THE SUM OF UNITARY SIMILARITY ORBITS CONTAINING ONLY SPECIAL OPERATORS

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Dedicated to Professor Shmuel Friedland.

Abstract

Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operator acting on a Hilbert space $\mathcal{H}$ (over the complex or real field). Characterization is given to $A_1, \ldots, A_k \in \mathcal{B}(\mathcal{H})$ such that for any unitary operators $U_1, \ldots, U_k$, $\sum_{j=1}^k U_j^* A_j U_j$ is always in a special class $\mathcal{S}$ of operators such as normal operators, self-adjoint operators, unitary operators. As corollaries, characterizations are given to $A \in \mathcal{B}(\mathcal{H})$ such that complex, real or nonnegative linear combinations of operators in its unitary orbit $\mathcal{U}(A) = \{U^*AU : U \text{ unitary}\}$ always lie in $\mathcal{S}$.

AMS Subject Classification 47B15, 15A48, 15A57.

Keywords Unitary orbit, normal, unitary, self-adjoint, symmetric, skew-symmetric operator.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators acting on a Hilbert space $\mathcal{H}$ over $\mathbf{C}$. If $\dim \mathcal{H} = n$, we identify $\mathcal{B}(\mathcal{H})$ with the algebra M_n of $n \times n$ complex matrices. Denote by

$$\mathcal{U}(A) = \{U^*AU : U^*U = UU^* = I\}$$

the unitary orbit of $A \in \mathcal{B}(\mathcal{H})$. Clearly, $\mathcal{U}(A)$ contains representations of the same operator under different orthonormal bases. So, it is useful in the study of the operator A. For instance, it is of interest to see whether A can be triangularized by an orthonormal basis; if it does, then a lot of information of A can be obtained from such a representation [12]. For instance, in the finite dimensional case, the triangular matrix is actually a diagonal (real diagonal) matrix if and only if $A \in M_n$ is normal (Hermitian); furthermore, $\mathcal{U}(A)$ is the equivalence class of A under the equivalence relation (Lie group action) of unitary similarities so that $\mathcal{U}(A)$ is a nice differentiable manifold and has nice geometrical properties; see [1] and its references.

In connection to many branches of pure and applied topics such as algebraic combinatorics, representation theory, quantum computing and quantum control, there is considerable interest in studying the properties of operators from the sum (or nonnegative linear combinations) of two or more unitary orbits; for example, see [3, 10] and their references. For $A_1, \ldots, A_k \in \mathcal{B}(\mathcal{H})$, let

$$\sum_{j=1}^k \mathcal{U}(A_j) = \left\{ \sum_{j=1}^k X_j : X_j \in \mathcal{U}(A_j),\ j = 1, \ldots, k \right\}.$$

Research of Li was partially supported by an NSF grant and the William and Mary Plumeri Award. He is an honorary professor of the University of Hong Kong.

In the finite dimensional case, researchers determined the ranks, determinants, eigenvalues and singular values of matrices in $\sum_{j=1}^k \mathcal{U}(A_j)$; see [8, 9, 11, 13] and their references. When $A_1, \ldots, A_k \in M_n$ are self-adjoint, researchers determined all n-tuples of real numbers that can be the eigenvalues of matrices in $\sum_{j=1}^k \mathcal{U}(A_j)$, and extended the result to compact self-adjoint operators $A_1, \ldots, A_k \in \mathcal{B}(\mathcal{H})$; see [2, 3, 4, 6, 7] and their references.

In this paper, we characterize $A_1, \ldots, A_k \in \mathcal{B}(\mathcal{H})$ such that $\sum_{j=1}^k \mathcal{U}(A_j)$ is a subset of a certain special class of operators such as the normal operators, self-adjoint operators, positive semidefinite operators, unitary operators, or scalar operators. A key step of our proofs is to characterize $A_1, \ldots, A_k$ such that $\sum_{j=1}^k \mathcal{U}(A_j)$ is a subset of normal operators. This is done in Section 2. We then characterize $A_1, \ldots, A_k \in \mathcal{B}(\mathcal{H})$ such that $\sum_{j=1}^k \mathcal{U}(A_j)$ contains only special operators in Sections 3 and 4. In Section 5, we characterize $A \in \mathcal{B}(\mathcal{H})$ such that complex, real or nonnegative linear combinations of operators in $\mathcal{U}(A)$ have special structure.

In our study, we also consider real Hilbert spaces $\mathcal{H}$. In such case, a self-adjoint operator $A \in \mathcal{B}(\mathcal{H})$ satisfying $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in \mathcal{H}$ is also called a *symmetric operator*; and a skew-adjoint operator $A \in \mathcal{B}(\mathcal{H})$ satisfying $\langle Ax, y\rangle = -\langle x, Ay\rangle$ is also called a *skew-symmetric operator*. Unitary operators and unitary orbits are also referred to as orthogonal operators and orthogonal orbits.

If $\mathcal{H}$ is complex and finite dimensional, then we can use the Schur triangularization theorem for M_n to give a different proof of the main result. However, for real or infinite dimensional cases, the Schur triangularization theorem does not hold. We will give a unified proof that covers all cases.

2. Normal Operators

In this section, we prove the main theorem of our paper. It is worth mentioning that our proofs rely on the basic fact that $A \in \mathcal{B}(\mathcal{H})$ is normal if and only if $\|Ax\| = \|A^*x\|$ for all $x \in \mathcal{H}$. Deeper results such as the spectral decomposition of normal operators are not used (and do not seem to be useful).

In our discussion, we say that $A \in \mathcal{B}(\mathcal{H})$ is *essentially self-adjoint* if there are $\alpha, \gamma \in \mathbf{C}$ with $|\gamma| = 1$ such that $\alpha I + \gamma A$ is self-adjoint. If $\mathbf{F} = \mathbf{R}$, we say that $A \in \mathcal{B}(\mathcal{H})$ is *essentially skew-symmetric* if there is $\alpha, \gamma \in \mathbf{R}$ with $|\gamma| = 1$ such that $\alpha I + \gamma A$ is skew-symmetric.

Theorem 2.1. *Let $A_1, \ldots, A_k \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space over $\mathbf{F} = \mathbf{C}$ or $\mathbf{R}$. Then every operator in $\sum_{j=1}^k \mathcal{U}(A_j)$ is normal if and only if one of the following holds.*

(1) *One of the operators $A_1, \ldots, A_k$ is normal, and the rest are scalar operators.*
(2) *There are $\alpha_1, \ldots, \alpha_k$, $\gamma \in \mathbf{F}$ with $|\gamma| = 1$, and self-adjoint operators $H_1, \ldots, H_k \in \mathcal{B}(\mathcal{H})$ such that $A_j = \alpha_j I + \gamma H_j$ for $j = 1, \ldots, k$.*
(3) *$\mathbf{F} = \mathbf{R}$, there are skew-symmetric operators $G_1, \ldots, G_k \in \mathcal{B}(\mathcal{H})$ and $\alpha_1, \ldots, \alpha_k \in \mathbf{R}$ such that $A_j = \alpha_j I + G_j$ for $j = 1, \ldots, k$.*

Proof. The implication ($\Leftarrow$) is clear. We consider the converse.

First we consider the case when $k = 2$. For notational convenience, let $(A_1, A_2) = (A, B)$. Suppose $U^*AU + V^*BV$ is normal for any unitary $U, V \in \mathcal{B}(\mathcal{H})$. Assume that condition (1) of Theorem 1 does not hold. Then neither A nor B can be a scalar operator. We show that (2) or (3) must hold.

Suppose $\dim \mathcal{H} = 2$. In the complex case, there are unitary $U, V \in M_2$ such that

$$U^*AU = \alpha I + \gamma \begin{pmatrix} 0 & a_1 \\ a_2 & 0 \end{pmatrix} \quad \text{and} \quad V^*BV = \beta I + \gamma \begin{pmatrix} 0 & b_1 \\ b_2 & 0 \end{pmatrix}$$

such that $|\gamma| = 1$, $a_1 \geq a_2 \geq 0$ and $b_1 \geq |b_2| \geq 0$; see [5, Theorem 1.3.4]. Since $U^*AU + V^*BV$ is normal, we see that

$$a_1 + b_1 = |a_2 + b_2| \leq a_2 + |b_2| \leq a_1 + b_1.$$

It follows that $(a_1, b_1) = (a_2, b_2)$, and condition (2) holds.

In the real case, let $U, V \in M_2$ be orthogonal such that $U^*(A^* + A)U$ have equal diagonal entries, and $V^*(B^* + B)V$ have equal diagonal entries. Then

$$U^*AU = \alpha I + \begin{pmatrix} 0 & a_1 \\ a_2 & 0 \end{pmatrix} \quad \text{and} \quad B = \beta I + \begin{pmatrix} 0 & b_1 \\ b_2 & 0 \end{pmatrix}$$

for some $\alpha, \beta \in \mathbf{R}$. We may assume that $a_1 \geq |a_2|$ and $b_1 \geq |b_2|$. Otherwise, adjust U (respectively, V) by switching its rows, or multiplying its first column by -1. Since $U^*AU + V^*BV$ is normal, we see that

$$a_1 + b_1 = |a_2 + b_2| \leq |a_2| + |b_2| \leq a_1 + b_1.$$

Again, we see that $(a_1, b_1) = \pm(a_2, b_2)$ Thus condition (2) or (3) holds.

Suppose $\dim \mathcal{H} > 2$. Since A is not a scalar operator, there is a unit vector $u \in \mathcal{H}$ such that Au is not a multiple of u. Suppose $Au = a_{11}u + a_{21}\hat{u}$ for a unit vector $\hat{u} \in u^\perp$. By a suitable choice of orthonormal basis and identifying A with its operator matrix, we may assume that

$$A = \begin{pmatrix} a_{11} & f \\ x & A_{22} \end{pmatrix}$$

such that $f = (a_{12}, a_{13}, 0, 0, \dots)$ and $x = (a_{21}, 0, 0, \dots)^t$. In the complex case, we may replace (A, B) by $(\gamma A, \gamma B)$ for a suitable complex unit γ and assume that $a_{12}a_{21} \geq 0$. Replace A by U^*AU with $U = [\mu] \oplus I$ such that $a_{21}\mu = |a_{21}| > 0$. So, we may assume that

1) $a_{21} > 0$ and $a_{12} \geq 0$ if $\mathbf{F} = \mathbf{C}$, and 2) $a_{21} > 0$ if $\mathbf{F} = \mathbf{R}$.

For a unitary $U \in \mathcal{B}(\mathcal{H})$, let

$$U^*BU = \begin{pmatrix} b_{11} & g \\ y & B_{22} \end{pmatrix}. \tag{2.1}$$

We **claim** that one of the following holds.

a) $\mathbf{F} = \mathbf{C}$ and for all unitary $U \in \mathcal{B}(\mathcal{H})$, $g = y^*$ in (2.1).

b) $\mathbf{F} = \mathbf{R}$, $a_{12} > 0$ and for all unitary $U \in \mathcal{B}(\mathcal{H})$, $g = y^*$ in (2.1).

c) $\mathbf{F} = \mathbf{R}$, $a_{12} < 0$, and for all unitary $U \in \mathcal{B}(\mathcal{H})$, $g = -y^*$ in (2.1).

Once the claim is established, we can show that (2) or (3) in Theorem 2.1 follows from a), b) or c) as follows.

Suppose a) or b) holds. Then for every unitary operator $U \in \mathcal{B}(\mathcal{H})$, $U^*BU - U^*B^*U$ has the form $\begin{pmatrix} c & 0 \\ 0 & C_{22} \end{pmatrix}$. It follows that $(B - B^*)u$ is a multiple of u for any unit vector $u \in \mathcal{H}$, and hence $C = B - B^*$ is a scalar operator. So, if a) holds, then $B = ibI + K$ for a self-adjoint K. Now, interchanging the roles of A and B, we see that $A = iaI + H$ for a self-adjoint H. The result for the complex case follows.

If b) holds, then B is a symmetric operator. Now, interchanging the roles of A and B, we see that A is a symmetric operator.

Suppose c) holds. Then for every unitary operator $U \in \mathcal{B}(\mathcal{H})$, $U^*BU + U^*B^*U$ has the form $\begin{pmatrix} c & 0 \\ 0 & C_{22} \end{pmatrix}$. It follows that $(B + B^*)u$ is a multiple of u for any unit vector $u \in \mathcal{H}$, and hence $C = B + B^*$ is a scalar operator. Hence, $B = bI + K$ for a skew-symmetric operator K. Now, interchanging the roles of A and B, we see that $A = aI + H$ for a skew-symmetric operator H.

Hence, (2) or (3) holds once the claim is established. Now, we turn to the proof of conditions a), b) or c).

For a unitary $U \in \mathcal{B}(\mathcal{H})$, let $U^*BU = \begin{pmatrix} b_{11} & g \\ y & B_{22} \end{pmatrix}$. If $y = 0$, then let u be the first column of U. For every unitary operator V on $u^\perp$, since $A + (1 \oplus V)^* U^*BU (1 \oplus V)$ is normal, we have $\|x\| = \|f^* + V^*g^*\|$. Therefore, $g = 0$ and a), b) or c) holds.

If $y \neq 0$, we may assume that $g = (b_{12}, b_{13}, 0, 0, \dots)$ and $y = (b_{21}, 0, 0, \dots)^t$, where $b_{21} > 0$. Let $V = [1] \oplus W^* \oplus I \in \mathcal{B}(\mathcal{H})$ be unitary, where $W \in M_2$ is unitary. Since $A + V^*U^*BUV$ is always normal,

$$\|(a_{21}, 0)^* + W(b_{21}, 0)^*\| = \|(a_{12}, a_{13})^* + W(b_{12}, b_{13})^*\|.$$

Hence,

$$\begin{aligned} & |a_{12}|^2 + |a_{13}|^2 + |b_{12}|^2 + |b_{13}|^2 - |a_{21}|^2 - |b_{21}|^2 \\ = \;& 2\mathrm{Re}[(a_{21}, 0)W(b_{21}, 0)^* - (a_{12}, a_{13})W(b_{12}, b_{13})^*] \\ = \;& 2\mathrm{Re}\,\mathrm{tr}\, W[(b_{21}, 0)^*(a_{21}, 0) - (b_{12}, b_{13})^*(a_{12}, a_{13})]. \end{aligned}$$

Since this is true for all unitary $W \in M_2$, we see that

$$(b_{21}, 0)^*(a_{21}, 0) = (b_{12}, b_{13})^*(a_{12}, a_{13})$$

and

$$|a_{12}|^2 + |a_{13}|^2 + |b_{12}|^2 + |b_{13}|^2 = |a_{21}|^2 + |b_{21}|^2.$$

From the first inequality, we have

$$\bar{b}_{12}a_{12} = b_{21}a_{21} \quad \text{and} \quad a_{13} = b_{13} = 0.$$

We are going to show that

$$(a_{21} - |a_{12}|)(b_{21} - |b_{12}|) = 0\,. \tag{2.2}$$

In the complex case, $a_{12}a_{21} \geq 0$ and $b_{21} > 0$, we see that $b_{12} > 0$ and $a_{12} > 0$. This would lead to $(a_{21}, b_{21}) = (a_{12}, b_{12})$ if $\mathbf{F} = \mathbf{C}$. Hence, a) holds.

In the real case, we see that $a_{12}b_{12} = b_{21}a_{21} > 0$, and hence $(a_{21}, b_{21}) = \pm(a_{12}, b_{12})$. Thus, b) or c) holds.

To prove (2.2), suppose $(a_{21} - |a_{12}|)(b_{21} - |b_{12}|) > 0$. If $a_{12} > |a_{21}|$ and $b_{12} > |b_{21}|$, then since $A + B$ is normal,

$$|a_{12} + b_{12}| = a_{21} + b_{21} > |a_{12}| + |b_{12}| \geq |a_{12} + b_{12}|,$$

a contradiction.

Similarly, we can get a contradiction if $|a_{12}| < a_{21}$ and $|b_{12}| < b_{21}$.

Suppose $(a_{21} - |a_{12}|)(b_{21} - |b_{12}|) < 0$. Let $V = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} \oplus I \in \mathcal{B}(\mathcal{H})$, where $b_{12}\varepsilon = |b_{12}|$. Since $A + V^*BV$ is normal,

$$|a_{12}| + b_{21} = a_{21} + |b_{12}|,$$

which contradicts the assumption that $(a_{21} - |a_{12}|)(b_{21} - |b_{12}|) < 0$.

By the above arguments, the proof for the case $k = 2$ is complete.

Let $k > 2$, and $A_1, \ldots, A_k \in \mathcal{B}(\mathcal{H})$. If there is at most one non-scalar operators among $A_1, \ldots, A_k$, then condition (1) holds. Suppose there are at least two non-scalar operators, say, A_1 and A_2. We can apply the result for $k = 2$ to (A_1, A_2) to conclude that A_1, A_2 satisfy condition (2) or (3). If there is $j > 3$ such that A_j is not a scalar, then we can apply the result for $k = 2$ to $(A_1, A_2), (A_1, A_j)$ and (A_2, A_j) and conclude that A_1, A_2, A_j satisfy condition (2) or (3). Since this is true for any non-scalar operator A_j for $j > 3$, we see that $A_1, \ldots, A_k$ satisfy (2) or (3). □

The following corollaries are immediate.

Corollary 2.2. *Let $A_1, \ldots, A_k \in \mathcal{B}(\mathcal{H})$. Then every operator in $\sum_{j=1}^{k} \mathcal{U}(A_j)$ is normal if and only if every nonnegative (or real) linear combination of operators in $\mathcal{U}(A_1) \cup \cdots \cup \mathcal{U}(A_k)$ is normal.*

Corollary 2.3. *Let $\mathcal{H}$ be a complex Hilbert space, and let $A_1, \ldots, A_k \in \mathcal{B}(\mathcal{H})$. The following conditions are equivalent.*

(a) *Any complex linear combination of operators in $\mathcal{U}(A_1) \cup \cdots \cup \mathcal{U}(A_k)$ is normal.*
(b) *Any complex linear combination of operators in $\mathcal{U}(A_1) \cup \cdots \cup \mathcal{U}(A_k)$ is a scalar operator.*
(c) *Each A_j is scalar operator.*

Proof. The implications (c) ⇒ (b) ⇒ (a) are clear. Suppose (a) holds. Assume one of the operator A_j is non-scalar. Then there are unitary U, V such that $U^*A_jU + V^*(iA_j)V$ is not normal by Theorem 2.1. Thus, we have (a) ⇒ (c). □

3. Self-Adjoint and Skew-Self-Adjoint Operators

Using the result in Section 2, we have the following.

Proposition 3.1. *Let $A_1, \ldots, A_k \in \mathcal{B}(\mathcal{H})$. The following conditions are equivalent.*

(a) *Every operator in* $\sum_{j=1}^k \mathcal{U}(A_j)$ *is essentially self-adjoint.*

(b) *All real linear combinations of operators in* $\mathcal{U}(A_1)\cdots\mathcal{U}(A_k)$ *are essentially self-adjoint operators.*

(c) *Either*

(c.1) *one of the operators* $A_1,\dots,A_k$ *is essentially self-adjoint, and the rest are scalar operators, or*

(c.2) *there exist self-adjoint operators* $H_1,\dots,H_k \in \mathcal{B}(\mathcal{H})$ *and* $\alpha_1,\dots,\alpha_k,\gamma \in \mathbf{F}$ *with* $|\gamma| = 1$ *such that* $A_j = \alpha_j I + \gamma H_j$ *for* $j = 1,\dots,k$.

Proof. The implications (c) $\Rightarrow$ (b) $\Rightarrow$ (a) are clear.

Suppose (a) holds. That is, $\sum_{j=1}^k \mathcal{U}(A_j)$ is a subset of self-adjoint operators. Then one of the conditions (1) – (3) of Theorem 1 holds. Clearly, condition (3) should be ruled out. If (1) holds, we get condition (c.1); if (2) holds, we get condition (c.2). □

Proposition 3.2. *Let* $A_1,\dots,A_k \in \mathcal{B}(\mathcal{H})$. *The following conditions are equivalent.*

(a) *Every operator in* $\sum_{j=1}^k \mathcal{U}(A_j)$ *is positive semidefinite.*

(b) *Either*

(b.1) *one of the operators* $A_1,\dots,A_k$ *is essentially self-adjoint of the form* $\alpha I + H$ *with* $H = H^*$, *and the rest are scalar operators summing up to* βI *such that* $\alpha + \beta + \inf\sigma(H)$ *is a nonnegative real number, or*

(b.2) *There are* $\alpha_1,\dots,\alpha_k \in \mathbf{F}$ *and* $\gamma \in \mathbf{R}$ *with* $|\gamma| = 1$, *and self-adjoint operators* $H_1,\dots,H_k \in \mathcal{B}(\mathcal{H})$ *such that* $A_j = \alpha_j I + \gamma H_j$ *for* $j = 1,\dots,k$, *and* $\sum_{j=1}^k (\alpha_j + \inf\sigma(\gamma H_j))$ *is a nonnegative real number.*

Proof. Let $W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H},\ \|x\| = 1\}$ be the numerical range of $A \in \mathcal{B}(\mathcal{H})$. It is well known and not hard to show that $W(\alpha I + \beta A) = \alpha + \beta W(A)$, and for a self-adjoint operator $H \in \mathcal{B}(\mathcal{H})$ the closure of $W(H) = [m, M]$, where $m = \inf\sigma(H)$ and $M = \sup\sigma(A)$. Thus,

$$\left\{\langle Bx, x\rangle : x \in \mathcal{H},\ \|x\| = 1,\ B \in \sum_{j=1}^k \mathcal{U}(A_j)\right\} = \sum_{j=1}^k W(A_j).$$

The implication (b) $\Rightarrow$ (a) is clear. Suppose (a) holds. That is, every operator in $\sum_{j=1}^k \mathcal{U}(A_j)$ is positive semidefinite. Furthermore, assume that (b.1) does not hold. By Proposition 3.1, there are self-adjoint operators $H_1,\dots,H_k \in \mathcal{B}(\mathcal{H})$ and $\alpha_1,\dots,\alpha_k,\gamma \in \mathbf{F}$ with $|\gamma| = 1$ such that $A_j = \alpha_j I + \gamma H_j$ for $j = 1,\dots,k$. We may assume that H_1 is non-scalar. Note that

$$\left\{\langle Bx, x\rangle : x \in \mathcal{H},\ \|x\| = 1,\ B \in \sum_{j=1}^k \mathcal{U}(A_j)\right\} = \sum_{j=1}^k \alpha_j + \gamma\left(\sum_{j=1}^k W(H_j)\right) \subset [0,\infty).$$

Since H_1 is non-scalar, $W(H_1)$ contains at least 2 points, and hence $\gamma \in \mathbf{R}$. Furthermore, we have

$$\inf\left\{\gamma w : w \in \sum_{j=1}^{k} W(H_j)\right\} = \sum_{j=1}^{k} \inf\{r : r \in W(\gamma H_j)\} = \sum_{j=1}^{k} \inf \sigma(\gamma H_j).$$

By the fact that $\gamma \in \mathbf{R}$ and each H_j is self-adjoint, we see that condition (b.2) holds. □

The following corollary is immediate.

Corollary 3.3. *Let $A_1, \dots, A_k \in \mathcal{B}(\mathcal{H})$. Then every operator in $\sum_{j=1}^{k} \mathcal{U}(A_j)$ is a scalar operator if and only if A_j is a scalar operator for each $j = 1, \dots, k$.*

Proposition 3.4. *Suppose $\mathbf{F} = \mathbf{R}$ and $A_1, \dots, A_k \in \mathcal{B}(\mathcal{H})$. The following conditions are equivalent.*

(a) *Every operator in $\sum_{j=1}^{k} \mathcal{U}(A_j)$ is essentially skew-symmetric.*
(b) *All real linear combinations of operators in $\mathcal{U}(A_1) \cup \cdots \cup \mathcal{U}(A_k)$ is essentially skew-symmetric.*
(c) *Either*
(c.1) *one of the operators $A_1, \dots, A_k$ is essentially skew-symmetric, and the rest are scalar operators, or*
(c.2) *there are $\alpha_1, \dots, \alpha_k, \gamma \in \mathbf{F}$ with $|\gamma| = 1$, and skew-symmetric operators $G_1, \dots, G_k \in \mathcal{B}(\mathcal{H})$ such that $A_j = \alpha_j I + \gamma G_j$ for $j = 1, \dots, k$.*

Proof. Similar to that of Proposition 3.1. □

4. Unitary operators

An operator $A \in \mathcal{B}(\mathcal{H})$ is *essentially unitary* if there is $\alpha, \gamma \in \mathbf{C}$ with $\gamma \neq 0$ such that $\alpha I + \gamma A$ is unitary. Clearly, A is essentially unitary if and only if A is normal with its spectrum lying in a circle.

Theorem 4.1. *Let $A_1, \dots, A_k \in \mathcal{B}(\mathcal{H})$. Then all operators in $\sum_{j=1}^{k} \mathcal{U}(A_j)$ are essentially unitary if and only if one of the following conditions holds.*

(a) *One of the operators $A_1, \dots, A_k$ is essentially unitary and the other operators are scalar operators.*
(b) *$\dim \mathcal{H} = 2$, and there exists $\gamma \in \mathbf{C}$ with $|\gamma| = 1$ such that $\gamma(A_j - (\operatorname{tr} A_j) I_2/2)$ is self-ajoint for all $j = 1, \dots, k$.*
(c) *$\mathbf{F} = \mathbf{R}$, $\dim \mathcal{H} = 2$, and $A_j - (\operatorname{tr} A_j) I_2/2$ is skew-symmetric for all $j = 1, \dots, k$.*

Proof. If (a), (b) or (c) holds, then clearly every $A \in \sum_{j=1}^{k} \mathcal{U}(A_j)$ is essentially unitary.

Conversely, if all operators in $\sum_{j=1}^{k} \mathcal{U}(A_j)$ are essentially unitary, then one of the conditions (1) – (3) of Theorem 2.1 holds.

If condition (1) of Theorem 2.1 holds, then the non-scalar operator among $A_1, \dots, A_k$ is clearly essentially unitary.

Suppose condition (2) of Theorem 2.1 holds. If $\dim \mathcal{H} = 2$, then condition (b) holds.

Claim Suppose $\dim \mathcal{H} > 2$. It is impossible to have two nonscalar operators, say, A_1 and A_2, among $A_1, \ldots, A_k$.

Assume our claim is not true. By Corollary 3.3, there are unitary operators $U_2, \ldots, U_k$ and $\gamma \in \mathbf{F}$ with $|\gamma| = 1$ such that $A = A_1 = \alpha I + \gamma H_1$ and $B = \sum_{j=2}^k U_j^* A_j U_j = \beta I + \gamma H_2$, where H_1 and H_2 are non-scalar self-adjoint operators. We may replace $(A_1, \ldots, A_k)$ by $(A_1, \ldots, A_k)/\gamma$ and assume that $\gamma = 1$. We may further replace (A, B) by $(A - \alpha I, B - \beta I)$ and assume that $A = H_1$ and $B = H_2$ are both self-adjoint. For any unitary $X, Y \in \mathcal{B}(\mathcal{H})$, since the self-adjoint operator $X^*AX + Y^*BY$ is essentially unitary, its spectrum always have at most two distinct real values. By Corollary 3.3, there are unitary U such that $A + U^*BU$ is non-scalar and has eigenvalues $c_1 > c_2$. We may replace (A, B) by $2(A - cI, B)/(c_1 - c_2)$ so that $T = A + U^*BU$ has eigenvalues $1 > -1$, and is unitary.

Since $A + U^*BU$ is essentially self-adjoint and also essentially unitary, its spectrum is a subset of the intersection of the real line and a circle. Since $A + U^*BU$ is non-scalar, its spectrum consists of two distinct points. Thus, $A + U^*BU = \mu_1 I_{\mathcal{H}_1} \oplus \mu_2 I_{\mathcal{H}_2}$, where $\mathcal{H}$ is an orthogonal sum $\mathcal{H}_1 \oplus \mathcal{H}_2$. Hence A and B have operator matrices

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mu_1 I_{\mathcal{H}_1} - A_{11} & -A_{12} \\ -A_{12}^* & \mu_2 I_{\mathcal{H}_2} - A_{22} \end{pmatrix}.$$

Now, let $V = I_{\mathcal{H}_1} \oplus -I_{\mathcal{H}_2}$. Then $A + V^*BV$ has operator matrix

$$\begin{pmatrix} \mu_1 I_{\mathcal{H}_1} & 2A_{12} \\ 2A_{12}^* & \mu_2 I_{\mathcal{H}_2} \end{pmatrix}.$$

Since $A + V^*BV$ is essentially unitary, there is $\nu \in \mathbf{F}$ such that $A + V^*BV - \nu I_{\mathcal{H}}$ is a multiple of a unitary operator. It follows that both

$$|\mu_1 - \nu|^2 I_{\mathcal{H}_1} + 4A_{12}A_{12}^* \quad \text{and} \quad 4A_{12}^*A_{12} + |\mu_2 - \nu|^2 I_{\mathcal{H}_2}$$

are scalar operators. Hence, we may assume that $\mathcal{H}_1 = \mathcal{H}_2$ and A_{12} is a multiple of a unitary. We may further assume that A_{12} is a scalar operator. We can clearly do that if $A_{12} = 0$. Suppose $A_{12} \neq 0$. Let $X = \|A_{12}\|^{-1} A_{12} \oplus I_{\mathcal{H}_1}$. Then X is unitary such that

$$X^*AX = \begin{pmatrix} \widetilde{A}_{11} & \|A_{12}\| I \\ \|A_{21}\| I & A_{22} \end{pmatrix} \quad \text{and} \quad X^*BX = \begin{pmatrix} \widetilde{B}_{11} & -\|A_{12}\| I \\ -\|A_{21}\| I & B_{22} \end{pmatrix},$$

where $\widetilde{A}_{11} + \widetilde{B}_{11} = \mu_1 I$. We may replace (A, B) by (X^*AX, X^*BX) and assume that $X = I_{\mathcal{H}}$. Then $(\widetilde{A}_{11}, \widetilde{B}_{11}) = (A_{11}, B_{11})$.

Next, we show that A_{11} and B_{11} are scalar operators. Assume that it is not true. Then the spectrum of A_{11} has at least two elements. We consider two cases.

If A_{11} has two distinct eigenvalues, say $a_1 > a_2$, then we may assume that $A_{11} = \operatorname{diag}(a_1, a_2) \oplus A_0$. Since $A_{11} + B_{11} = \mu_1 I_{\mathcal{H}_1}$, we see that $B = \operatorname{diag}(b_1, b_2) \oplus B_0$ with $b_1 < b_2$. Then there is a unitary $V \in M_2$ such that

$$C_0 = \operatorname{diag}(a_1, a_2) + V^* \operatorname{diag}(b_1, b_2) V = \begin{pmatrix} a_1 + b_1 \cos^2 t + b_2 \sin^2 t & (b_1 - b_2) \cos t \sin t \\ (b_1 - b_2) \cos t \sin t & a_2 + b_2 \cos^2 t + b_1 \sin^2 t \end{pmatrix}$$

has two distinct eigenvalues different from μ_1. Let $X = V \oplus I \in \mathcal{B}(\mathcal{H}_1)$, and $\widetilde{X} = X \oplus X$. Then the self-adjoint operator

$$A + \widetilde{X}^* B \widetilde{X} = C_0 \oplus (A_0 + B_0) \oplus \widetilde{C}_2$$

has at least three different eigenvalues and is not essentially unitary.

Suppose A_{11} has one or no eigenvalue. Let a_1 and a_2 be the supremum and infimum of the set S obtained from $\sigma(A_{11})$ by removing the eigenvalue if it exists. Then there are mutually orthonormal sequences $\{u_n : n \geq 1\}$ and $\{v_n : n \geq 1\}$ in $\mathcal{H}_1$ such that $\|A_{11}u_n - a_1u_n\| \to 0$ and $\|A_{11}v_n - a_2v_n\| \to 0$. Since $A_{11} + B_{11} = \mu_1 I_{\mathcal{H}_1}$, we have $\|B_{11}u_n - b_1u_n\| \to 0$ and $\|B_{11}v_n - b_2v_n\| \to 0$ with $(b_1, b_2) = (\mu_1 - a_1, \mu_1 - a_2)$. Let $X \in \mathcal{B}(\mathcal{H}_1)$ be unitary such that $Xu_{3n} = u_{3n}$, $Xu_{3n+1} = v_{3n+1}$, and $Xv_{3n+2} = u_{3n+2}$ for $n \geq 1$. Then

$$\|(A_{11} + X^*B_{11}X)u_{3n} - (a_1 + b_1)u_{3n}\| \to 0, \qquad \|(A_{11} + X^*B_{11}X)u_{3n+1} - (a_1 + b_2)u_{3n+1}\| \to 0,$$

and

$$\|(A_{11} + X^*B_{11}X)v_{3n+2} - (a_2 + b_1)v_{3n+2}\| \to 0.$$

Thus, the self-adjoint operator $A + X^*BX$ has at least three distinct elements $a_1 + b_2 > a_1 + b_1 > a_2 + b_1$ in the spectrum, and is not essentially unitary.

Similarly, we can prove that A_{22} and B_{22} are scalar operator. Thus,

$$A = \begin{pmatrix} a_1 I_{\mathcal{H}_1} & \|A_{12}\| I_{\mathcal{H}_1} \\ \|A_{12}\| I_{\mathcal{H}_1} & a_2 I_{\mathcal{H}_1} \end{pmatrix}$$

has discrete spectrum $\{\alpha_1, \alpha_2\}$. Similarly, B has discrete spectrum $\{\beta_1, \beta_2\}$. But then one can easily construct unitary $X \in \mathcal{B}(\mathcal{H})$ such that $A + X^*BX$ has distinct eigenvalues $\alpha_1 + \beta_1, \alpha_1 + \beta_2, \alpha_2 + \beta_2$, so that the self-adjoint operator $A + X^*BX$ is not essentially unitary.

Now, suppose condition (3) of Theorem 2.1 holds. If only one of the operators $A_1, \ldots, A_k$ is non-scalar, then condition (a) holds. Suppose there are at least two, say, A_1 and A_2, are non-scalar operators among $A_1, \ldots, A_k$. If $\dim \mathcal{H} = 2$, then condition (b) holds.

Claim Assume that $\dim \mathcal{H} > 2$. The assumption that A_1 and A_2 are non-scalar is impossible.

By Corollary 3.3, there are unitary operators $U_2, \ldots, U_k$ such that $A = A_1 = \alpha I + G_1$ and $B = \sum_{j=2}^k U_j^* A_j U_j = \beta I + G_2$, where G_1 and G_2 are non-scalar skew-symmetric operators. We may further replace (A_1, A_2) by $(A_1 - \alpha I, A_2 - \beta I)$ and assume that $A = G_1$ and $B = G_2$. For any orthogonal operators $X, Y \in \mathcal{B}(\mathcal{H})$, the skew-symmetric operator $C = X^*AX + Y^*BY$ is essentially orthogonal. So, there exist $a, b \in \mathbf{R}$ such that

$$b^2 I = (C - aI)^*(C - aI) = C^*C - a(C + C^*) + a^2 I = C^*C + a^2 I.$$

It follows that $C^*C = (b^2 - a^2)I$, and hence C is always a multiple of an orthogonal operator. We consider two cases.

Case 1. Suppose $\dim \mathcal{H} = n$ is finite. Then there are orthogonal matrices $X, Y \in M_n$ such that

$$X^t A X = A_1 \oplus \cdots \oplus A_p \oplus 0_{n-2p} \quad \text{and} \quad Y^t B Y = B_1 \oplus \cdots \oplus B_q \oplus 0_{n-2q},$$

where

$$A_j = \begin{pmatrix} 0 & a_j \\ -a_j & 0 \end{pmatrix} \quad \text{with } a_1 \geq \cdots \geq a_p > 0$$

and

$$B_j = \begin{pmatrix} 0 & b_j \\ -b_j & 0 \end{pmatrix} \quad \text{with } b_1 \geq \cdots \geq b_q > 0.$$

Since $X^tAX + Y^tBY$ is a multiple of an orthogonal matrix, we see that $n = 2p = 2q$ and $a_1 = \cdots = a_q = b_1 = \cdots = b_q$. But then if Z is obtained from Y by switching its first two columns, then $X^tAX + Z^tBZ$ is not a multiple of an orthogonal operator, which is a contradiction.

Case 2. Suppose $\dim \mathcal{H}$ is infinite. By Corollary 3.3, we can choose orthogonal operators $X, Y \in \mathcal{B}(\mathcal{H})$ so that $C = X^*AX + Y^*BY$ is non-scalar. Let $C^*C = rI$. We may replace (A, B) by $(X^*AX, Y^*BY)/\sqrt{r}$ so that $C = A + B$ is orthogonal. Let $x \in \mathcal{B}(\mathcal{H})$ be a unit vector. Since $-C = C^*$ is acting on a real Hilbert space, we see that

$$\langle Cx, x\rangle = \langle x, C^*x\rangle = -\langle x, Cx\rangle = -\langle Cx, x\rangle.$$

Thus, $\langle Cx, x\rangle = 0$. Since C is orthogonal, $Cx = y$ for some $y \in x^\perp$. Note also that $x = C^*(Cx) = C^*y = -Cy$. Thus, $\operatorname{span}\{x, y\}$ is a reducing subspace of C. As a result, C can be written as $C_1 \oplus \tilde{C}_1$ where $C_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Now applying the argument to $\tilde{C}_1$, we can further decompose C as $C_1 \oplus C_1 \oplus \widetilde{C}_0$ so that $\tilde{C}_0$ is orthogonal. If $Ax = 0$ (respectively, $Bx = 0$) in the first step of the above decomposition, we should choose a unit vector $u \in \{x, y\}^\perp$ so that $Au = v \neq 0$ (respectively, $Bu = v \neq 0$) in the second step of the decomposition. Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ -A_{12}^* & A_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & -A_{12} \\ A_{12}^* & B_{22} \end{pmatrix}$$

with $A_{11} + B_{11} = C_1 \oplus C_1$ and $A_{22} + B_{22} = \widetilde{C}_0$. We claim that $A_{12} = 0$. If it is not the case, then there are orthogonal operators U, V such that the $(1, 1)$ entry of the operator matrix $U^*A_{12}V$ equals $a \neq 0$. Let $Z = (U \oplus V) \in \mathcal{B}(\mathcal{H})$. Then

$$Z^*CZ = U^*(C_1 \oplus C_1)U \oplus V^*\widetilde{C}_0V.$$

Moreover,

$$\hat{C} = Z^*AZ + ([-1] \oplus I)Z^*BZ([-1] \oplus I)$$

is again a multiple of an orthogonal operator, which can be obtained from Z^*CZ by a rank 2 perturbation because the two operator matrices differ only in the first row and the first column. Clearly a finite rank perturbation cannot change the Z^*CZ to a different multiple of orthogonal operator. Thus, $\hat{C}$ is itself an orthogonal operator. Recall that the $(1, 1)$ entry of the operator matrix $U^*A_{12}V$ equals $a \neq 0$. Comparing the fifth columns of the two operator matrices of Z^*CZ and $\hat{C}$, we see that the former has length 1 and the latter has length $\sqrt{1 + (2a)^2}$, which contradicts the fact that $\hat{C}$ is orthogonal. Now, $A_{11}, B_{11} \in M_4$ are skew-symmetric. By our choice of the vectors x, y, u, v for decomposing C as $C_1 \oplus C_1 \oplus \tilde{C}_0$, we see that neither A_{11} nor B_{11} is the zero

operator. Now, using the result in the finite dimensional case, there are orthogonal $R, S \in M_4$ such that $R^*A_{11}R + S^*B_{11}S$ is not a multiple of an orthogonal matrix. Then

$$(R \oplus I)^*A(R \oplus I) + (S \oplus I)^*B(S \oplus I) = R^*A_{11}R \oplus A_{22} + S^*B_{11}S \oplus B_{22}$$

is not a multiple of an orthogonal operator, which is a contradiction. □

Corollary 4.2. *Let $A_1, \dots, A_k \in \mathcal{B}(\mathcal{H})$. Then every operator in $\sum_{j=1}^k \mathcal{U}(A_j)$ is unitary if and only if at least $k-1$ of the operators $A_1, \dots, A_k$ are scalar operators and $\sum_{j=1}^k A_j$ is unitary.*

Corollary 4.3. *Let $A_1, \dots, A_k \in \mathcal{B}(\mathcal{H})$. Then every (nonnegative, real or complex) linear combination of operators in $\mathcal{U}(A_j)$ are multiple of unitaries if and only if one of the following conditions holds.*

(a) *All operators $A_1, \dots, A_k$ are scalar.*
(b) $\dim \mathcal{H} = 2$ *and there is $\gamma \in \mathbf{F}$ with $|\gamma| = 1$ such that γA_j is a trace zero matrix for each j.*
(c) $\mathbf{F} = \mathbf{R}$, $\dim \mathcal{H} = 2$ *and each A_j is skew-symmetric for each j.*

5. Sum of operators from a single unitary orbit

We can use the results in the previous sections to characterize $A \in \mathcal{B}(\mathcal{H})$ such that the nonnegative (or real) linear combinations of operators in $\mathcal{U}(A)$ have special structure.

Proposition 5.1. *Suppose $\mathcal{H}$ is a complex Hilbert space. Then $A \in \mathcal{B}(\mathcal{H})$ is essentially self-adjoint if and only if any one of the following equivalent conditions holds.*

(a) *There is a positive integer $k \geq 2$ such that the sum of any k operators in $\mathcal{U}(A)$ is normal.*
(b) *There is a positive integer $k \geq 2$ such that the sum of any k operators in $\mathcal{U}(A)$ is essentially self-adjoint.*
(c) *Any nonnegative (or real) linear combinations of operators in $\mathcal{U}(A)$ is essentially self-adjoint.*

Proposition 5.2. *Let $\mathcal{H}$ be a real or complex Hilbert space and $A \in \mathcal{B}(\mathcal{H})$. Then the following conditions are equivalent.*

(a) *There is a positive integer $k \geq 2$ such that the sum of any k operators in $\mathcal{U}(A)$ is a multiple of a unitary operator.*
(b) *Any (real or complex) linear combination of operators in $\mathcal{U}(A)$ is a multiple of a unitary operator.*
(c) *Either A is a scalar operator or* $(\dim \mathcal{H}, \operatorname{tr} A, AA^* - A^*A) = (2, 0, 0_2)$.

Proposition 5.3. *Suppose $\mathcal{H}$ is a real Hilbert space. Let $A \in \mathcal{B}(\mathcal{H})$ be a non-scalar operator, and let $k \geq 2$ be a positive integer.*

(a) *The operator A is essentially symmetric or essentially skew-symmetric if and only if the sum of any k operators in $\mathcal{U}(A)$ is normal.*

(b) *The operator A is essentially symmetric if and only if the sum of any k operators in $\mathcal{U}(A)$ is essentially symmetric. Equivalently, any nonnegative (or real) linear combination of operators in $\mathcal{U}(A)$ is essentially symmetric.*

(c) *The operator A is essentially skew-symmetric if and only if the sum of any k operators in $\mathcal{U}(A)$ is essentially skew-symmetric. Equivalently, any nonnegative (or real) linear combination of operators in $\mathcal{U}(A)$ is essentially skew-symmetric.*

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