## A Partition Problem for Sets of Permutation Matrices

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# 1 Introduction

Consider a set $\mathcal{P}$ of permutation matrices of order $n$. What is the smallest integer $m$ such that $\mathcal{P}$ can be partitioned into subsets $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m$ such that

$$\sum\{P : P \in \mathcal{P}_i\}, \quad (i = 1, 2, \ldots, m)$$

are (0,1)-matrices? Let $G(\mathcal{P})$ be a graph with vertex set $\mathcal{P}$ with an edge joining two permutation matrices $P, Q \in \mathcal{P}$ provided $P$ and $Q$ have a 1 in common (that is, a 1 in the same position). The integer $m$ equals the chromatic number $\chi(G(\mathcal{P}))$. Natural sets $\mathcal{P}$ of permutation matrices arise by choosing $A = [a_{ij}]$ to be a $(0,1)$-matrix and

$$\mathcal{P} = \mathcal{P}_A = \{P : P \leq A, P \text{ is a permutation matrix}\}. \tag{1}$$

(Here the inequality $P \leq A$ is interpreted entrywise.) In this case the sets $\mathcal{P}_i$ in the partition must satisfy

$$\sum\{P : P \in \mathcal{P}_i\} \leq A.$$

A more restrictive problem requires that

$$\sum\{P : P \in \mathcal{P}_i\} = A \quad (i = 1, 2, \ldots, m). \tag{2}$$

If (2) holds, then

$$\sum\{P : P \in \mathcal{P}_A\} = mA,$$

and we say that $\mathcal{P}_A$ has a *perfect partition.* The cardinality of the set $\mathcal{P}_A$ equals the *permanent* of $A$ defined, as usual, by:

$$\text{per}(A) = \sum_{(i_1, i_2, \ldots, i_n) \in \mathcal{S}_n} a_{1i_1} a_{2i_2} \cdots a_{ni_n},$$

where the summation is over the symmetric group $\mathcal{S}_n$ of all permutations of $\{1, 2, \ldots, n\}$.

Suppose that $\mathcal{P}_A$ has a perfect partition. Then there are two consequences for the structure of $A$. First, there is an integer $k$ such that all row and column sums of $A$ equal $k$, and this integer $k$ satisfies the equation $\text{per}(A) = mk$. Second, a perfect partition implies that each 1 of $A$ belongs to $m$ permutation matrices $P \le A$, and hence, where $A(i,j)$ denotes the submatrix of $A$ obtained by deleting row $i$ and column $j$, that

$$\mathrm{per}A(i,j) = m \text{ if } a_{ij} = 1,$$

that is, the permanental minors of the 1's of $A$ all equal the same constant $m$.

Let $G_A = G(\mathcal{P}_A)$. Since the chromatic number of $G_A$ equals the minimal number of independent sets into which $\mathcal{P}_A$ can be partitioned, we have

$$\chi(G_A) \ge \frac{\mathrm{per}(A)}{\alpha(G_A)}, \tag{3}$$

where $\alpha(G_A)$ is the maximal size of an independent set of $G_A$. We can have equality in (3) only if $\alpha(G_A)|\mathrm{per}(A)$. If $\mathcal{P}_A$ has a *perfect partition*, then the integer $m$ in (2) equals $\chi(G_A)$. Since $\chi(G_A)$ is an integer, (3) implies that

$$\chi(G_A) \ge \left\lceil \frac{\mathrm{per}(A)}{\alpha(G_A)} \right\rceil. \tag{4}$$

By a theorem of Folkman and Fulkerson [2] (see also Theorem 6.4.3 in [1]), the independence number $\alpha(G_A)$ equals

$$\min\left\{\frac{\mathrm{sum}(A_{kl})}{k+l-n} : k+l > n\right\}$$

where the minimum is taken over all pairs of integers $k$ and $l$ with $n < k+l \le 2n$ and $k \times l$ submatrices $A_{kl}$ of $A$, and $\mathrm{sum}(A_{kl})$ is the sum of the entries of $A_{kl}$.

There is a geometrical interpretation of the perfect partition problem. Recall that a necessary condition for the existence of a perfect partition for $\mathcal{P}_A$ is that the sum of matrices in $\mathcal{P}_A$ is a multiple of $A$. Thus, the average of $\mathcal{P}_A$, which can also be viewed as the centroid of the convex hull of $\mathcal{P}_A$, has the form $\gamma A$. Clearly, every element in $\mathcal{P}_A$ is an extreme point of the convex hull of $\mathcal{P}_A$. (To see this, note that every element $X$ in $\mathcal{P}_A$ has the same Frobenius norm $(\mathrm{trace}\, XX^t)^{1/2}$ and therefore cannot be written as a convex combination of the others.) If $A$ has row sums and column sums all equal to $k$, then one needs at least $k$ elements in $\mathcal{P}_A$ whose average (regarded as the centroid of the convex hull of the $k$ elements) is equal to $\gamma A$; if the desired partition is a partition of $\mathcal{P}_A$ in $k$-element sets, then each of them has the same average as that of $\mathcal{P}_A$.

In the subsequent discussion, let $J_n$ be the $n \times n$ matrix of all 1's. In the next section we consider perfect partitions of $\mathcal{S}_n = \mathcal{P}_{J_n}$ (where we now regard $\mathcal{S}_n$ as the set of $n \times n$ permutation matrices) and the alternating group $\mathcal{A}_n$ of all $n \times n$ even permutation matrices (permutation matrices with determinant equal to 1). In Section 3, we consider the set $\mathcal{D}_n = \mathcal{P}_{J_n - I_n}$ of $n \times n$ *derangement permutation matrices*; we present some partial results and open problems. Additional open questions are discussed in the final section.

## 2 Partitioning $\mathcal{S}_n$ and $\mathcal{A}_n$

We have $\alpha(J_n) = n$ and $\mathrm{per}(J_n) = n!$, and it is easy to show that $\sum_{X\in\mathcal{S}_n} X = (n-1)!J_n$. Can we partition $\mathcal{S}_n$ into $(n-1)!$ subsets so that the sum of the matrices in each subset is $J_n$? The answer is affirmative.

**Proposition 2.1** *The set $\mathcal{S}_n = \mathcal{P}_{J_n}$ is a disjoint union of $(n-1)!$ subsets such that the sum of the matrices in each subset is $J_n$. Hence $\mathcal{S}_n$ has a perfect partition.*

*Proof.* Let $H = \{I_n, P, \ldots, P^{n-1}\}$ where $P$ is the basic $n \times n$ circulant matrix

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{5}$$

Then $H$ is a cyclic group with $n$ elements whose sum is the matrix $J_n$. There are $(n-1)!$ cosets of $H$ in $\mathcal{S}_n$. Each coset has the form $QH = \{QP^j : j = 0, \ldots, n-1\}$ for some $Q \in \mathcal{S}_n$. Clearly, the sum of the matrices in each coset is also the matrix $J_n$. □

Now we consider the group $\mathcal{A}_n$ of even permutation matrices. We have $|\mathcal{A}_n| = n!/2$, and it is not hard to show that $\sum_{X\in\mathcal{A}_n} X = [(n-1)!/2]J_n$ if $n \geq 3$. Can we partition $\mathcal{A}_n$ into $(n-1)!/2$ subsets so that the sum of the matrices in each subset is $J_n$? We have the following result.

**Proposition 2.2** *Suppose $n \geq 3$. The set $\mathcal{A}_n$ can be partitioned into $(n-1)!/2$ subsets so that the sum of the matrices in each subset is $J_n$.*

*Proof.* We consider three cases according to $n$.

**Case 1.** If $n \geq 3$ is odd, then the basic circulant matrix $P$ is in $\mathcal{A}_n$. Thus $H = \langle P \rangle$ is a subgroup of $\mathcal{A}_n$ with $(n-1)!/2$ cosets, and the sum of the matrices in each coset is $J_n$.

**Case 2.** If $n = 4k$ for some positive integer $k$, we can prove by induction that:

There is a subgroup $H$ in $\mathcal{A}_n$ with $n$ elements whose sum equals $J_n$, and hence the cosets of the group $H$ will be a desired partition.

When $k = 1$, let $H_4$ be the subgroup of $\mathcal{A}_4$ containing all the elements of order 2 or 0 ($H_4$ is the 2-Sylow subgroup of $\mathcal{A}_4$). One can readily check that the the sum of the matrices in $H_4$ sum up to $J_4$.

Now, suppose the result is true for $n = 4k$ for some $k \geq 1$. Consider the case when $n = 4(k+1)$. By the induction assumption, there is a group $H_{4k}$ of $\mathcal{A}_{4k}$ such that the sum of the matrices in $H_{4k}$ is $J_{4k}$. Let $H = \{A \otimes B : A \in H_4,\ B \in H_{4k}\}$, where $X \otimes Y = (x_{ij}Y)$ denotes the usual tensor product of two matrices. Then $H$ is a subgroup of $\mathcal{A}_n$ with $n = 4(k+1)$ elements whose sum is the matrix $J_n$. By induction, our claim is proved.

**Case 3.** Let $n = 2m$ for some odd integer $m$. We consider the subgroup $K$ of $\mathcal{A}_n$ consisting of matrices of the form $A \oplus B$, where $A$ and $B$ are $m \times m$ permutation matrices. There are $(m!)^2/2$ such matrices. To see this, if we allow $A$ and $B$ to be arbitrary matrices in $\mathcal{S}_m$, there will be $(m!)^2$ such matrices in $\mathcal{S}_n$. Since half of them are odd permutations, we see that $K$ has $(m!)^2/2$ elements as asserted.

We claim that $K$ can be partitioned into $m((m-1)!)^2/2$ subsets such that each subset has $m$ elements summing up to $J_m \oplus J_m$. To this end, let $P \in \mathcal{S}_m$ be the basic circulant. Let $G = \langle P \rangle$, and let $Q_1G, \ldots, Q_rG$ be the cosets of $G$ in $\mathcal{S}_m$, where $r = (m-1)!$, $Q_1, \ldots, Q_{r/2} \in \mathcal{A}_m$ and $Q_j \notin \mathcal{A}_m$ for $j > r/2$.

For each $i, j = 1, \ldots, r/2$, consider the following $m$-element subsets of $\mathcal{A}_n$:

$$\mathcal{S}_{ij1} = \{(Q_i \oplus Q_j)(P \oplus P)^k : k = 0, \ldots, m-1\},$$

$$\mathcal{S}_{ij2} = \{X(I_m \oplus P) : X \in \mathcal{S}_{ij1}\}, \quad \mathcal{S}_{ij3} = \{X(I_m \oplus P^2) : X \in \mathcal{S}_{ij1}\}, \quad \ldots,$$

$$\ldots, \quad \mathcal{S}_{ijm} = \{X(I_m \oplus P^{m-1}) : X \in \mathcal{S}_{ij1}\}.$$

We get $m(r/2)^2$ disjoint $m$-element subsets of $K$.

Next, for each $i, j = r/2+1, \ldots, r$, consider

$$\mathcal{S}_{ij1} = \{(Q_i \oplus Q_j)(P \oplus P)^k : k = 0, \ldots, m-1\},$$

$$\mathcal{S}_{ij2} = \{X(I_m \oplus P) : X \in \mathcal{S}_{ij1}\}, \quad \mathcal{S}_{ij3} = \{X(I_m \oplus P^2) : X \in \mathcal{S}_{ij1}\}, \quad \ldots,$$

$$\ldots, \quad \mathcal{S}_{ijm} = \{X(I_m \oplus P^{m-1}) : X \in \mathcal{S}_{ij1}\}.$$

We get another $m(r/2)^2$ disjoint $m$-element subsets of $K$.

Consequently, we get $mr^2/2 = m((m-1)!)^2/2$ disjoint $m$-element subsets of $K$. Moreover, the matrices in each subset sum up to $J_m \oplus J_m$ as desired.

Now, consider the matrix $R$ obtained by switching the first two rows of $\begin{pmatrix} 0_m & I_m \\ I_m & 0_m \end{pmatrix}$. Then $R \in \mathcal{A}_m$. Let

$$H = K \cup \{RX : X \in K\}.$$

One easily checks that $H$ is the subgroup of $\mathcal{A}_n$ consisting of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}.$$

Moreover, for each set $\mathcal{S}_{ijk}$ defined above, we may construct

$$T_{ijk} = \mathcal{S}_{ijk} \cup \{RX : X \in \mathcal{S}_{ijk}\}.$$

Then each $T_{ijk}$ will have $n = 2m$ elements summing up to $J_n$, and these $T_{ijk}$ form a partition of the subgroup $H$.

Now, let $H, W_1H, W_2H, \ldots, W_tH$ be the cosets of $H$ in $\mathcal{A}_n$, where $t+1 = |\mathcal{A}_n|/|H|$. Each coset $W_sH$ is a disjoint union of $W_sT_{ijk}$'s, and each $W_sT_{ijk}$ has $n$ elements summing up to $J_n$. □

**Corollary 2.3** *The set $\mathcal{S}_n$ has a perfect partition in which each part of the partition consists of all even permutation matrices or all odd permutation matrices.*

*Proof.* As in the proof of Proposition 2.1, the coset of odd permutations also can be partitioned into sets summing to $J_n$. $\square$

# 3 Partitioning $\mathcal{D}_n = \mathcal{P}_{J_n - I_n}$: Partial Result

Let $L_n = J_n - I_n$. For $n = 2, \ldots, 5$ we show that $\mathcal{D}_n = \mathcal{P}_{L_n}$ can be partitioned into subsets each with $n-1$ matrices that sum to $L_n$.

In the following discussion, we identify a permutation $\sigma$ in disjoint cycle representation with the corresponding permutation matrix in $\mathcal{S}_n$. For example, $(1,2)(3,4)$ represents the permutation obtained from the identity matrix by interchanging the first and second rows, and also the third and fourth rows. Then $\sigma \in \mathcal{S}_n$ is a derangement if and only if $\sigma(i) \neq i$ for $i = 1, \ldots, n$. Moreover, the elements in a set of derangements $\{\sigma_1, \ldots, \sigma_{n-1}\} \subseteq \mathcal{D}_n$ sum to $L_n$ if and only if $\sigma_r(i) \neq \sigma_s(i)$ for $r \neq s$ and for all $i = 1, \ldots, n$. We have the following partial result for the partition problem of $\mathcal{P}_{L_n}$.

**Proposition 3.1** *The set $\mathcal{D}_n$ has a perfect partition if $n \leq 5$.*

*Proof.* If $n = 2$, then $\mathcal{D}_n = \{L_n\}$ is a singleton. If $n = 3$, then the members of $\mathcal{D}_n = \{(1,2,3), (1,3,2)\}$ sum to $L_n$.

For $n = 4$, a permutation belongs to $\mathcal{D}_n$ if and only if it is a 4-cycle or a product of two disjoint transpositions. If

$$F_1 = \{(1,2)(3,4), (1,3,2,4), (1,4,2,3)\},$$

$$F_2 = \{(1,3)(2,4), (1,2,3,4), (1,4,3,2)\},$$

$$F_3 = \{(1,4)(2,3), (1,2,4,3), (1,3,4,2)\},$$

then $\mathcal{D}_4 = \cup_{k=1}^3 F_k$ and the members of each $F_k$ sum to $L_4$.

For $n = 5$, a permutation belongs to $\mathcal{D}_n$ if and only if it is of the form $(i_1, i_2, i_3, i_4, i_5)$ or $(i_1, i_2)(i_3, i_4, i_5)$. Let $\mathcal{D}_5' \subset \mathcal{D}_5$ be the set of derangements of the form $(i_1, i_2, i_3, i_4, i_5)$ and let $\mathcal{D}_5'' \subset \mathcal{D}_5$ be the set of derangements of the form $(i_1, i_2)(i_3, i_4, i_5)$. Observe that $|\mathcal{D}_5'| = 24$ and $|\mathcal{D}_5''| = 20$. We show that $\mathcal{D}_5'$ and $\mathcal{D}_5''$ can be partitioned into 6 and 5 subsets, respectively, such that the members of each subset sum to $L_5$.

Let $\tau_1 = (1,2,3,4,5)$, $\tau_2 = (1,2,3,5,4)$, $\tau_3 = (1,2,4,3,5)$, $\tau_4 = (1,2,4,5,3)$, $\tau_5 = (1,2,5,3,4)$, and $\tau_6 = (1,2,5,4,3)$. If $T_k = \{\tau_k, \tau_k^2, \tau_k^3, \tau_k^4\}$, then the collection of subsets $T_1, \ldots, T_6$ forms a partition of $\mathcal{D}_5'$ such that the members of each $T_k$ sum to $L_5$. Now, consider the following subsets of $\mathcal{D}_5''$:

$$R_1 = \{(1,2)(3,4,5), (1,3)(2,5,4), (1,4)(2,3,5), (1,5)(2,4,3)\}$$

$$R_2 = \{(2,1)(3,5,4), (2,3)(1,4,5), (2,4)(1,5,3), (2,5)(1,3,4)\}$$
$$R_3 = \{(3,1)(2,4,5), (3,2)(1,5,4), (3,4)(1,2,5), (3,5)(1,4,2)\}$$
$$R_4 = \{(4,1)(2,5,3), (4,2)(1,3,5), (4,3)(1,5,2), (4,5)(1,2,3)\}$$
$$R_5 = \{(5,1)(2,3,4), (5,2)(1,4,3), (5,3)(1,2,4), (5,4)(1,3,2)\}.$$

Then the collection of subsets $R_1, \ldots, R_5$ forms a partition of $\mathcal{D}_5''$ such that the members of each $R_k$ sum to $L_5$. This completes the partition of $\mathcal{D}_5$. □

The problem of partitioning $\mathcal{D}_n$ with $n \geq 6$ is more difficult. In the following, we describe several different approaches we considered.

First, we divide the set $\mathcal{D}_n$ into subsets according to different cycle decompositions, and we attempt to show that each of these subsets admits a partition into $(n-1)$-element subsets with elements summing to $L_n$. In particular, when $n = 5$, the partition was done in this way. When we apply this idea to $\mathcal{D}_6$, we get the following subsets:

$T_1$: the set of length-6 cycles – 120 elements;
$T_2$: the set of permutations obtained by the product of a 2-cycle and a 4-cycle – 90 elements;
$T_3$: the set of permutations obtained by the product of two 3-cycles – 40 elements;
$T_4$: the set of permutations obtained by the product of three 2-cycles – 15 elements.

For each subset, the sum of its elements (say, denoted by X) will be a multiple of $L_n$ because all of the diagonal entries of $X$ are zeroes and $PXP^t = X$ for every permutation matrix $P$. However, this approach to partitioning $\mathcal{D}_n$ fails when $n = 6$. One can check that the set $T_4$ cannot be partitioned into three 5-element subsets such that the elements in each subset sum up to $L_6$.

An alternative idea is to select 15 elements $\tau_1, \ldots, \tau_{15}$ from $T_1$ and construct disjoint subsets

$$U_i = \{\tau_i^j : j = 1, \ldots, 5\}$$

so that each of them has elements summing up to $L_6$. Note that each $U_i$ will have two elements in $T_1$, two elements in $T_3$, and one element in $T_4$. If this is done, then we are left with 90 elements in $T_1$, the entire set $T_2$, and 10 elements in $T_3$.

Another scheme is to select one element in $T_4$ and four elements in $T_1$ of the form $\tau_1, \tau_1^{-1}, \tau_2, \tau_2^{-1}$ to form a set whose elements sum up to $L_6$. Here is an example:

$$(1,2)(3,4)(5,6), (1,3,5,2,6,4), (1,4,6,2,5,3), (1,6,3,2,4,5), (1,5,4,2,3,6).$$

In fact, one can construct 15 sets of such form and use up the 15 elements in $T_4$ together with 60 elements in $T_1$.

It is also possible to use two elements in $T_3$ and three elements in $T_1$ to form a set whose elements sum up to $L_6$. Here is an example:

$$(1,2,3)(4,5,6), (1,3,2)(4,6,5), (1,4,2,5,3,6), (1,5,2,6,3,4), (1,6,2,4,3,5).$$

One can actually construct 20 subsets of this form and use up the 40 elements in $T_3$ together with 60 elements in $T_1$.

One may want to use the two schemes in the last two paragraphs to exhaust the elements in $T_1$, $T_3$, and $T_4$, but this strategy seems to be impossible. Of course, even if it can be done, one must still partition the elements in $T_2$ into 18 sets, each of which has elements summing up to $L_6$. Here is an example of such a set:

$$(1,2)(3,4,5,6), (1,3)(2,4,6,5), (1,4)(2,5,3,6), (1,5)(2,6,4,3), (1,6)(2,3,5,4).$$

It is unclear whether one can construct 18 disjoint subsets of $T_2$ with the desired property.

Thus, the problem of finding a perfect partition for $\mathcal{P}_{L_n}$ seems difficult. We close this section with a statement of the problem and some related questions:

**Problem 3.2** For $n \geq 6$, is there a perfect partition for $\mathcal{P}_{L_n}$ or $\mathcal{P}_{L_n} \cap \mathcal{A}_n$?

**Problem 3.3** For $n \geq 6$, is there a perfect partition for the collection of permutations in $\mathcal{P}_{L_n}$ with some specific cycle decomposition?

For example, can the set of permutations obtained by the product of a 2-cycle and an $(n-2)$-cycle be partitioned into subsets such that the elements of each subset sum up to $L_n$? The answer is no for $n = 4$, yes for $n = 5$, and unknown for $n \geq 6$.

# 4 Additional Problems

We continue to use $P$ to denote the basic circulant as defined in (5). Note that $J_n = \sum_{k=0}^{n-1} P^k$ and $L_n = \sum_{k=1}^{n-1} P^k$. For any subsets $K \subseteq \{0, 1, \ldots, n-1\}$, let

$$P_K = \sum_{k \in K} P^k.$$

A general question is:

**Problem 4.1** Determine $K \subseteq \{0, 1, \ldots, n-1\}$ so that $\mathcal{P}_{P_K}$ (respectively, $\mathcal{P}_{P_K} \cap \mathcal{A}_n$) has a perfect partition.

By the results in the previous sections, we see that both problems in Problem 4.1 have affirmative answers if $|K| = n$. If $|K| = 1$, then both problems also have affirmative answers trivially. If $|K| = 2$, then we have the following proposition.

**Proposition 4.2** *Let $A = I_n + P^k$ with $0 < k < n$. Then $\mathcal{P}_A$ admits a perfect partition.*

*Proof.* Write $P^k$ in disjoint cycle notation. There are two cases.

**Case 1.** If $(n, k)$ is relatively prime, then $P^k$ is just one long cycle, and $I_n$ and $P^k$ are the only two elements in $\mathcal{P}_A$, which admits a trivial perfect partition.

**Case 2.** If $d > 1$ is the greatest common divisor of $n$ and $k$, and $m = n/d$, then $P^k$ is the product of $d$ cycles of length $m$. Now, we can rewrite $A = I_n + P^k$ as the direct sum of $d$

$m\times m$ matrices, each of which is $I_m+Q$, where $Q$ is the $m\times m$ basic circulant. In this form, one readily checks that $X\in\mathcal{P}_A$ if and only if $X = X_1\oplus\cdots\oplus X_d$ such that $X_j\in\{I_m, Q\}$. Thus, there are $2^d$ matrices in $\mathcal{P}_A$. Moreover, $\mathcal{P}_A$ has a perfect partition consisting of sets of the form $\{X, A-X\}$ with $X\in\mathcal{P}_A$. □

If $|K| = n-1$, we basically have the $\mathcal{P}_{L_n}$ problem, and we only have partial results. If $|K| = 3$, even the necessary condition for a perfect partition may not hold. Here is an example which can be verified readily.

**Example 4.3** *For $n = 5$ there are 13 matrices in $\mathcal{P}_A$ for $A = I_n+P+P^2$ or $A = I_n+P^2+P^3$. In either case, a perfect partition is impossible.*

Note that in general, if $|K| = n-2$, then $P_K = J_n - P^r - P^s$. Replacing $P_K$ by $P^jP_K$ for a suitable $j\in\{0,\dots,n-1\}$, we may assume that $(r,s) = (-l,l)$ with $1\le l\le n/2$, or $(r,s) = (0,1)$. For example, for $n = 5$, we only need to consider the cases in Example 4.3.

If $n$ is even and $(r,s) = (-l,l)$ with $1\le l\le n/2$, then up to a permutation equivalence, i.e., replace $A$ by $RAS$ for some suitable $R,S\in\mathcal{S}_n$, we can assume that $P_K = J_n-(I_{n/2}\otimes J_2)$, which can be viewed as a generalization of $L_n$. In general, if $n = km$, we consider $L_{n,k} = J_n - (I_m\otimes J_k)$. We have the following.

**Proposition 4.4** *Suppose $n\ge 4$ and $n = km$. Then $\operatorname{per}(L_{n,k})$ is a multiple of $(n-k)$. Moreover, if $n > k\ge 2$, then $\left|\mathcal{P}_{L_{n,k}}\cap\mathcal{A}_n\right| = \operatorname{per}(L_{n,k})/2$ is also a multiple of $(n-k)$.*

*Proof.* Use Laplace expansion about the first row of $L_{n,k}$. Note that all of the submatrices of $L_{n,k}$ obtained by deleting the first row and $j$th column with $k<j\le n$ are permutationally equivalent and have the same permanent, say, $r$. Thus, $\operatorname{per}(L_{n,k}) = (n-k)r$.

Next, suppose $n>k\ge 2$. Then for each $\sigma\in\mathcal{P}_{L_{n,k}}$, we have $(1,2)\sigma\in\mathcal{P}_{L_{n,k}}$, and either $\sigma$ or $(1,2)\sigma$ is an even permutation. Thus, half of the elements in $\mathcal{P}_{L_{n,k}}$ belong to $\mathcal{A}_n$. Next, consider the Laplace expansion of $\operatorname{per}(L_{n,k})$ as in the first paragraph of the proof. We claim that $r$ is even. To this end, suppose $A$ is obtained from $L_{n,k}$ by deleting its first row and $(k+1)$st column. Note that for any permutation $\sigma\in\mathcal{P}_A$, we have $\sigma(1,2)\in\mathcal{P}_A$, and either $\sigma$ or $\sigma(1,2)$ is an even permutation (in $\mathcal{S}_{n-1}$). Thus, $|\mathcal{P}_A| = r$ is even. Consequently, $\left|\mathcal{P}_{L_{n,k}}\cap\mathcal{A}_n\right| = \operatorname{per}(L_{n,k})/2 = (n-k)(r/2)$ is also a multiple of $(n-k)$. □

Note that the second assertion of the above proposition is not valid for $\mathcal{P}_{L_n}$. As shown in Section 3, the number of even and odd permutations in $\mathcal{P}_{L_n}$ may be different:

| n : | 3 | 4 | 5 | 6 |
|---|---|---|---|---|
| $\lvert P_{L_n}\rvert$ : | 2 | 9 | 44 | 265 |
| $\lvert P_{L_n}\cap\mathcal{A}_n\rvert$ : | 2 | 3 | 24 | 130 |

Nevertheless, for $(n, k) = (3,1), (4,1), (5,1)$ it is not hard to find a perfect partition for $\mathcal{P}_{L_n} \cap \mathcal{A}_n$; see the results in the last section. In general, we have the following.

**Problem 4.5** Determine whether there is a perfect partition for $\mathcal{P}_{L_{n,k}}$ (respectively, $\mathcal{P}_{L_{n,k}} \cap \mathcal{A}_n$).

Notice that finding a perfect partition for $\mathcal{P}_{L_{n,n/2}}$ is the same as finding a perfect partition for $\mathcal{P}_A$ with $A = J_{n/2} \oplus J_{n/2}$. Examining Case 3 in the proof of Proposition 2.2, we have the following.

**Proposition 4.6** *Suppose $n$ is even. There is always a perfect partition for $\mathcal{P}_{L_{n,n/2}}$ (respectively, $\mathcal{P}_{L_{n,n/2}} \cap \mathcal{A}_n$).*

Answering Problem 4.5 for other values of $(n, k)$ is not so easy. For $(n, k) = (6, 2)$, we have an affirmative answer.

**Proposition 4.7** *There is a perfect partition for $\mathcal{P}_{L_{6,2}}$.*

*Proof.* Let $L_{6,2} = (A_{ij})_{1 \le i,j \le 3}$, where $A_{ii} = 0_2$ for $i = 1, 2, 3$, and $A_{ij} = J_2$ for $i \ne j$. We first show that $|\mathcal{P}_{L_{6,2}}| = 80$. Every permutation matrix in $\mathcal{P}_{L_{6,2}}$ is determined by selecting exactly one nonzero entry from each row and column of $L_{6,2}$. Consider the number of ways to construct a matrix $X \in \mathcal{P}_{L_{6,2}}$ if the 1 in the $(1, 3)$ position of $L_{6,2}$ is selected to be in $X$. In the following discussion, a nonzero entry of $L_{6,2}$ is said to be *available* if no other nonzero entry in its row or column has been selected to be in $X$. We consider two cases, depending on the nonzero entry selected from the second row of $L_{6,2}$.
**Case 1.** If the $(2, 4)$ entry of $L_{6,2}$ is selected to be in $X$, then the remaining four nonzero entries of $X$ must be obtained by selecting two nonzero entries each from the $A_{23}$ and $A_{31}$ submatrices of $L_{6,2}$. The nonzero entries from each of these two submatrices can be selected in one of two ways: either entirely on the submatrix diagonal or entirely off of the submatrix diagonal. Thus, there are $2 \times 2 = 4$ possible ways to construct $X$ in this case.
**Case 2.** If the $(2, 4)$ entry of $L_{6,2}$ is not selected, then there are two available nonzero entries in the second row of $L_{6,2}$ that can be selected; both choices lie in $A_{13}$. Each choice sequentially forces the selection of one of two available nonzero entries each from the $A_{23}$, $A_{21}$, and $A_{31}$ submatrices, thereby determining the final selection of the only available nonzero entry from the $A_{32}$ submatrix. Thus, there are $2^4 = 16$ ways to construct $X$ in this case.

Combining the two preceding cases, there are $4 + 16 = 20$ matrices in $\mathcal{P}_{L_{6,2}}$ with a 1 in the $(1, 3)$ position. By analogous arguments, one can show that there are 20 matrices in $\mathcal{P}_{L_{6,2}}$ with a 1 in the $(1, k)$ position for $k = 4, 5, 6$. Thus, $|\mathcal{P}_{L_{6,2}}| = 4 \times 20 = 80$.

Next, we show that $P_{L_{6,2}}$ admits a perfect partition. Let $T_2 \in \mathcal{S}_2$ correspond to the permutation $(1, 2)$, and let $R_3 \in \mathcal{S}_3$ correspond to the permutation $(1, 3, 2)$. Let

$$W = \{R_3 \otimes I_2, R_3 \otimes T_2, R_3^t \otimes I_2, R_3^t \otimes T_2\}.$$

Then $W$, $(3,4)W$, $(5,6)W$, and $(3,4)(5,6)W$ are disjoint subsets of $\mathcal{P}_{L_{6,2}}$ such that the matrices in each subset sum up to $L_{6,2}$, and each of the four subsets contains exactly one matrix from Case 1 above.

The remaining 64 matrices in $\mathcal{P}_{L_{6,2}}$ can be partitioned as follows. Recall that each of the 16 matrices $X_1, \ldots, X_{16}$ from Case 2 above has exactly one nonzero entry from every $A_{ij}$ in $L_{6,2}$ with $i \neq j$. Now, we associate each matrix $X_r$ from Case 2 with three other matrices $X_{r,2}$, $X_{r,3}$, and $X_{r,4}$ in $\mathcal{P}_{L_{6,2}}$ as determined in the following manner:

$X_{r,2}$: From each nonzero $A_{ij}$, select the entry horizontally adjacent to the entry that was selected to be in $X_r$.

$X_{r,3}$: From each nonzero $A_{ij}$, select the entry vertically adjacent to the entry that was selected to be in $X_r$.

$X_{r,4}$: From each nonzero $A_{ij}$, select the entry diagonal to the entry that was selected to be in $X_r$.

Then we have 16 disjoint sets of the form $\{X_r, X_{r,2}, X_{r,3}, X_{r,4}\}$ such the matrices in each set sum up to $L_{6,2}$. For example, the following four matrices in $\mathcal{P}_{L_{6,2}}$ constitute a set in the partition:

$$X_r = \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&1&0&0&0&0 \end{pmatrix}, \qquad X_{r,2} = \begin{pmatrix} 0&0&0&1&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix},$$

$$X_{r,3} = \begin{pmatrix} 0&0&0&0&1&0\\ 0&0&1&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix}, \qquad X_{r,4} = \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&1&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 1&0&0&0&0&0\\ 0&0&1&0&0&0 \end{pmatrix}.$$

None of the 64 matrices partitioned into sets of the form $\{X_r, X_{r,2}, X_{r,3}, X_{r,4}\}$ were previously used up in sets of the form $\sigma W$, because all matrices belonging to sets of the form $\sigma W$ in the partition have either two or zero entries from each nonzero $A_{ij}$ in $L_{6,2}$. We thus have a perfect partition of $\mathcal{P}_{L_{6,2}}$. □

We close the paper with some general questions.

**Problem 4.8** For which $n \times n$ (0,1)-matrices $A$ does $\mathcal{P}_A$ have a perfect partition?

Note that such matrices $A$ must be regular, and if $k$ is the constant row and column sum, $k$ must be a factor of the permanent of $A$. In addition, the permanental minors of the 1's of $A$ are constant.

**Problem 4.9** Determine a good upper bound on the chromatic number $\chi(G_A)$ of the permutation graph of a regular matrix $A$. More specifically, find a constant $c_n$ such that

$$\chi(G_A) \leq c_n \left\lceil \frac{\text{per}(A)}{k} \right\rceil.$$

An even more general problem is the following.

**Problem 4.10** Let $\mathcal{P} = \{P_i : i \in I\}$ be a set of permutation matrices of order $n$. Let $\mathcal{A}$ be a multiset of $(0,1)$-matrices of order $n$. When is there a partition of $I$ into sets $I_1, I_2, \ldots, I_m$ such that the matrices $\sum\{P_j : j \in I_i\}$, $(i = 1, 2, \ldots, m)$, are the matrices in $\mathcal{A}$, including multiplicities?

The problem discussed in this paper concerns sets of permutation matrices $\mathcal{P}_A$ where $A$ is a $(0,1)$-matrix and $\mathcal{A}$ is the multiset consisting of $A$ with a certain multiplicity.

# References


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