus iI_q)^* = \exp \begin{pmatrix} 0_p & \tilde{L} \\ \tilde{L}^t & 0_q \end{pmatrix},$$

where $\tilde{L}$ is the $p \times q$ matrix with ℓ_j at the (j,j) entry whenever $d_j > 0$, and zeros elsewhere. Let $U = YX$ and

$$\begin{pmatrix} 0 & L \\ -L^* & 0 \end{pmatrix} = Y \begin{pmatrix} 0 & \tilde{L} \\ -\tilde{L}^t & 0 \end{pmatrix} Y^*.$$

We get condition (b). Conversely, suppose (b) holds. Consider a singular value decomposition of $L = V_1^* \tilde{L} V_2$, where $V_1 \in \mathcal{U}_p$ and $V_2 \in \mathcal{U}_q$. Let $V = V_1 \oplus V_2 \in \mathcal{U}(p,q)$. Then

$$(VU)AV^* = \exp\begin{pmatrix} 0 & \tilde{L} \\ \tilde{L}^t & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{I_p + DD^t} & D \\ D^t & \sqrt{I_q + D^t D} \end{pmatrix}$$

has the form (2.1) with $X = VU$ and $Y = V^*$.

Next, we turn to the equivalent condition (c). Suppose A is J-unitary, and XAY has the form (2.1). Note that

$$\begin{pmatrix} \sqrt{1+d_j^2} & d_j \\ d_j & \sqrt{1+d_j^2} \end{pmatrix} = \begin{pmatrix} 1 & k_j \\ k_j & 1 \end{pmatrix} \begin{pmatrix} 1 & -k_j \\ -k_j & 1 \end{pmatrix}^{-1},$$

where $k_j \in (0,1)$ satisfying $2k_j/(1-k_j^2) = d_j$. Hence,

$$XAY = \begin{pmatrix} I_p & \tilde{K} \\ \tilde{K}^t & I_q \end{pmatrix} \begin{pmatrix} I_p & -\tilde{K} \\ -\tilde{K}^t & I_q \end{pmatrix}^{-1},$$

where $\tilde{K}$ is the $p \times q$ matrix with k_j at the (j,j) entry whenever $d_j > 0$, and zeros elsewhere. Let $U = YX$ and

$$\begin{pmatrix} 0 & K \\ K^* & 0 \end{pmatrix} = Y \begin{pmatrix} 0 & \tilde{K} \\ \tilde{K}^t & 0 \end{pmatrix} Y^*.$$

We get condition (b). Conversely, suppose (c) holds. Putting $M = 2(I_p - KK^*)^{-1}K$, we see that (c) implies (a). Thus, A is J-unitary.

Finally, we consider the equivalent condition (d). Suppose A is J-unitary and condition (a) holds with $U = U_1 \oplus U_2$, where $U_1 \in \mathcal{U}_p$ and $U_2 \in \mathcal{U}_q$, i.e.,

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} U_1^* & 0 \\ 0 & U_2^* \end{pmatrix} \begin{pmatrix} \sqrt{I_p + MM^*} & M \\ M^* & \sqrt{I_q + M^*M} \end{pmatrix}.$$

Then

$$X = A_{11}(I_p + A_{21}^* A_{21})^{-1/2} = U_1^* \sqrt{I_p + MM^*}(I_p + MM^*)^{-1/2} = U_1^*,$$

$$Y = A_{22}(I_q + A_{12}^* A_{12})^{-1/2} = U_2^* \sqrt{I_q + M^*M}(I_p + M^*M)^{-1/2} = U_2^*,$$

and

$$X^* A_{12} = U_1(U_1^* M) = M = (MU_2)U_2^* = A_{21}^* Y.$$

Thus, condition (d) holds. Conversely, suppose condition (d) holds. Putting $M = X^* A_{12}$, we see that condition (a) holds with $U = X^* \oplus Y^* \in \mathcal{U}(p,q)$. So, A is J-unitary. ∎

Recall that (see [22]) that the rows of $(X_0, Y_0) \in M_{r\times p} \times M_{r\times q}$ (respectively, $(X_1 \;\; Y_1) \in M_{r\times p} \times M_{r\times q}$) are *initial vectors* (respectively, *final vectors*) of the J-unitary matrix A if

$$(X_0 \,|\, Y_0)A = (X_1 \,|\, Y_1).$$

There is an interest in determining a J-unitary matrix in terms of its initial and final vectors, see [22]. In this connection, we can use Theorem 2.4 (c) to get the following corollary.

Corollary 2.5. *Suppose A is a J-unitary matrix expressed as in Theorem* 2.4(*c*) *with $U = U_1 \oplus U_2$. Let $(X_0 \,|\, Y_0), (X_1 \,|\, Y_1) \in M_{r\times p} \times M_{r\times q}$ be initial vectors and final vectors of A, respectively. Then*

$$X_0 U_1^*(I_p - KK^*) = X_1(I_p + KK^*) - 2Y_1K^*$$

and

$$Y_0 U_2^*(I_q - K^*K) = Y_1(I_q + K^*K) - 2X_1K.$$

In particular, if $r = p$ and $\det(X_0) \neq 0$ *then*

$$X_0^{-1}[X_1(I_p + KK^*) - 2Y_1K^*]$$

is a constant matrix, i.e., independent of $(X_0 \,|\, Y_0)$ and of $(X_1 \,|\, Y_1)$. Similarly, if $r = q$ and $\det(Y_0) \neq 0$ *then*

$$Y_0^{-1}[Y_1(I_q + K^*K) - 2X_1K]$$

is a constant matrix.

By the canonical form in Theorem 2.4 (a), we have the following.

Corollary 2.6. *The group of J-unitary matrices is homeomorphic to $\mathcal{U}_p \times \mathcal{U}_q \times \mathbb{F}^{pq}$. In the real case, it is a real analytic manifold consisting of four arc-wise connected components, and the identity component is locally isomorphic to $\mathbb{R}^{p(p-1)/2} \times \mathbb{R}^{q(q-1)/2} \times \mathbb{R}^{pq}$. In the complex case, it is a real analytic manifold consisting of one arc-wise connected component which is locally isomorphic to $\mathbb{R}^{p^2} \times \mathbb{R}^{q^2} \times \mathbb{R}^{2pq}$.*

Corollary 2.7. *The group $\mathcal{U}(p,q)$ is a maximal bounded subgroup of $\mathcal{U}_{\mathbb{F}}^J$.*

Proof. The group $\mathcal{U}(p,q)$ is clearly bounded. Let $\mathcal{G}$ be a subgroup of $\mathcal{U}_{\mathbb{F}}^J$ that strictly contains $\mathcal{U}(p,q)$, and let $A \in \mathcal{G} \setminus \mathcal{U}(p,q)$. Then in the representation (2.1) of A, we clearly have $D \neq 0$, and

$$\begin{pmatrix} \sqrt{I + DD^t} & D \\ D^t & \sqrt{I + D^tD} \end{pmatrix} \in \mathcal{G}.$$

But since $D \neq 0$, the cyclic subgroup generated by $\begin{pmatrix} \sqrt{I + DD^t} & D \\ D^t & \sqrt{I + D^tD} \end{pmatrix}$ is not bounded. ∎

In connection with Corollary 2.7 observe that there exist bounded cyclic subgroups of $\mathcal{U}_{\mathbb{F}}^J$ which are not contained in $\mathcal{U}(p,q)$ (see Theorem 4.9).

Suppose $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_n$ is a J-unitary matrix with $A_{11} \in M_p$ and $A_{22} \in M_q$. By Theorem 2.3, A_{11} and A_{22} are invertible. For $\mathbb{F} = \mathbb{R}$, define

$$\sigma_+(A) = \begin{cases} 1 & \text{if } \det A_{11} > 0, \\ -1 & \text{if } \det A_{11} < 0, \end{cases} \quad , \quad \sigma_-(A) = \begin{cases} 1 & \text{if } \det A_{22} > 0, \\ -1 & \text{if } \det A_{22} < 0 \end{cases} \quad . \tag{2.2}$$

We can use Theorem 2.4 (a) to deduce the following corollary (see, e. g., [8]).

Corollary 2.8. ($\mathbb{F} = \mathbb{R}$). *For any J-unitary matrices $A, B \in M_n(\mathbb{R})$,*

$$\sigma_+(A)\sigma_+(B) = \sigma_+(AB) \qquad \text{and} \qquad \sigma_-(A)\sigma_-(B) = \sigma_-(AB).$$

Proof. From Theorem 2.4 (a),

$$A = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} \sqrt{I_p + MM^t} & M \\ M^t & \sqrt{I_q + M^tM} \end{pmatrix},$$

$$B = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \begin{pmatrix} \sqrt{I_p + NN^t} & N \\ N^t & \sqrt{I_q + N^tN} \end{pmatrix},$$

for some real $p \times q$ matrices M and N. Thus, $\sigma_+(A) = \det U_1$, $\sigma_+(B) = \det V_1$, and

$$\sigma_+(AB) = \operatorname{sign}\{\det(U_1\sqrt{I_p + MM^t}V_1\sqrt{I_p + NN^t} + U_1MV_1N^t)\}. \tag{2.3}$$

Let $M_x = xM$, $N_x = xN$, $0 \le x \le 1$. By the comment before the corollary and Theorem 2.4 (a), the matrix

$$W_x := U_1\sqrt{I_p + M_xM_x^t}V_1\sqrt{I_p + N_xN_x^t} + U_1M_xV_2N_x^t$$

is invertible for every $x \in [0, 1]$. Therefore, the sign of the determinant of W_x does not depend on x, and we have

$$\begin{aligned} \operatorname{sign}\{\det W_1) = \operatorname{sign}\{\det W_0\} &= \operatorname{sign}\{\det(U_1V_1)\} \\ = \operatorname{sign}\{\det U_1\}\operatorname{sign}\{\det V_1\} &= \sigma_+(A)\sigma_+(B). \end{aligned}$$

In view of (2.3) the result for σ_+ follows. The proof for σ_- is similar. ∎

We conclude this section with the following well known result that H-unitary matrices can be obtained by applying linear fractional transforms to H-skewadjoint matrices. A matrix $K \in M_n$ is called *H-skewadjoint* if $[Kx, y] = -[x, Ky]$ for every $x, y, \in \mathbb{F}^n$, i.e., $HK = -K^*H$. Here we just assume that $H \in M_n$ is an invertible Hermitian (symmetric in the real case) matrix.

Proposition 2.9. *Suppose A is H-unitary, and $\mu, \xi \in \mathbb{F}$ satisfy $|\mu| = 1$ with* $\det(A - \mu I) \neq 0$ *and $-\bar{\xi} \neq \xi$ if $\mathbb{F} = \mathbb{C}$. Then the operator $K = (\xi A + \mu\bar{\xi}I)(A - \mu I)^{-1}$ is H-skewadjoint such that* $\det(K - \xi I) \neq 0$. *Conversely, suppose $K \in M_n$ is H-skewadjoint, and $\mu, \xi \in \mathbb{F}$ satisfy $|\mu| = 1$,* $\det(K - \xi I) \neq 0$, *and $-\bar{\xi} \neq \xi$ if $\mathbb{F} = \mathbb{C}$. Then $A = \mu(K + \bar{\xi}I)(K - \xi I)^{-1}$ is H-unitary such that* $\det(A - \mu I) \neq 0$.

For a proof see, e.g., [4, pp.38-39] or [9]; the proposition is also easy to verify directly using algebraic manipulations.

3. Diagonalizable H-unitary matrices. In this section we assume that $H = H^* \in M_n(\mathbb{F})$ is indefinite and invertible but not necessarily equal to J, as in the previous section.

Evidently, A is H-unitary if and only if $S^{-1}AS$ is S^*HS-unitary, for any invertible matrix S. In the complex case, a canonical form under this transformation is described in [11], [12], see also [9]; other canonical forms in both real and complex cases are given in [20]. We will not present these forms in full generality, and consider in Sections 3.1

and 3.2 only the diagonalizable cases, which will suffice for the applications presented in later sections.

Let $J_j(\lambda)$ denote the upper triangular $j \times j$ Jordan block with eigenvalue λ. In the real case, we let $J_{2k}(\lambda \pm i\mu)$ be the almost upper triangular $2k \times 2k$ real Jordan block with a pair of nonreal complex conjugate eigenvalues $\lambda \pm i\mu$ (here λ and μ are real and $\mu \neq 0$):

$$J_{2k}(\lambda \pm i\mu) = \begin{pmatrix} J_2(\lambda \pm i\mu) & I_2 & 0_2 & \dots & 0_2 \\ 0_2 & J_2(\lambda \pm i\mu) & I_2 & \dots & 0_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_2 & 0_2 & 0_2 & \dots & J_2(\lambda \pm i\mu) \end{pmatrix}, \quad J_2(\lambda \pm i\mu) = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix}.$$

We use also the following notation: G_j is the $j \times j$ matrix with 1's on the top-right - left-bottom diagonal, and zeros in all other positions.

Proposition 3.1. *$\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$. A matrix $A \in M_n(\mathbb{F})$ is H-unitary for some invertible Hermitian matrix $H \in M_n(\mathbb{F})$ if and only if A is invertible and similar (over $\mathbb{F}$) to $(A^{-1})^*$, and the following condition holds in the real case: Each Jordan block of eigenvalue 1 having even size (if it exists) appears an even number of times in the Jordan form of A, and Jordan block of eigenvalue -1 having even size (if it exists) appears an even number of times in the Jordan form of A.*

Proof. Consider the complex case first. "Only if" is clear: $A^*HA = H$ implies $(A^{-1})^* = HAH^{-1}$. For the "if" part, observe that without loss of generality (using the transformation $(A, H) \mapsto (S^{-1}AS, S^*HS)$ for a suitable invertible S), we may assume that A is in the Jordan form. Considering separately every Jordan block of A with a unimodular eigenvalue, and collecting together every pair of Jordan blocks of equal size with eigenvalues λ and μ such that $\lambda\overline{\mu} = 1$, we reduce the proof to two cases:

(i) $A = J_j(\lambda)$, $|\lambda| = 1$;
(ii) $A = J_j(\lambda) \oplus J_j(\mu)$, $\lambda\overline{\mu} = 1$, $|\lambda| \neq 1$.

In case (ii), by making a similarity transformation, we may assume that $A = J_j(\lambda) \oplus \left(\overline{J_j(\lambda)}\right)^{-1}$; then a calculation shows that A is G_{2j}-unitary. In case (i), by making a similarity transformation, assume (cf. [9, Section 2.3])

$$A = \lambda \begin{pmatrix} 1 & 2i & 2i^2 & \dots & 2i^{j-1} \\ 0 & 1 & 2i & \dots & 2i^{j-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix};$$

then A is G_j-unitary.

Let now $\mathbb{F} = \mathbb{R}$. Consider the "if" part. As in the complex case, we may assume that A is in the real Jordan form (see [10, Chapter 12]), and furthermore that A has one of the following four forms:

(a) $A = J_m(1)$ or $A = J_m(-1)$, where m is odd;
(b) $A = J_{2k}(\lambda \pm i\mu)$, where $\lambda^2 + \mu^2 = 1$, $\mu > 0$;
(c) $A = J_k(\lambda) \oplus J_k(\lambda^{-1})$, where λ is real and $|\lambda| \geq 1$, and in cases $\lambda = \pm 1$ the size k is even;
(d) $A = J_k(\lambda \pm i\mu) \oplus J_k(\lambda' \pm i\mu')$, where k is even, $\lambda^2 + \mu^2 > 1$ and $\lambda' + i\mu' = (\lambda - i\mu)^{-1}$.

In the cases (c) and (d) A is similar to a matrix of the form $\begin{pmatrix} B & 0 \\ 0 & (B^t)^{-1} \end{pmatrix}$, where $B \in M_k(\mathbb{R})$, and the matrix $\begin{pmatrix} B & 0 \\ 0 & (B^t)^{-1} \end{pmatrix}$ is $\begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix}$-unitary, as one verifies easily. The case (b) is reduced to the already proven complex case by using the identification (1.1).

Consider the case (a). The case when $A = J_m(-1)$ is easily reduced, by replacing A with $-A$, and by making a similarity transformation, to the case when $A = J_m(1)$; thus, we assume $A = J_m(1)$. The matrix $K = (I - A)(I + A)^{-1}$ has the Jordan form $J_m(0)$ and by Proposition 2.9 is H-skewadjoint if and only if A is H-unitary. Thus, it suffices to find an invertible real symmetric H such that $J_m(0)$ is H-skewadjoint, i.e., $HJ_m(0) = -(J_m(0))^t H$. One such H is given by $H = [h_{j,k}]_{j,k=1}^m \in M_m(\mathbb{R})$ with the entries $h_{j,m+1-j} = (-1)^{j+1}$, $j = 1, \ldots, m$, and all other entries being zero.

We now prove the "only if" part in the real case. Let $A \in M_n(\mathbb{R})$ be H-unitary, and assume first that one, but not both, of the numbers 1 and -1 are eigenvalues of A (if $1, -1 \notin \operatorname{Spec}(A)$, we are done.) Say, $-1 \notin \operatorname{Spec}(A)$. By Proposition 2.9, the matrix $K = (I - A)(I + A)^{-1}$ is H-skewadjoint. Since the derivative of the function $f(z) = \dfrac{1-z}{1+z}$ is $f'(z) = \dfrac{-2}{(1+z)^2}$, which is nonzero for $z \in \operatorname{Spec}(A)$, the calculus of functions of the matrix A (which can be found in many graduate texts on linear algebra, see, for example, [18] or [15, Chapter 6]) shows that the Jordan blocks of K with eigenvalue 0 of K have sizes equal to the sizes of corresponding Jordan blocks of A with eigenvalue 1. Now $K = H^{-1}(HK)$ is a product of an invertible symmetric matrix H^{-1} and a skewsymmetric matrix HK, and therefore every nilpotent Jordan block of even size in the Jordan form of K appears an even number of times (see [21, Lemma 2.2]). Hence the same property holds for the Jordan blocks of A corresponding to the eigenvalue 1.

We leave aside the more difficult case when both 1 and -1 are eigenvalues of A. This case can be dealt with using the proof of [6, Theorem 9, Section I.5], upon replacing there the operation of transposition $A \mapsto A^t$ by the operation of H-adjoint: $A \mapsto H^{-1}A^tH$, and making use of the already mentioned fact that every nilpotent Jordan block of even size in the Jordan form of an H-skewadjoint matrix appears an even number of times. All arguments in the proof go through, and we omit the details. ■

Thus, in the complex case, if λ is an eigenvalue of an H-unitary matrix A, then so is $\overline{\lambda}^{-1}$, with the same algebraic and geometric multiplicities as λ; a similar statement applies to pairs of complex conjugate eigenvalues of real H-unitary matrices.

3.1. The complex case. We assume $\mathbb{F} = \mathbb{C}$ in this subsection. Denote by $\mathcal{R}_\lambda(A) = \operatorname{Ker}(A - \lambda I)^n$ the root subspace corresponding to the eigenvalue λ of an $n \times n$ matrix A. We need orthogonality properties of the root subspaces and certain eigenvectors.

Lemma 3.2. *Let A be H-unitary.*

(a) *If $v \in \mathcal{R}_\lambda(A)$, $w \in \mathcal{R}_\mu(A)$, where $\lambda\overline{\mu} \neq 1$, then v and w are H-orthogonal:*

$$[v, w] = 0. \tag{3.1}$$

(b) *If x is an eigenvector of A corresponding to the eigenvalue λ, and if EITHER $|\lambda| \neq 1$ OR $|\lambda| = 1$ and $(A - \lambda I)y = x$ for some vector y, then $[x, x] = 0$.*

Proof. (a) We have

$$(A-\lambda I)^p v = 0, \qquad (A-\mu I)^q w = 0, \tag{3.2}$$

for some positive integers p and q. We prove (3.1) by induction on $p+q$ (see [11, Lemma 3]). If $p=q=1$, then $(A-\lambda I)v = (A-\mu I)w = 0$, and therefore

$$\lambda\overline{\mu}[v,w] = [\lambda v, \mu w] = [Av, Aw] = [v,w],$$

which implies (in view of $\lambda\overline{\mu}\neq 1$) $[v,w]=0$. Assume now (3.2) holds, and assume that $[v',w']=0$ for v', w' satisfying (3.2) with smaller values of $p+q$. We let $v'=(A-\lambda I)v$, $w'=(A-\mu I)w$, and then

$$\begin{aligned}\lambda\overline{\mu}[v,w] &= [Av-v', Aw-w'] \\ &= [Av,Aw]-[v',Aw]-[Av,w']+[v',w'] \\ &= [Av,Aw]-[v',w']-[v',\mu w]-[v',w']-[\lambda v,w']+[v',w'] \\ &= [Av,Aw]=[v,w],\end{aligned}$$

where the last but one equality follows by the induction hypothesis. So, the desired conclusion $[v,w]=0$ is obtained.

Part (b) under the hypothesis that $|\lambda|\neq 1$ follows from (a) (take $\mu=\lambda$). Assume now

$$(A-\lambda I)x = 0, \quad (A-\lambda I)y = x, \quad x\neq 0, \quad |\lambda|=1.$$

Arguing by contradiction suppose that $[x,x]\neq 0$. Then, adding to y a suitable multiple of x, we may assume without loss of generality that $[y,x]=0$. Now

$$\begin{aligned}[x,x] &= y^*(A-\lambda I)^*H(A-\lambda I)y \\ &= y^*(A^*HA-\overline{\lambda}HA-\lambda A^*H+H)y \quad (\text{using } A^*HA=H) \\ &= y^*(H-\overline{\lambda}HA-\lambda A^*H+H)y \\ &= -\overline{\lambda}y^*H(A-\lambda I)y-\lambda y^*(A-\lambda I)^*Hy \\ &= -\overline{\lambda}y^*Hx-\lambda x^*Hy \\ &= 0,\end{aligned}$$

a contradiction. ■

THEOREM 3.3. *A diagonalizable matrix $A\in M_n(\mathbb{C})$ is H-unitary if and only if there exists an invertible matrix S such that $(S^{-1}AS, S^*HS)$ equals*

$$\left(U_1\oplus\ldots\oplus U_m\oplus U_{m+1}\oplus\ldots\oplus U_{m+q}, \epsilon_1\oplus\ldots\oplus\epsilon_m\oplus\begin{pmatrix}0&1\\1&0\end{pmatrix}\oplus\ldots\oplus\begin{pmatrix}0&1\\1&0\end{pmatrix}\right), \tag{3.3}$$

where $\epsilon_j=\pm 1$ ($j=1,\ldots,m$), the complex numbers U_j for $j=1,\ldots,m$ are unimodular, and the 2×2 matrices U_j for $j=m+1,\ldots,m+q$ are of the form $U_j=\begin{pmatrix}\lambda_j&0\\0&(\overline{\lambda_j})^{-1}\end{pmatrix}$, $|\lambda_j|\neq 1$.

Moreover, the representation of an H-unitary matrix A as in the right hand side of (3.3) is unique up to a simultaneous permutation of pairs (U_j,ϵ_j), $j=1,\ldots,m$, and up to a permutation of blocks $U_{m+1},\ldots,U_{m+q}$.

Proof. The part "if" being obvious, consider the "only if" part. In view of Lemma 3.2, we need to consider only the case when A has either only one (possibly of high multiplicity) unimodular eigenvalue λ, or only one pair $\lambda, (\overline{\lambda})^{-1}$ of non-unimodular eigenvalues (again, possibly of high multiplicity). Since A is diagonalizable, in the first case $A = \lambda I$, and using a congruence transformation $H \mapsto S^*HS$ we put H in the diagonal form, as required. In the second case, we may assume $A = \lambda I \oplus (\overline{\lambda})^{-1} I$, and (by Lemma 3.2) $H = \begin{pmatrix} 0 & Q \\ Q^* & 0 \end{pmatrix}$ for some (necessarily invertible) matrix Q. A transformation

$$\begin{pmatrix} 0 & Q \\ Q^* & 0 \end{pmatrix} \mapsto \begin{pmatrix} I & 0 \\ 0 & (Q^{-1})^* \end{pmatrix} \begin{pmatrix} 0 & Q \\ Q^* & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & Q^{-1} \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

shows that A is $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$-unitary, and a simultaneous permutation of rows and columns in A and in $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ yields the desired form. ∎

3.2. The real case. In this subsection $\mathbb{F} = \mathbb{R}$. We say that a matrix $A \in M_n(\mathbb{R})$ is diagonalizable if A is similar to a diagonal matrix (over the complex field). Thus, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is diagonalizable, and $\operatorname{Spec}(A) = \{i, -i\}$. If $\lambda \in \operatorname{Spec}(A)$ is real, we let $\mathcal{R}_\lambda(A) = \operatorname{Ker}(A - \lambda I)^n \subseteq \mathbb{R}^n$. If $\lambda \pm \mu i$ is a pair of nonreal complex conjugate eigenvalues of A, we let

$$\mathcal{R}_{\lambda \pm \mu i}(A) = \operatorname{Ker}\left(A^2 - 2\lambda A + (\lambda^2 + \mu^2) I\right)^n \subseteq \mathbb{R}^n.$$

Then we have a direct sum (see, e.g., [10, Section 12.2])

$$\mathbb{R}^n = \sum_{j=1}^{k} \mathcal{R}_{\lambda_j}(A) + \sum_{j=1}^{\ell} \mathcal{R}_{\lambda_j \pm i\mu_j}(A)$$

where $\lambda_1, \ldots, \lambda_k$ are all distinct real eigenvalues of A (if any), and $\lambda_1 \pm i\mu_1, \ldots, \lambda_\ell \pm i\mu_\ell$ are all distinct pairs of nonreal complex conjugate eigenvalues of A (if any), where it is assumed that $\mu_j > 0$.

If A is H-unitary, then by Proposition 3.1 the eigenvalues of A can be collected into sets of the following four structures (for a particular A, some of these sets may be absent):

(i) $\lambda = \pm 1 \in \operatorname{Spec}(A)$;
(ii) $\{\lambda, \overline{\lambda}\} \subseteq \operatorname{Spec}(A)$, where $|\lambda| = 1$ and the imaginary part of λ is positive;
(iii) $\{\lambda, \lambda^{-1}\} \subseteq \operatorname{Spec}(A)$, where $\lambda \in \mathbb{R}$, $|\lambda| > 1$;
(iv) $\{\lambda, \overline{\lambda}, \lambda^{-1}, \overline{\lambda}^{-1}\} \subseteq \operatorname{Spec}(A)$, where λ has positive imaginary part and $|\lambda| > 1$.

According to these four structures, we let

$$\mathcal{R}\mathcal{R}_\lambda(A) := \begin{cases} \mathcal{R}_\lambda(A) & \text{if } \lambda = 1 \text{ or } \lambda = -1 \\ \mathcal{R}_{\lambda,\overline{\lambda}}(A) & \text{if } |\lambda| = 1 \text{ and the imaginary part of } \lambda \text{ is positive} \\ \mathcal{R}_\lambda(A) \dot{+} \mathcal{R}_{\lambda^{-1}}(A) & \text{if } \lambda \in \mathbb{R},\ |\lambda| > 1 \\ \mathcal{R}_{\lambda,\overline{\lambda}} \dot{+} \mathcal{R}_{\lambda^{-1},\overline{\lambda}^{-1}}(A) & \text{if } \lambda \text{ has positive imaginary part and } |\lambda| > 1 \end{cases}$$

With this notation, we can now state an orthogonality result analogous to Lemma 3.2.

LEMMA 3.4. *Let A be H-unitary.*

(a) *If $v \in \mathcal{RR}_\lambda(A)$, $w \in \mathcal{RR}_\mu(A)$, where $\lambda \neq \mu$, then v and w are H-orthogonal:*

$$[v, w] = 0. \tag{3.4}$$

(b) *If $x \in \mathbb{R}^n$ is an eigenvector corresponding to a real eigenvalue λ of A, and EITHER $|\lambda| \neq 1$ OR $\lambda = \pm 1$ and there exists $y \in \mathbb{R}^n$ such that $(A - \lambda I)y = x$, then $[x, x] = 0$.*

(c) *If $\lambda = \mu + i\nu$ is a nonreal eigenvalue of A with positive imaginary part ν and with $|\lambda| \neq 1$, and if $(A^2 - 2\mu A + (\mu^2 + \nu^2)I)x = 0$, then $[x, x] = 0$.*

(d) *If $\lambda = \mu + i\nu$ is a nonreal eigenvalue of A with positive imaginary part ν and with $|\lambda| = 1$, and if*

$$(A^2 - 2\mu A + I)x = 0, \quad (A^2 - 2\mu A + I)y = x \tag{3.5}$$

for some $y \in \mathbb{R}^n$, then $[x, x] = 0$.

Proof. Part (a) follows from Lemma 3.2 by considering a complexification of A, i.e., considering A as a complex matrix representing a linear transformation in $\mathbb{C}^n$. The same complexification takes care of statement (c). Part (b) is proved in exactly the same way as part (b) of Lemma 3.2.

It remains to prove part (d). Assume (3.5) holds, and arguing by contradiction, suppose $[x, x] \neq 0$. Let

$$y_N = y + \alpha x + \beta Ax, \tag{3.6}$$

where $\alpha, \beta \in \mathbb{R}$ are chosen so that

$$[y_N, x] = [Ay_N, x] = 0. \tag{3.7}$$

This choice of α and β is possible. Indeed, (3.7) amount to the following system of linear equations for α and β:

$$\alpha[x, x] + \beta[Ax, x] = -[y, x],$$

$$\alpha[Ax, x] + \beta[A^2x, x] = \alpha[Ax, x] + \beta(2\mu[Ax, x] - [x, x]) = -[Ay, x].$$

The determinant of the system is

$$-[Ax, x]^2 - [x, x]^2 + 2\mu[x, x][Ax, x],$$

which is negative since $[x, x] \neq 0$ and $-1 < \mu < 1$. Clearly, $(A^2 - 2\mu A + I)y_N = x$, and using (3.7), we obtain

$$\begin{aligned}
[x, x] &= [(A^2 - 2\mu A + I)y_N, x] = [A^2 y_N, x] = [A^2 y_N, (A^2 - 2\mu A + I)y_N] \\
&= [A^2 y_N, A^2 y_N] - 2\mu[Ay_N, y_N] + [A^2 y_N, y_N] = \text{ (because } [Ay_N, Ay_N] = [y_N, y_N]\text{)} \\
&= [y_N, y_N] - 2\mu[Ay_N, y_N] + [x + 2\mu Ay_N - y_N, y_N] \\
&= [y_N, y_N] - 2\mu[Ay_N, y_N] + [x, y_N] + 2\mu[Ay_N, y_N] - [y_N, y_N] = 0,
\end{aligned}$$

a contradiction with our supposition. ∎

Theorem 3.5. *Let H be a real symmetric invertible $n \times n$ matrix. A diagonalizable matrix $A \in M_n(\mathbb{R})$ is H-unitary if and only if there exists an invertible matrix $S \in M_n(\mathbb{R})$ such that $S^{-1}AS$ equals*

$$U_0 \oplus U_1 \oplus \ldots \oplus U_q \oplus U_{q+1} \oplus \ldots \oplus U_{q+r} \oplus U_{q+r+1} \oplus \ldots \oplus U_{q+r+s}, \tag{3.8}$$

and S^tHS equals

$$H_0 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \epsilon_{q+1} I_2 \oplus \ldots \oplus \epsilon_{q+r} I_2 \oplus \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \tag{3.9}$$

where the constituents of (3.8) *and* (3.9) *are as follows:*

(i) *$U_0, H_0 \in M_m(\mathbb{R})$ are diagonal orthogonal matrices;*

(ii) *For $j = 1, \ldots, q$: the 2×2 matrices U_j are of the form $U_j = \begin{pmatrix} \lambda_j & 0 \\ 0 & \lambda_j^{-1} \end{pmatrix}$, where $\lambda_j \in \mathbb{R}$, $|\lambda_j| > 1$;*

(iii) *For $j = q+1, \ldots, q+r$: the 2×2 matrices U_j are of the form $U_j = \begin{pmatrix} \lambda_j & \mu_j \\ -\mu_j & \lambda_j \end{pmatrix}$, where $\lambda_j^2 + \mu_j^2 = 1$ and $\mu_j > 0$, and the ϵ_j's are ± 1;*

(iv) *For $j = q+r+1, \ldots, q+r+s$: the 4×4 matrices U_j are of the form*

$$U_j = \begin{pmatrix} \lambda_j & \mu_j & 0 & 0 \\ -\mu_j & \lambda_j & 0 & 0 \\ 0 & 0 & \lambda_j' & \mu_j' \\ 0 & 0 & -\mu_j' & \lambda_j' \end{pmatrix}, \qquad \lambda_j^2 + \mu_j^2 > 1, \quad \mu_j > 0, \quad \lambda_j' + \mu_j' i = (\lambda_j - \mu_j i)^{-1}.$$

One or more of types (i) - (iv) may be absent in (3.8) *and* (3.9).

Moreover, the representation of an H-unitary matrix A as in (3.8), (3.9) *is unique up to a simultaneous permutation of constituent pairs.*

Proof. We prove the (nontrivial) "only if" part. By Lemma 3.4, we may assume that one of the following four cases (a) - (d) happens: (a) $\operatorname{Spec}(A) = 1$ or $\operatorname{Spec}(A) = -1$; (b) $\operatorname{Spec}(A) = \{\lambda, \lambda^{-1}\}$, $|\lambda| > 1$, λ real; (c) $\operatorname{Spec}(A) = \{\lambda \pm i\mu\}$, $\lambda^2 + \mu^2 = 1$, $\mu > 0$; (d) $\operatorname{Spec}(A) = \{\lambda \pm i\mu, (\overline{\lambda \pm i\mu})^{-1}\}$, $\lambda^2 + \mu^2 > 1$, $\mu > 0$. In the cases (a) and (b), one argues as in the proof of Theorem 3.3. Consider the case (c). Applying the transformation $A \mapsto S^{-1}AS$, $H \mapsto S^tHS$, where S is a real invertible matrix, we can assume that that A is in the real Jordan form, i.e., since A is diagonalizable,

$$A = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} \in M_{2m}(\mathbb{R}).$$

Partition H: $H = [H_{j,k}]_{j,k=1}^m$, where $H_{j,k}$ is 2×2. It turns out that (since A is H-unitary) $H_{j,k} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, where a, b are real numbers (which depend on j and k). Indeed, fix j and k, and let $H_{j,k} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in \mathbb{R}$. Equation

$$\begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

may be rewritten as a system of 4 homogeneous linear equations with unknowns a, b, c, d:

$$\begin{pmatrix} \lambda^2 - 1 & -\mu\lambda & -\mu\lambda & \mu^2 \\ \mu\lambda & \lambda^2 - 1 & -\mu^2 & -\mu\lambda \\ \mu\lambda & -\mu^2 & \lambda^2 - 1 & -\mu\lambda \\ \mu^2 & \mu\lambda & \mu\lambda & \lambda^2 - 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0. \tag{3.10}$$

It is easy to see, using $\lambda^2+\mu^2=1$, that the general solution of (3.10) is $\{(a,b,c,d)^t : a=d,\, b=-c\}$, and hence H has the required form. Now the proof of Theorem 3.5 in the case (c) reduces to the complex case via the identification (1.1).

Finally, consider the case (d). As in the proof of case (c), the proof of Theorem 3.5 in case (d) boils down to the following claim: Let λ and μ be real numbers such that $\lambda^2+\mu^2>1$ and $\mu>0$, let $\lambda'+i\mu'=(\lambda-i\mu)^{-1}$, and assume that

$$\begin{pmatrix} \lambda & -\mu & 0 & 0 \\ \mu & \lambda & 0 & 0 \\ 0 & 0 & \lambda' & -\mu' \\ 0 & 0 & \mu' & \lambda' \end{pmatrix} H \begin{pmatrix} \lambda & \mu & 0 & 0 \\ -\mu & \lambda & 0 & 0 \\ 0 & 0 & \lambda' & \mu' \\ 0 & 0 & -\mu' & \lambda' \end{pmatrix} = H, \tag{3.11}$$

where

$$H = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ e & f & 0 & 0 \\ g & h & 0 & 0 \end{pmatrix} \in M_4(\mathbb{R}); \tag{3.12}$$

then in fact

$$H = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & -b & a \\ e & f & 0 & 0 \\ -f & e & 0 & 0 \end{pmatrix}. \tag{3.13}$$

(It follows from Lemma 3.4 that the 2×2 top left and bottom right corners of H in (3.12) are zeros.) Rewrite (3.11) as a system of linear equations

$$\begin{pmatrix} \lambda\lambda'-1 & -\mu'\lambda & -\mu\lambda' & \mu\mu' \\ \mu'\lambda & \lambda\lambda'-1 & -\mu\mu' & -\mu\lambda' \\ \mu\lambda' & -\mu\mu' & \lambda\lambda'-1 & -\mu'\lambda \\ \mu\mu' & \mu'\lambda & \mu\lambda' & \lambda\lambda'-1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0, \tag{3.14}$$

and an analogous system for e, f, g, h. Since $\lambda'+i\mu'=(\lambda-i\mu)^{-1}$, we have $\mu\lambda'=\mu'\lambda$ and $\lambda\lambda'+\mu\mu'=1$, and our claim follows easily. ∎

4. Applications. In this section we present several applications and consequences of the canonical forms given in Sections 2 and 3.

4.1. Connected components of H-unitary similarity orbit. Since the group of H-unitary matrices is connected in the complex case, the H-unitary orbit

$$\mathcal{U}_{\mathbb{F}}^H(A) = \{U^{-1}AU : U \in \mathcal{U}_{\mathbb{F}}^H\}$$

is also (arcwise) connected. In the real case, this need not be true. Using Theorem 3.5, we sort out the number of connected components in the H-unitary orbit of an H-unitary diagonalizable matrix A, in the real case.

We assume from now on in this section that $\mathbb{F}=\mathbb{R}$. Let $\mathcal{U}_{\mathbb{R},0}^H$ be the connected component of $\mathcal{U}_{\mathbb{R}}^H$ containing the identity. Since by Corollaries 2.6 and 2.8 the factor group $\mathcal{U}_{\mathbb{R}}^H/\mathcal{U}_{\mathbb{R},0}^H$ is isomorphic to $\{1,-1\}\times\{1,-1\}$, the H-unitary orbit may have one, two, or four connected components. The proof of the following lemma is obvious.

LEMMA 4.1. *Let $A \in M_n(\mathbb{R})$. The orbit $\mathcal{U}_{\mathbb{R}}^H(A)$ has one, two, or four connected components if and only if the group*

$$\{U \in \mathcal{U}_{\mathbb{R}}^H \, : \, AU = UA\} \tag{4.1}$$

intersects all connected components of $\mathcal{U}_{\mathbb{R}}^H$, intersects only two connected components of $\mathcal{U}_{\mathbb{R}}^H$, or is contained in $\mathcal{U}_{\mathbb{R},0}^H$, respectively.

LEMMA 4.2. *The orbit $\mathcal{U}_{\mathbb{R}}^H(A)$ and the orbit $\mathcal{U}_{\mathbb{R}}^{S^tHS}(S^{-1}AS)$ have the same number of connected components, for every invertible $S \in M_n(\mathbb{R})$.*

Proof. Notice that $\mathcal{U}_{\mathbb{R}}^{S^tHS}(S^{-1}AS) = S^{-1}\left(\mathcal{U}_{\mathbb{R}}^H(A)\right)S$.

LEMMA 4.3. *If A is diagonalizable, H-unitary, and has no real eigenvalues, then $\mathcal{U}_{\mathbb{R}}^H(A)$ has four connected components.*

Proof. By Lemma 4.1 we have to prove that the group (4.1) is connected, and by Lemma 4.2 and Theorem 3.5 we may assume that A and H are given by

$$A = U_{q+1} \oplus \ldots \oplus U_{q+r} \oplus U_{q+r+1} \oplus \ldots \oplus U_{q+r+s} \in M_{2m}(\mathbb{R}),$$

$$H = \epsilon_{q+1} I_2 \oplus \ldots \oplus \epsilon_{q+r} I_2 \oplus \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix},$$

where the U_j's and ϵ_j's are as in Theorem 3.5. Consider the map

$$X \in M_m(\mathbb{C}) \;\mapsto\; \phi(X) \in M_{2m}(\mathbb{R}),$$

defined entrywise by $\phi(x+iy) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$, $x, y \in \mathbb{R}$. We obviously have $A = \phi(\widehat{A})$, $H = \phi(\widehat{H})$, where $\widehat{A}$ is $\widehat{H}$-unitary. Since every real matrix S commuting with A has the form $S = [S_{j,k}]$, with the 2×2 blocks $S_{j,k} = \begin{pmatrix} \alpha_{j,k} & \beta_{j,k} \\ -\beta_{j,k} & \alpha_{j,k} \end{pmatrix}$, $\alpha_{j,k}, \beta_{j,k} \in \mathbb{R}$ ([10, Theorem 12.4.2]), we have

$$\phi(\{\widehat{U} \in M_m(\mathbb{C}) \, : \, \widehat{A}\widehat{U} = \widehat{U}\widehat{A},\, \widehat{U}^*\widehat{H}\widehat{U} = \widehat{H}\}) = \{U \in M_{2m}(\mathbb{R}) \, : \, AU = UA,\; U^tHU = H\}.$$

Now $\widehat{A}$ is diagonalizable, and therefore the canonical form of Theorem 3.3 together with the connectedness of the H-unitary group in the complex case guarantee that the group

$$\{\widehat{U} \in M_m(\mathbb{C}) \, : \, \widehat{A}\widehat{U} = \widehat{U}\widehat{A}, \;\; \widehat{U}^*\widehat{H}\widehat{U} = \widehat{H}\}$$

is connected. Since ϕ is continuous, the group $\{U \in M_{2m}(\mathbb{R}) \, : \, AU = UA, \;\; U^tHU = H\}$ is connected as well. ∎

LEMMA 4.4. *If A is diagonalizable, H-unitary, and all its eigenvalues are real and different from ± 1, then $\mathcal{U}_{\mathbb{R}}^H(A)$ has two connected components:*

$$\{U^{-1}AU : U \in \mathcal{U}_{\mathbb{R}}^H,\; \det U = 1\} \quad \text{and} \quad \{U^{-1}AU : U \in \mathcal{U}_{\mathbb{R}}^H,\; \det U = -1\}.$$

Proof. We have to prove that the group (4.1) intersects exactly two components of the H-unitary group, and all elements of (4.1) have determinant one. In view of Lemma 4.2 and Theorem 3.5, we may assume that

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad m \text{ times}, \quad |\lambda| > 1, \quad \lambda \in \mathbb{R},$$

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad m \text{ times.}$$

It will be convenient to apply a simultaneous row and column permutation to represent A and H in the form

$$A = \begin{pmatrix} \lambda I_m & 0 \\ 0 & \lambda^{-1} I_m \end{pmatrix}, \quad H = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}.$$

Now a matrix A belongs to (4.1) if and only if $A = \begin{pmatrix} U & 0 \\ 0 & (U^t)^{-1} \end{pmatrix}$, where $U \in M_m(\mathbb{R})$ is invertible. Clearly, $\det(A) = 1$. To see that the group (4.1) intersects the component of the H-unitary group defined by $\sigma_+ = \sigma_- = -1$ (see (2.2)), we transform A and H again:

$$\frac{1}{2}\begin{pmatrix} I & I \\ I & -I \end{pmatrix} A \begin{pmatrix} I & I \\ I & -I \end{pmatrix} = \frac{1}{2}\begin{pmatrix} U + (U^t)^{-1} & U - (U^t)^{-1} \\ U - (U^t)^{-1} & U + (U^t)^{-1} \end{pmatrix},$$

$$\begin{pmatrix} I & I \\ I & -I \end{pmatrix} H \begin{pmatrix} I & I \\ I & -I \end{pmatrix} = \begin{pmatrix} 2I & 0 \\ 0 & -2I \end{pmatrix}.$$

Clearly, there exist invertible $U \in M_m(\mathbb{R})$ such that $\det(U + (U^t)^{-1}) < 0$. ■

The proof of the following theorem is obtained by using the preceding lemmas, and arguing analogously in the case when eigenvalues ± 1 are present.

THEOREM 4.5. *Let $A \in M_n(\mathbb{R})$ be H-unitary and diagonalizable, where $H \in M_n(\mathbb{R})$ is symmetric and invertible (recall the standing assumption that H is indefinite). Then the orbit $\mathcal{U}^H_{\mathbb{R}}(A)$ has four connected components if and only if A has no real eigenvalues, and has two connected components*

$$\{U^{-1}AU : U \in \mathcal{U}^H_{\mathbb{R}},\ \det U = 1\} \quad \textit{and} \quad \{U^{-1}AU : U \in \mathcal{U}^H_{\mathbb{R}},\ \det U = -1\}.$$

if and only if A has real eigenvalues but all of them are different from ± 1.

Assume now that 1 or -1 (or both) belongs to $\operatorname{Spec}(A)$. *If the quadratic form $x^t H x$, $x \in \operatorname{Ker}(A^2 - I)$, is indefinite, then $\mathcal{U}^H_{\mathbb{R}}(A)$ is connected. If the quadratic form $x^t H x$, $x \in \operatorname{Ker}(A^2 - I)$, is (positive or negative) definite, then in fact*

$$\mathcal{U}^H_{\mathbb{R}}(A) = \{U^{-1}AU : U \in \mathcal{U}^H_{\mathbb{R}},\ \det U = 1\},$$

and the orbit $\mathcal{U}^H_{\mathbb{R}}(A)$ is connected if A has real eigenvalues different from ± 1, and has two connected components otherwise.

4.2. Products of positive definite J-unitary matrices. Let $J = I_p \oplus -I_q$. Consider the problem of characterizing those J-unitary matrices that can be written as the product

$$\begin{pmatrix} \sqrt{I_p + XX^*} & X \\ X^* & \sqrt{I_q + X^*X} \end{pmatrix} \begin{pmatrix} \sqrt{I_p + YY^*} & N \\ Y^* & \sqrt{I_q + Y^*Y} \end{pmatrix}, \quad X, Y \in M_{p\times q}(\mathbb{F}). \tag{4.2}$$

A related question has been considered by van Wyk [22] for the case $H = [-1] \oplus I_3$.

THEOREM 4.6. $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{F} = \mathbb{C}$. *The following statements are equivalent for a J-unitary matrix A:*

(a) *A is J-unitarily similar to a matrix of the form (4.2), i.e., $A = U^{-1}BU$ for some J-unitary U and some B of the form (4.2).*

(b) *A is J-unitarily similar to a matrix of the form*

$$\begin{pmatrix} \sqrt{I_p + CC^t} & C \\ C^t & \sqrt{I_q + C^tC} \end{pmatrix},$$

where $C = [c_{ij}]$ with $c_{11} \geq \cdots \geq c_{ss} > 0$ for some $s \leq \min\{p, q\}$ and all other entries of C are zero.

(c) *The eigenvalues of A are positive and semisimple, i.e., no Jordan blocks of size bigger than 1 in the Jordan form of A.*

(d) *A is of the form (4.2).*

Proof. (b) $\Rightarrow$ (a) is obvious, whereas (a) $\Rightarrow$ (c) follows because (4.2) is a product of two positive definite matrices, and every product of two positive definite matrices has positive and semisimple eigenvalues. Assume (c) holds. By Theorems 3.3 and 3.5, we have

$$S^{-1}AS = I_r \oplus U_{m+1} \oplus \ldots \oplus U_{m+s}, \quad S^*JS = I_{r_+} \oplus -I_{r_-} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

for some invertible $S \in M_n(\mathbb{F})$, where $U_j = \begin{pmatrix} \lambda_j & 0 \\ 0 & \lambda_j^{-1} \end{pmatrix}$, $\lambda_j \in \mathbb{R}$, $|\lambda_j| > 1$. Applying a suitable matrix transformation T, we obtain $T^*(S^*JS)T = J$ and

$$T^{-1}(S^{-1}AS)T = \begin{pmatrix} D_1 & C \\ C^t & D_2 \end{pmatrix}, \tag{4.3}$$

where

$$D_1 = \operatorname{diag}\left(\frac{1}{2}(\lambda_1 + \lambda_1^{-1}), \ldots, \frac{1}{2}(\lambda_s + \lambda_s^{-1}), \overbrace{1, \ldots, 1}^{r_+ \text{ times}}\right),$$

$$D_2 = \operatorname{diag}\left(\frac{1}{2}(\lambda_1 + \lambda_1^{-1}), \ldots, \frac{1}{2}(\lambda_s + \lambda_s^{-1}), \overbrace{1, \ldots, 1}^{r_- \text{ times}}\right),$$

and $C = [c_{ij}]$ with $c_{jj} = \frac{1}{2}(\lambda_j - \lambda_j^{-1})$ for $j = 1, \ldots, s$, and all other c_{ij} equal to zero. Thus, the right hand side of (4.3) has the form as required in (b).

Finally, suppose (a) holds true. Let $A = U^{-1}CDU$, where C and D are the two matrices as in (4.2), and U is J-unitary. Then

$$A = U^{-1}C(U^{-1})^*U^*DU.$$

Now both $C_1 = U^{-1}C(U^{-1})^*$ and $D_1 = U^*DU$ are J-unitary and positive definite. By the uniqueness of polar decomposition of invertible matrices, and applying Theorem 2.4(a) with A replaced with C_1 and with D_1, we see that $A = C_1D_1$ is of the form (4.2). It is obvious that (d) implies (a). ∎

One can check that for a given J-unitary matrix A with positive and semisimple eigenvalues, a representation in the form (4.2) is not unique, as observed in [22].

4.3. Products of reflections. In $M_n(\mathrm{IR})$, a matrix of the form $T_v = I - 2vv^t$ is called a reflection, and $T_v(x) = x - 2(v^tx)v$ is just a reflection of the vector x about the plane $v^\perp$. We can extend the definition using the indefinite inner product $[x, y] = y^tHx$, and define

$$I - 2vv^tH/(v^tHv), \quad \text{where} \quad v^tHv \neq 0. \tag{4.4}$$

One readily checks that T_v is an H-unitary matrix such that $T_v(x) = x-2[x, v]v/[v, v]$. Assuming that $H = J = I_p \oplus -I_q$, if v is in the linear span of two basic vectors e_i and e_j, then we say that T_v is an *elementary reflection.* Thus, an elementary reflection is a direct sum of a two by two matrix and I_{n-2} (or a diagonal matrix in the degenerate case) with determinant -1. We have the following result.

THEOREM 4.7. *A matrix $A \in M_n(\mathrm{IR})$ is J-unitary if and only if it is a product of at most $f(p, q) = p(p-1) + q(q-1) + \min\{p, q\} + 4$ so many elementary reflections.*

Proof. The ($\Leftarrow$) is clear. Conversely, suppose A is J-unitary. By Theorem 2.3, there exist $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ with $X_1, Y_1 \in \mathcal{U}_p$ and $X_2, Y_2 \in \mathcal{U}_q$ such that

$$XAY = \begin{pmatrix} -\sqrt{I_p + DD^t} & D \\ -D^t & \sqrt{I_q + D^tD} \end{pmatrix}, \tag{4.5}$$

with D as in Theorem 2.3. First, we show that X_1 is a product of no more than $p(p-1)/2+1$ elementary reflections. This can be proved by simple inductive arguments as follows. Suppose $X_1 = [u_1|\cdots|u_p]$. By elementary considerations or by [13, p. 226], there are elementary reflections $T_1, \ldots, T_{p-1}$ such that $T_1 \cdots T_{p-1}u_1 = e_1$. Hence $T_1 \cdots T_{p-1}X_1 = [1] \oplus \tilde{X}_1$. Repeat the arguments to $\tilde{X}_1$ and so on until we get a two by two matrix, which is either an elementary reflection or the product of two elementary reflections. Thus, we can write X_1 as a product of no more than

$$[(p-1) + (p-2) + \cdots + 1] + 1 = p(p-1)/2 + 1$$

so many elementary reflections. We can apply similar arguments to X_2, Y_1, Y_2, and conclude each of the matrices X and Y can be written as the product of at most $p(p-1)/2+q(q-1)/2+2$ elementary reflections. Finally, we deal with XAY. Denote by $E_{ij} = e_ie_j^t \in M_n(\mathrm{IR})$ for $i, j \in \{1, \ldots, n\}$. Evidently, $XAY = B_1 \cdots B_m$, where

$$B_j = \sqrt{1 + d_j^2}(-E_{jj} + E_{p+j,p+j}) + d_j(E_{j,p+j} - E_{p+j,j}) + \sum_{k \neq i,j} E_{kk}, \ j = 1, \ldots, m.$$

To show that B_j is a matrix of the form (4.4), we only need to deal with the 2×2 matrix

$$\begin{pmatrix} -\sqrt{1 + d^2} & d \\ -d & \sqrt{1 + d^2} \end{pmatrix} \quad \text{with } d \geq 0,$$

here $J = [1] \oplus [-1]$. To this end, let $f(\theta) = 2\sec\theta\tan\theta$ with $\theta \in [0, \pi/2)$. Then f maps $[0, \pi/2)$ to $[0, \infty)$. So, there exists $\theta \in [0, \pi/2)$ such that $f(\theta) = d$. Let $v = (\sec\theta, \tan\theta)^t$. Then $v^tJv = 1$, and

$$R = I_2 - 2vv^tJ = \begin{pmatrix} -1 - 2\tan^2\theta & 2\sec\theta\tan\theta \\ -2\sec\theta\tan\theta & 1 + 2\tan^2\theta \end{pmatrix} = \begin{pmatrix} -\sqrt{1 + d^2} & d \\ -d & \sqrt{1 + d^2} \end{pmatrix};$$

here we have used $1 + 2\tan^2\theta = \sqrt{1+d^2} = \sqrt{1+(2\sec\theta\tan\theta)^2}$. The result follows. ■

If one uses (general) reflections instead of elementary reflections, then the number of reflections needed to represent every J-unitary matrix as a product of reflections can be considerably improved comparing with Theorem 4.7: Every J-unitary matrix, in the real as well as in the complex case, can be written as a product of no more than n reflections, a result that goes back to [5].

4.4. Stability and robust stability of J-unitary matrices. In the complex case the results of this section are given in [9] (see also references there).

In applications, one often needs conditions for powers of a matrix to be bounded. We say that a matrix $A \in M_n(\mathbb{F})$ is *forward stable* if the set $\{A^m\}_{m=0}^{\infty}$ is bounded, and is *backward stable* if A is invertible and the set $\{A^m\}_{m=-\infty}^{0}$ is bounded.

Theorem 4.8. *$\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$.*
The following statements are equivalent for an H-unitary matrix A:
(a) *A is forward stable;*
(b) *A is backward stable;*
(c) *A is diagonalizable and has only unimodular eigenvalues.*

Proof. It is well-known (and easy to see from the Jordan form of A) that (c) is equivalent to both forward and backward stability of A, whereas (a) (resp., (b)) is equivalent to A having all its eigenvalues inside the closed unit circle (resp., outside of the open unit circle), with unimodular eigenvalues, if any, being semisimple, i.e, their geometric multiplicity coincides with their algebraic multiplicity. (This remark applies to any $A \in M_n(\mathbb{F})$, not necessarily H-unitary.) It remains to observe that if $\lambda \in \sigma(A)$ then $\overline{\lambda}^{-1} \in \sigma(A)$, and so (a) and (b) are equivalent for H-unitary matrices. ■

In view of Theorem 4.8, we say that an H-unitary matrix is *stable* if it is backward or forward stable. For A H-unitary, we say that A is *robustly stable* if there is $\epsilon > 0$ such that every G-unitary matrix B is stable, provided G is Hermitian and

$$\|G - H\| + \|B - A\| < \epsilon.$$

Here $\|\cdot\|$ is any fixed norm in $M_n(\mathbb{F})$. Note that by taking ϵ sufficiently small, the invertibility of G is guaranteed.

Theorem 4.9. $\mathbb{F} = \mathbb{C}$. *An H-unitary matrix A is robustly stable if and only if A is diagonalizable with only unimodular eigenvalues and every eigenvector is H-definite:*

$$Ax = \lambda x, \quad x \neq 0 \quad \Longrightarrow \quad x^*Hx \neq 0. \tag{4.6}$$

Proof. By Theorem 4.8, we can assume to start with that A is diagonalizable with only unimodular eigenvalues. By Theorem 3.3 we may further assume that

$$A = U_1 \oplus \ldots \oplus U_m, \quad H = \epsilon_1 \oplus \ldots \oplus \epsilon_m, \tag{4.7}$$

where U_j and ϵ_j are as in (3.3).

Assume first that (4.6) does not hold. Then there exist indices $j \neq k$ such that $U_j = U_k = \lambda$ and $\epsilon_j \neq \epsilon_k$. For notational convenience assume $j = 1$, $k = 2$, and

$\epsilon_1 = 1$. Let q be any complex number different from $-\lambda$. Then a straightforward computation shows that

$$(4.8)\quad A(q) := \begin{pmatrix} \frac{1}{2}(\lambda+q)+\frac{1}{2}(\overline{\lambda+q})^{-1} & \frac{1}{2}(\lambda+q)-\frac{1}{2}(\overline{\lambda+q})^{-1} \\ \frac{1}{2}(\lambda+q)-\frac{1}{2}(\overline{\lambda+q})^{-1} & \frac{1}{2}(\lambda+q)+\frac{1}{2}(\overline{\lambda+q})^{-1} \end{pmatrix} \oplus U_3 \ldots \oplus U_m$$

is H-unitary, as close to A as we wish (if q is sufficiently close to zero), and has non-unimodular eigenvalues $\lambda+q$, $\overline{\lambda+q}^{-1}$ (if q is chosen so that $|\lambda+q| \neq 1$). For such a choice of q, the matrix $A(q)$ cannot be stable. This proves the "only if" part.

"If" part. Assume that (4.6) holds true. Let $\lambda_1, \ldots, \lambda_k$ be all the distinct eigenvalues of A, and let $\delta > 0$ be so small that each disk $\{z \in \mathbb{C} : |z-\lambda_j| \leq \delta\}$ does not contain any eigenvalues of A besides λ_j.

To continue the proof, we need the well–known notion of the gap between subspaces. If $\mathcal{M}, \mathcal{N}$ are subspaces in $\mathbb{C}^n$, the *gap* $\operatorname{gap}(\mathcal{M}, \mathcal{N})$ is defined as $\|P_{\mathcal{M}} - P_{\mathcal{N}}\|_{\mathrm{op}}$, where $P_{\mathcal{M}}$ (resp., $P_{\mathcal{N}}$) is the orthogonal projection onto $\mathcal{M}$ (resp., $\mathcal{N}$), and $\|\cdot\|_{\mathrm{op}}$ is the operator norm (i.e. the largest singular value). We refer the reader to [10] for many basic properties of the gap. Returning to our proof, we need the following property (see [10, Section 15.2]):

$$\forall\, \epsilon_2 > 0\, \exists\, \epsilon_1 > 0 \text{ such that } \|B-A\| < \epsilon_1 \implies \max_{j=1}^{k}\left(\operatorname{gap}(\mathcal{R}_{\Omega_j}(B), \mathcal{R}_{\lambda_j}(A))\right) < \epsilon_2. \quad (4.9)$$

Here $\mathcal{R}_{\Omega_j}(B)$ is the sum of all root subspaces of B corresponding to the eigenvalues of B in the disk $\Omega_j := \{z \in \mathbb{C} : |z-\lambda_j| \leq \delta\}$. Taking $\epsilon_2 < 1$ we guarantee that for every j, the dimensions of $\mathcal{R}_{\Omega_j}(B)$ and of $\mathcal{R}_{\lambda_j}(A)$ coincide, and in particular, B cannot have eigenvalues outside of $\cup_{j=1}^{k}\Omega_j$.

On the other hand, since H is definite on each $\mathcal{R}_{\lambda_j}(A)$, and since the property of being definite is preserved under sufficiently small perturbations of H and sufficiently small perturbations of $\mathcal{R}_{\lambda_j}(A)$ (with respect to the gap), there exists $\epsilon_3 > 0$ (which depend on H and A only) such that a Hermitian matrix G is invertible and definite on each $\mathcal{R}_{\Omega_j}(B)$ $(j = 1, \ldots, k)$ as long as $\|G-H\| < \epsilon_3$ and $\operatorname{gap}(\mathcal{R}_{\Omega_j}(B), \mathcal{R}_{\lambda_j}(A)) < \epsilon_3$. Take $\epsilon_2 = \min\{1, \epsilon_3\}$ in (4.9); as a result, letting $\epsilon = \min\{\epsilon_3, \epsilon_1\}$, we obtain that G is definite on each $\mathcal{R}_{\Omega_j}(B)$ provided that

$$\|G-H\| + \|B-A\| < \epsilon.$$

If B is, in addition, G-unitary, then by Lemma 3.2 (b), B is diagonalizable with only unimodular eigenvalues, i.e., B is stable. ∎

We consider now robustly stable H-unitary matrices in the real case.

THEOREM 4.10. $\mathbb{F} = \mathbb{R}$. *An H-unitary matrix A is robustly stable if and only if A is diagonalizable with only unimodular eigenvalues and the following conditions hold: For eigenvalues ± 1 of A (if any):*

$$(4.10)\qquad Ax = \pm x, \quad x \in \mathbb{R}^n \setminus \{0\} \implies x^t H x \neq 0;$$

for every pair of complex conjugate eigenvalues $\mu \pm i\nu$, $\mu^2 + \nu^2 = 1$, of A (if any):

$$(4.11)\qquad (A^2 - 2\mu A + I)x = 0, \quad x \in \mathbb{R}^n \setminus \{0\} \implies x^t H x \neq 0.$$

Proof. The "if" follows part follows from the complex result (Theorem 4.9). For the "only if" part, we will prove that if A is diagonalizable with only unimodular eigenvalues and at least one of the conditions (4.10) and (4.11) does not hold, then there exists a real H-unitary matrix B as close as we wish to A which is not stable. We may assume, using the canonical form Theorem 3.5, that either $A = \pm I_2$, $H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or

$$A = \begin{pmatrix} \mu & \nu & 0 & 0 \\ -\nu & \mu & 0 & 0 \\ 0 & 0 & \mu & \nu \\ 0 & 0 & -\nu & \mu \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \mu^2 + \nu^2 = 1, \quad \nu > 0.$$

In the former case, use

$$B = \begin{pmatrix} \frac{1}{2}(\pm 1 + q) + \frac{1}{2}(\pm 1 + q)^{-1} & \frac{1}{2}(\pm 1 + q) - \frac{1}{2}(\pm 1 + q)^{-1} \\ \frac{1}{2}(\pm 1 + q) - \frac{1}{2}(\pm 1 + q)^{-1} & \frac{1}{2}(\pm 1 + q) + \frac{1}{2}(\pm 1 + q)^{-1} \end{pmatrix}, \qquad q \in \mathbb{R} \text{ close to zero.}$$

In the latter case, use (4.8) with $\lambda = \mu + i\nu$, and take advantage of the identification (1.1).

■

4.5. Stability and robust stability of differential equations. The results of the preceding section have immediate applications to systems of differential equations. Consider the system

$$E\frac{dx}{dt} = iH(t)x, \quad t \in \mathbb{R}, \tag{4.12}$$

where $H(t)$ is a given piecewise continuous function that takes values in the set of $n \times n$ Hermitian matrices, E is a fixed (constant) invertible $n \times n$ Hermitian matrix, and $x(t)$ is a $\mathbb{C}^n$-valued function of t to be found. We assume in addition that $H(t)$ is periodic with a period $\omega \neq 0$: $H(t + \omega) = H(t)$ for all $t \in \mathbb{R}$.

The *matrizant* $X(t)$ of equation (4.12) is defined as the unique $n \times n$ matrix valued solution of the initial value problem

$$E\frac{dX}{dt} = iH(t)X, \quad X(0) = I. \tag{4.13}$$

If $X(t)$ is the matrizant, then differentiating the function X^*EX with respect to t, and using (4.13) and the property of E and H being Hermitian, we obtain

$$\frac{d}{dt}(X(t)^*EX(t)) = \frac{dX^*}{dt}EX + X^*E\frac{dX}{dt} = -iX^*HE^{-1}EX + X^*(iH)X = 0, \tag{4.14}$$

thus X^*EX is constant. Evaluating X^*EX at $t = 0$, we obtain $X(t)^*EX(t) = E$ for all $t \in \mathbb{R}$, in other words, the matrizant is E-unitary valued. Furthermore, since $H(t)$ is periodic with period ω, it is easy to see (because of the uniqueness of the solution of the initial value problem) that $X(t + \omega) = X(t)X(\omega)$, $t \in \mathbb{R}$, and by repeatedly applying this equality we obtain $X(t+m\omega) = X(t)(X(\omega))^m$, m any integer. Therefore, the equation (4.12) is *forward stable*, i.e., all solutions are bounded when $t \to +\infty$, precisely when the set $\{X(\omega)^m\}_{m=0}^{\infty}$ is bounded, and the equation (4.12) is *backward stable*, i.e., all solutions are bounded when $t \to -\infty$, precisely when the set $\{X(\omega)^m\}_{m=-\infty}^{0}$ is bounded. Recalling Theorem 4.8, we have:

THEOREM 4.11. *The following conditions are equivalent:*

(a) *Equation* (4.12) *is forward stable.*
(b) *Equation* (4.12) *is backward stable.*
(c) *The matrix* $X(\omega)$*, where* $X(t)$ *is the matrizant, is diagonalizable and has only unimodular eigenvalues.*

Thus, we say that (4.12) is *stable* if it is backward or forward stable. We say that (4.12) is *robustly stable* if there exists $\epsilon > 0$ (which depends on E and $H(t)$ only) such that every system

$$\widetilde{E}\frac{dx}{dt} = i\widetilde{H}(t)x, \quad t \in \mathbb{R},$$

is stable provided that the Hermitian valued ω-periodic piecewise continuous function $\widetilde{H}(t)$ and the constant Hermitian matrix $\widetilde{E}$ are such that

$$\|\widetilde{E} - E\| + \max\{\|\widetilde{H}(t) - H(t)\| \,:\, 0 \le t < \omega\} < \epsilon.$$

Using the continuous dependence of the solutions of (4.12) on the data E and $H(t)$ (see, e.g., [9, Section II.1.1] for details), Theorem 4.9 yields:

THEOREM 4.12. *Equation* (4.12) *is robustly stable if and only the matrix* $X(\omega)$ *is diagonalizable, has only unimodular eigenvalues, and every eigenvector is* E*-definite.*

Theorems 4.11 and 4.12 (with $\widetilde{E} = E$) are given in [9]. The book also contains more advanced material concerning stability of (4.12), as well as references to the original literature. In particular, connected components of robustly stable systems (4.12) are described in [9]; in the real skew symmetric case the study of connected components of robustly stable periodic systems goes back to [7].

There are complete analogues of Theorems 4.11 and 4.12 in the real case, in which case the system of differential equations is:

$$E\frac{dx}{dt} = H(t)x, \;\; t \in \mathbb{R}, \quad E = E^t \in M_n(\mathbb{R}) \;\text{ invertible}, \quad H(t)^t = -H(t) \in M_n(\mathbb{R}). \tag{4.15}$$

We assume in addition that $H(t)$ is periodic with period $\omega \neq 0$. The matrizant $X(t)$ is defined again as the solution of the initial value problem $E\dfrac{dX}{dt} = H(t)X(t)$, $X(0) = I$. As in (4.14) one obtains that $X(t)^t EX(t)$ is constant, hence $X(t)$ is E-unitary valued. The definitions of stability (forward or backward, which turn out to be the same) and of robust stability of (4.15) are analogous to those given above for the complex case, with only real perturbations allowed for the robust stability. We have from Theorems 4.8 and 4.10:

THEOREM 4.13. *Equation* (4.15) *is stable if and only if the real matrix* $X(\omega)$*, where* $X(t)$ *is the matrizant of* (4.15)*, is diagonalizable and has only unimodular eigenvalues. Equation* (4.15) *is robustly stable if and only if it is stable, and in addition every eigenvector of* $X(\omega)$ *corresponding to an eigenvalue* ± 1 (*if any*) *is* E*-definite, and* $x^t Ex \neq 0$ *for every vector* $x \in \mathbb{R}^n \setminus \{0\}$ *such that*

$$(X(\omega)^2 - 2\mu X(\omega) + I)x = 0, \qquad \lambda \pm i\mu \in \operatorname{Spec}(X(\omega)), \quad \lambda^2 + \mu^2 = 1.$$

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