MAPS PRESERVING PERIPHERAL SPECTRUM OF JORDAN PRODUCTS OF OPERATORS

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be (not necessarily unital or closed) standard operator algebras on complex Banach spaces X and Y, respectively. For a bounded linear operator A on X, the peripheral spectrum $\sigma_\pi(A)$ of A is defined by $\sigma_\pi(A) = \{z \in \sigma(A) : |z| = \max_{w\in\sigma(A)} |w|\}$, where $\sigma(A)$ denotes the spectrum of A. Assume that $\Phi : \mathcal{A} \to \mathcal{B}$ is a map and the range of Φ contains all operators with rank at most two. It is proved that the map Φ satisfies the condition that $\sigma_\pi(\Phi(A)\Phi(B) + \Phi(B)\Phi(A)) = \sigma_\pi(AB + BA)$ for all $A, B \in \mathcal{A}$ if and only if either there exists an invertible operator $T \in \mathcal{B}(X, Y)$ such that $\Phi(A) = \varepsilon TAT^{-1}$ for every $A \in \mathcal{A}$; or X and Y are reflexive and there exists an invertible operator $T \in \mathcal{B}(X^*, Y)$ such that $\Phi(A) = \varepsilon TA^*T^{-1}$ for every $A \in \mathcal{A}$, where $\varepsilon \in \{1, -1\}$. Furthermore, the same conclusion holds if $\mathcal{A}$ and $\mathcal{B}$ are replaced by standard real Jordan algebras of self-adjoint operators on complex Hilbert spaces. If X and Y are complex Hilbert space, we characterize also maps preserving the peripheral spectrum of the product $AB^* + B^*A$, and prove that such maps are of the form $A \mapsto \gamma UAU^*$ or $A \mapsto \gamma UA^tU^*$, where $U \in \mathcal{B}(X, Y)$ is a unitary operator and $\gamma \in \mathbb{C}$ with $|\gamma| = 1$, A^t denotes the transpose of A for an arbitrary but fixed orthonormal basis of X.

1. Introduction

Surjective linear maps between Banach algebras which preserve the spectrum are extensively studied in connection with a longstanding open problem sometimes called Kaplansky's problem on invertibility preserving linear maps. A weaker version of that problem reads as follows. *Is it true that between semi-simple complex Banach algebras every surjective linear map which preserves the spectrum is a Jordan homomorphism?* For the algebra of all bounded linear operators acting on a Banach space this was proved to be true by Jafarian and Sourour in [18] (see, also [1]-[2], [6]-[7], [9]-[10], [13]). Clearly, general spectrum preserving transformations can be almost arbitrary. So, one has to impose some restrictions on the maps under consideration (see, for example, [4] and [8]). In [22], Molnár characterized surjective maps ϕ on bounded linear operators acting on a Hilbert space preserving the spectrum of the product of operators, i.e., AB and $\phi(A)\phi(B)$ always have the same spectrum. Recently, Hou, Li and Wong [16]-[17] studied maps Φ between certain operator algebras preserving the spectrum of a general product $T_1 * T_2 * \cdots * T_k$, namely, $T_1 * \cdots * T_k$ and $\Phi(T_1) * \cdots * \Phi(T_k)$ has the same spectrum. The general product $T_1 * \cdots * T_k$ covers the usual product T_1T_2 and the Jordan product $T_1T_2 + T_2T_1$, and the triple one $T_1T_2T_1$ and $T_1T_2T_3 + T_3T_2T_1$. The purpose of this paper is to study maps preserving the peripheral spectrum of Jordan product of operators.

2002 Mathematical Subject Classification. 47B49, 47A12.

Key words and phrases. Peripheral spectrum, Jordan products, Banach space, nonlinear preserver maps.

The first author was supported by National Natural Science Foundation of China (No.10871111) and the Specialized Research Fund for Doctoral Program of Higher Education (No.200800030059). The research was done while she was visiting the College of William and Mary in the academic year 2009-2010 supported by the China Scholarship Council. The second author was partially supported by USA NSF and the William and Mary Plumeri Award. He is an honorary professor of the University of Hong Kong.

Let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators on a complex Banach space X. Denote by $\sigma(T)$ and $r(T)$ the spectrum and the spectral radius of $T \in \mathcal{B}(X)$, respectively. The *peripheral spectrum* of T is defined by

$$\sigma_\pi(T) = \{z \in \sigma(T) : |z| = r(T)\}.$$

Since $\sigma(T)$ is compact, $\sigma_\pi(T)$ is a well-defined non-empty set. Brešar and Šemrl in [6] characterized linear maps preserving spectral radius on $\mathcal{B}(X)$, a key step was reducing spectral radius linear preservers to peripheral spectrum linear preservers. In [3], Bai and Hou generalized the result in [6] to additive spectral radius preservers on $\mathcal{B}(X)$. In [24], Luttman and Tonev studied maps preserving peripheral spectrum of the usual operator products on standard operator algebras. It was proved that, if such a map is surjective, then it must be a positive or negative multiple of an isomorphism or an anti-isomorphism. They studied also the corresponding problems in uniform algebras; see [20] and [21]. Recently, Takeshi and Dai generalized the result in [24], and characterized surjective maps ϕ and ψ satisfying $\sigma_\pi(\phi(T)\psi(S)) = \sigma_\pi(TS)$ on standard operator algebras (see [23]). In particular, the authors characterized surjective maps between standard operator algebras on Hilbert spaces that preserve the peripheral spectrum of skew products T^*S of operators.

In this paper, we characterize maps between standard operator algebras on Banach spaces preserving peripheral spectrum of Jordan products of operators under the mild assumption on the ranges of the maps containing operators with rank at most two. In particular, we will show that such a map is a positive or negative multiple of an isomorphism or an anti-isomorphism (see Theorem 2.1). We obtain similar results on the standard real Jordan algebra of Hilbert space self-adjoint operators and characterize also maps preserving peripheral spectrum of the product $AB^* + B^*A$ on Hilbert space (see Theorem 3.1 and Theorem 4.1).

Several remarks about our study and proofs are in order. In the study of preserver problems on standard operator algebras, one often reduces the problems to rank one preservers. To this end, one often has to obtain new characterizations of rank one operators using the concepts related to the preserver problems. For the Jordan product of self-adjoint operators, it is impossible to get such a characterization because the peripheral spectrum of a self-adjoint operator has at most two different points. Hence, new arguments are needed to show that the preservers indeed leave invariant rank one operators; see Lemma 3.4. For the Jordan product on standard operator algebras and the product $AB^* + B^*A$ on Hilbert space, we reduce our problem to rank one nilpotent preservers and use some existing results (Lemmas 2.5 and 2.6) in this study. This leads to a more efficient proof and a relaxation of the requirement of the range of Φ. In fact, if one reduces the problem to rank one idempotent preservers as in [16, 17], one has to assume that the range of Φ contains rank 3 or less operators, whereas our results only require that the range of Φ contains rank 2 or less operators. This improves the surjective assumption on the preservers as done in other study (for example, see [20, 21, 23, 24]). However, if one completely removes the assumption on the range of the maps, one may get non-standard peripheral spectrum preserving maps. For instance, for a Hilbert space H, the map $\Phi : B(H) \to B(H \oplus H)$ defined by $\Phi(A) = A \oplus f(A)N$ will preserve peripheral spectrum, where N is a nilpotent operator and $f : \mathcal{B}(H) \to \mathbb{C}$ is a functional. Nevertheless, it would be interesting to further improve our results by weakening the assumption on the range of Φ. It is also interesting to note that our proofs require only that the domain of the map is a linear space, and require only that the condition $\sigma_\pi(\Phi(A)\Phi(B) + \Phi(B)\Phi(A)) = \sigma_\pi(AB + BA)$ holds when A or B has rank at most two. For simplicity, we do not include these in the statements of our results. Finally, our study can be viewed as a step towards the study of the more challenging problem on spectral radius preservers. Our proofs may be useful or inspiring for the study of this and other preserver problems on Jordan product of operators such as the norm preservers and numerical radius preservers.

To conclude the introduction, we fix some notation for our discussion. Let X be a complex Banach space, and denote by X^* the dual space of X. Denote by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X. For $A \in \mathcal{B}(X)$, A^* denotes the adjoint operator of A from X^* to itself. For $x \in X$ and $f \in X^*$, a rank one operator $x \otimes f$ is defined by $(x \otimes f)z = f(z)x$ for every $z \in X$, and every rank one operator in $\mathcal{B}(X)$ can be written in this form. Recall that a standard operator algebra $\mathcal{A}$ on a Banach space X is a subalgebra of $\mathcal{B}(X)$ containing all finite rank linear operators on X. We do not assume $\mathcal{A}$ contains the identity operator I on X, or $\mathcal{A}$ is closed, however. For a finite rank operator T, $\operatorname{rank} T$ denotes the rank of T, that is, the dimension of the range of T. For $A \in \mathcal{B}(X)$, $\ker A$ denote the kernel of A. Let $\mathbb{C}$ and $\mathbb{R}$ denote respectively the complex field and real field. For a ring automorphism τ of $\mathbb{C}$, a transformation T on X is said to be τ-linear if $T(\lambda x) = \tau(\lambda)Tx$ for every $x \in X$ and $\lambda \in \mathbb{C}$.

2. Jordan products of operators on Banach spaces

In this section, we will study maps between standard operator algebras on complex Banach spaces preserving peripheral spectrum of Jordan products of operators. The main result is the following.

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras (not necessarily unital or closed) on complex Banach spaces X and Y, respectively. Assume that $\Phi : \mathcal{A} \to \mathcal{B}$ is a map of which range contains all operators with rank at most two. Then Φ satisfies*

$$\sigma_\pi(\Phi(A)\Phi(B) + \Phi(B)\Phi(A)) = \sigma_\pi(AB + BA) \qquad \text{for } A, B \in \mathcal{A}$$

if and only if one of the followings holds:
(1) *There exist $\varepsilon \in \{1, -1\}$ and an invertible operator $T \in \mathcal{B}(X, Y)$ such that*

$$\Phi(A) = \varepsilon TAT^{-1} \qquad \text{for every } A \in \mathcal{A}.$$

(2) *The spaces X and Y are reflexive, and there exist $\varepsilon \in \{1, -1\}$ and an invertible operator $T \in \mathcal{B}(X^*, Y)$ such that*

$$\Phi(A) = \varepsilon TA^*T^{-1} \qquad \text{for every } A \in \mathcal{A}.$$

If $\dim X \leq 2$, no assumption on the range of Φ is needed.

We assume always that $\dim X \geq 3$ in the following Lemma 2.2-Lemma 2.4.

Lemma 2.2. *Let $x \in X$ and $f \in X^*$. Then, for every $B \in \mathcal{A}$,*

$$\begin{aligned} &\sigma_\pi(Bx \otimes f + x \otimes fB) \\ = &\begin{cases} \{f(Bx)\} & \text{if } f(x) = 0 \text{ or } f(B^2x) = 0, \\ \{\pm\sqrt{f(B^2x)f(x)}\} & \text{if } f(x) \neq 0, f(Bx) = 0, f(B^2x) \neq 0, \\ \{\alpha\} & \text{if } f(x) \neq 0, f(Bx) \neq 0, f(B^2x) \neq 0, \end{cases} \end{aligned}$$

where the scalar α satisfies that

$$|\alpha| = \max\{|f(Bx) + \sqrt{f(B^2x)f(x)}|, |f(Bx) - \sqrt{f(B^2x)f(x)}|\}.$$

Proof. Let $x \in X$ and $f \in X^*$. For any $B \in \mathcal{A}$, $Bx \otimes f + x \otimes fB$ is of rank at most two. If $Bx \otimes f + x \otimes fB$ is nilpotent, then $\sigma_\pi(Bx \otimes f + x \otimes fB) = \{0\}$ and $f(x) = f(Bx) = 0$ or $f(Bx) = f(B^2x) = 0$. Now assume that there exist nonzero scalars $\alpha \in \sigma(Bx \otimes f + x \otimes fB)$. Then there exist nonzero vectors $z \in X$ such that $(Bx \otimes f + x \otimes fB)z = \alpha z$, that is,

$$f(z)Bx + f(Bz)x = \alpha z. \tag{2.1}$$

It follows that

$$f(z)f(Bx) + f(Bz)f(x) = \alpha f(z) \tag{2.2}$$

and

$$f(z)f(B^2x) + f(Bz)f(Bx) = \alpha f(Bz). \tag{2.3}$$

We consider the following several cases.

Case 1. $f(x) = 0$.

If $f(z) \neq 0$, it follows from (2.2) that $\alpha = f(Bx)$; if $f(z) = 0$, then Eq. (2.1) implies that $f(Bz) \neq 0$, and hence, it follows from (2.3) that $\alpha = f(Bx)$. So $\sigma(Bx \otimes f + x \otimes fB) = \{f(Bx), 0\}$, and $\sigma_\pi(Bx \otimes f + x \otimes fB) = \{f(Bx)\}$.

Case 2. $f(B^2x) = 0$.

If $f(Bz) \neq 0$, Eq. (2.3) implies that $\alpha = f(Bx)$; if $f(Bz) = 0$, then Eq. (2.1) implies that $f(z) \neq 0$, and hence, Eq. (2.2) infers that $\alpha = f(Bx)$. So $\sigma(Bx \otimes f + x \otimes fB) = \{f(Bx), 0\}$, and therefore, $\sigma_\pi(Bx \otimes f + x \otimes fB) = \{f(Bx)\}$.

Case 3. $f(x) \neq 0$ and $f(B^2x) \neq 0$.

In this case, there must be $f(Bz) \neq 0$ and $f(z) \neq 0$, and hence, it follows from Eqs. (2.2) and (2.3) that

$$(\alpha - f(Bx))^2 = f(B^2x)f(x),$$

which implies that $\alpha = f(Bx) \pm \sqrt{f(B^2x)f(x)}$, so

$$\sigma(Bx \otimes f + x \otimes fB) = \{f(Bx) \pm \sqrt{f(B^2x)f(x)}, 0\}.$$

Now the result follows from the above expression. □

Next we characterize rank one nilpotent operators by peripheral spectrum of Jordan products.

Lemma 2.3. *Let $A \in \mathcal{A}$ be nonzero. Then the following statements are equivalent.*

(1) *A is rank one nilpotent.*

(2) *For any $B \in \mathcal{A}$, $\sigma_\pi(AB + BA)$ is a singleton.*

(3) *For any $B \in \mathcal{A}$ with $\operatorname{rank} B \leq 2$, $\sigma_\pi(AB + BA)$ is a singleton.*

Proof. (1)⇒(2) follows from Lemma 2.2. (2)⇒ (3) is clear.

To prove (3)⇒ (1), assume, on the contrary, that $\operatorname{rank} A \geq 2$. Assume first that $A^2 = 0$. Then there exist linearly independent vectors $x_1, x_2 \in X$ such that Ax_1 and Ax_2 are linearly independent. It follows from $A^2 = 0$ that $\{x_1, x_2, Ax_1, Ax_2\}$ is a linearly independent set. Let N be a closed subspace of X such that

$$X = \operatorname{span}\{x_1, x_2, Ax_1, Ax_2\} \oplus N.$$

Then A has an operator matrix

$$A = \begin{pmatrix} 0_2 & 0_2 & * \\ I_2 & 0_2 & * \\ 0 & 0 & * \end{pmatrix}.$$

According to the corresponding space decomposition, take $B = \begin{pmatrix} 0 & B_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ with $B_{12} \in M_2$ (the set of all 2×2 complex matrices) and $\sigma(B_{12}) = \{-1, 1\}$. Then $\operatorname{rank} B = 2$ and

$$AB + BA = \begin{pmatrix} B_{12} & 0 & * \\ 0 & B_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

it follows that $\sigma(AB + BA) \setminus \{0\} = \sigma(B_{12}) = \{-1, 1\}$. So $\sigma_\pi(AB + BA) = \{-1, 1\}$, a contradiction.

Next assume that $A^2 \neq 0$. Consider the following two cases.

Case 1. There exists $x \in X$ such that $\{x, Ax, A^2x\}$ is a linearly independent set.

Write $x_1 = x$, $x_2 = Ax$ and $x_3 = A^2x$. Then $Ax_1 = x_2$ and $Ax_2 = x_3$. By Hahn-Banach Theorem, there exist $f_i \in X^*$ such that $f_i(x_j) = \delta_{ij}$ (Kronecker symbol), $i, j = 1, 2, 3$. Pick $B = x_1 \otimes f_2 - x_2 \otimes f_3$, then $\operatorname{rank} B = 2$ and $\sigma_\pi(AB + BA) = \{-1, 1\}$, a contradiction.

Case 2. For every $x \in X$, $\{x, Ax, A^2x\}$ is a linearly dependent set.

Then A is a locally algebraic operator, and hence Kaplansky Lemma (see, for example, [19]) tells us that A is an algebraic operator, so there exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that

$$\alpha A^2 + \beta A + \gamma I = 0, \qquad (\alpha, \beta, \gamma) \neq (0, 0, 0). \tag{2.4}$$

If $\alpha = 0$, then $\beta \neq 0$ and $\gamma \neq 0$, and therefore, A is a scalar operator. Take a rank two operator $B \in \mathcal{A}$ such that $\sigma(B) = \{1, -1, 0\}$, then $\sigma_\pi(AB + BA)$ contains two different points, a contradiction. Now assume that $\alpha \neq 0$ in Eq. (2.4). Since $A^2 \neq 0$, it follows that $(\beta, \gamma) \neq (0, 0)$, and $\sigma(A) = \{a, b\}$, where $a, b \in \mathbb{C}$.

Subcase 1°. $\sigma(A) = \{a, b\}$ with $a \neq 0$ and $b \neq 0$.

Then there exist linearly independent vectors x_1, $x_2 \in X$ such that $Ax_1 = ax_1$ and $Ax_2 = bx_2$. By Hahn-Banach Theorem, there exist $f_i \in X^*$ such that $f_i(x_j) = \delta_{ij}$ $(i, j = 1, 2)$. Take $B = \frac{1}{a}x_1 \otimes f_1 - \frac{1}{b}x_2 \otimes f_2$. Then $\operatorname{rank} B = 2$ and $\sigma_\pi(AB + BA) = \{-2, 2\}$, a contradiction.

Subcase 2°. $\sigma(A) = \{a, 0\}$ with $a \neq 0$.

There exist linearly independent vectors $x_1, x_2 \in X$ such that $Ax_1 = ax_1$ and $Ax_2 = 0$. So there exist $f_i \in X^*$ such that $f_i(x_j) = \delta_{ij}$ $(i, j = 1, 2)$. Pick $B = x_1 \otimes f_2 + x_2 \otimes f_1$. Then $\operatorname{rank} B = 2$ and $\sigma_\pi(AB + BA) = \{-a, a\}$, a contradiction. This completes the proof. □

Lemma 2.4. *Let Φ satisfy assumptions in Theorem 2.1. Then, for $A \in \mathcal{A}$, $\Phi(A) = 0$ if and only if $A = 0$.*

Proof. Let $\Phi(A) = 0$. For any $x \in X$ and $f \in X^*$ with $f(x) = 0$, it follows from Lemma 2.2 that

$$\{0\} = \sigma_\pi(\Phi(A)\Phi(x \otimes f) + \Phi(x \otimes f)\Phi(A)) = \sigma_\pi(Ax \otimes f + x \otimes fA) = \{f(Ax)\},$$

so there exists $\alpha \in \mathbb{C}$ such that $A = \alpha I$. Now take a rank one idempotent $y \otimes g$, then

$$\{0\} = \sigma_\pi(\Phi(A)\Phi(y \otimes g) + \Phi(y \otimes g)\Phi(A)) = \sigma_\pi(Ay \otimes g + y \otimes gA) = \{2\alpha\}$$

implies that $\alpha = 0$, and hence $A = 0$.

Next we prove that $\Phi(0) = 0$. For any $y \in X$ and $g \in X^*$ with $g(y) = 0$, since the range of Φ contains all operators with rank at most two, there exists $A \in \mathcal{A}$ such that $\Phi(A) = y \otimes g$. It follows from Lemma 2.2 that

$$\{g(\Phi(0)y)\} = \sigma_\pi(\Phi(0)y \otimes g + y \otimes g\Phi(0)) = \sigma_\pi(\Phi(0)\Phi(A) + \Phi(A)\Phi(0)) = \{0\},$$

and therefore, $\Phi(0) = \beta I$ for some $\beta \in \mathbb{C}$. Similar to the previous discussion, we have $\beta = 0$ and $\Phi(0) = 0$. $\square$

Let $\mathcal{N}_1(X)$ and $\mathcal{N}_1(Y)$ denote the set of all rank one nilpotent operators on Banach spaces X and Y, respectively. The following result was proved in [11, Lemma 2.2].

Lemma 2.5. *Let X and Y be complex Banach spaces with dimension at least 3. Suppose that $\Phi : \mathcal{N}_1(X) \to \mathcal{N}_1(Y)$ is a bijective map with the property that*

$$A + B \in \mathcal{N}_1(X) \Leftrightarrow \Phi(A) + \Phi(B) \in \mathcal{N}_1(Y)$$

for all $A, B \in \mathcal{N}_1(X)$. Then there exists a ring automorphism τ of $\mathbb{C}$ such that one of the followings holds.

(1) *There exists a τ-linear transformation $T : X \to Y$ satisfying $\Phi(N) = \lambda_N TNT^{-1}$ for all $N \in \mathcal{N}_1(X)$, where λ_N is a scalar depending on N.*

(2) *There exists a τ-linear transformation $T : X^* \to Y$ satisfying $\Phi(N) = \lambda_N TN^*T^{-1}$ for all $N \in \mathcal{N}_1(X)$, where λ_N is a scalar depending on N.*

If X is infinite dimensional, the transformation T is an invertible bounded linear or conjugate linear operator.

Let M_2' denote the set of trace zero matrices in M_2. The following result was proved in [14] (see also [5]).

Lemma 2.6. *A linear map $\Phi : M_2' \to M_2'$ preserves rank one nilpotent matrices if and only if there exist a nonzero scalar c and a nonsingular matrix $T \in M_2$ such that one of the followings holds:*

(1) $\Phi(A) = cTAT^{-1}$ *for all $A \in M_2'$.*

(2) $\Phi(A) = cTA^tT^{-1}$ *for all $A \in M_2'$, where A^t denotes the transpose of A.*

Proof of Theorem 2.1. The proof of the theorem will be completed after proving the following several claims.

Claim 1. *If* $\dim X \leq 2$ *or* $\dim Y \leq 2$, *then* $\dim X = \dim Y$.

Suppose $\dim X = m \leq 2$. If $\dim Y = n < m$, then we can find $A, B \in \mathcal{A}$ so that $\sigma_\pi(AB + BA)$ has m points. Since $\sigma_\pi(\Phi(A)\Phi(B) + \Phi(B)\Phi(A))$ has at most n points, we see that $\sigma_\pi(AB+BA) \neq \sigma_\pi(\Phi(A)\Phi(B)+\Phi(B)\Phi(A))$. Now, if $n > m$, then we can find $C, D \in \mathcal{B}$ with rank at most two such that $\sigma_\pi(CD + DC)$ has $m + 1$ points. But then we cannot find $A, B \in \mathcal{A}$ satisfying $\Phi(A) = C$ and $\Phi(B) = D$ such that $\sigma_\pi(AB + BA) = \sigma_\pi(CD + DC)$ has $m + 1$ points. Thus, $\dim X = \dim Y$. We can obtain the same conclusion by a similar argument if $\dim Y \leq 2$.

Next we divide the proof into two cases $\dim X \geq 3$ and $\dim X \leq 2$. First assume that $\dim X \geq 3$. Then, by Claim 1, there must be $\dim Y \geq 3$.

Claim 2. Φ preserves rank one nilpotent operators in both directions.

Assume that A is rank one nilpotent, then Lemma 2.4 ensures that $\Phi(A) \neq 0$. For any $B \in \mathcal{A}$, Lemma 2.3 implies that $\sigma_\pi(\Phi(A)\Phi(B) + \Phi(B)\Phi(A)) = \sigma_\pi(AB + BA)$ is a singleton. Since the range of Φ contains all operators with rank at most two, applying again Lemma 2.3, one has $\Phi(A)$ is rank one nilpotent. Conversely, a similar discussion infers that $\Phi(A)$ is rank one nilpotent implies that A is rank one nilpotent.

Claim 3. For any $A_1, A_2 \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{C}$, there exists $\delta_{A_1,A_2} \in \mathbb{C}$ such that

$$\Phi(\lambda A_1 + \mu A_2) = \lambda\Phi(A_1) + \mu\Phi(A_2) + \delta_{A_1,A_2} I.$$

In particular, for every rank one nilpotent element $A \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, $\Phi(\lambda A) = \lambda\Phi(A)$.

Let $A_1, A_2 \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{C}$ be arbitrary. For any $y \in X$ and $g \in X^*$ with $g(y) = 0$, Claim 2 implies that there exist $x \in X$ and $f \in X^*$ with $f(x) = 0$ such that $\Phi(x \otimes f) = y \otimes g$. It follows from Lemma 2.2 that for $i = 1, 2$,

$$\{g(\Phi(A_i)y)\} = \sigma_\pi(\Phi(A_i)\Phi(x \otimes f) + \Phi(x \otimes f)\Phi(A_i)) = \sigma_\pi(A_i x \otimes f + x \otimes f A_i) = \{f(A_i x)\}.$$

So

$$g((\lambda\Phi(A_1) + \mu\Phi(A_2))y) = f((\lambda A_1 + \mu A_2)x). \tag{2.5}$$

Also

$$\begin{aligned} &\{g(\Phi(\lambda A_1 + \mu A_2)y)\} = \sigma_\pi(\Phi(\lambda A_1 + \mu A_2)y \otimes g + y \otimes g\Phi(\lambda A_1 + \mu A_2)) \\ = \;& \sigma_\pi(\Phi(\lambda A_1 + \mu A_2)\Phi(x \otimes f) + \Phi(x \otimes f)\Phi(\lambda A_1 + \mu A_2)) \\ = \;& \{f((\lambda A_1 + \mu A_2)x)\}. \end{aligned} \tag{2.6}$$

Combining Eqs. (2.5) and (2.6), one has $g(\Phi(\lambda A_1 + \mu A_2)y) = g((\lambda\Phi(A_1) + \mu\Phi(A_2))y)$ for any rank one nilpotent $y \otimes g$, and consequently, there exists $\delta_{A_1,A_2} \in \mathbb{C}$ such that

$$\Phi(\lambda A_1 + \mu A_2) = \lambda\Phi(A_1) + \mu\Phi(A_2) + \delta_{A_1,A_2} I.$$

In particular, for any rank one nilpotent operator $A_1 \in \mathcal{A}$ and any $\lambda \in \mathbb{C}$, we have $\Phi(\lambda A_1) = \lambda\Phi(A_1) + \delta_{A_1} I$, note that $\Phi(\lambda A_1)$ and $\Phi(A_1)$ are of rank one, so $\delta_{A_1} = 0$. That is, Φ is homogeneous when restricted on the set of rank one nilpotent operators.

Claim 4. One of the following results holds:

(1) There exists a bounded invertible linear operator $T : X \to Y$ such that $\Phi(N) = \varepsilon TNT^{-1}$ for every $N \in \mathcal{N}_1(X)$, where $\varepsilon \in \{-1, 1\}$.

(2) There exists a bounded invertible linear operator $T : X^* \to Y$ such that $\Phi(N) = \varepsilon TN^*T^{-1}$ for every $N \in \mathcal{N}_1(X)$, where $\varepsilon \in \{-1, 1\}$.

Since Φ preserves rank one nilpotent operators in both directions, it follows that $\Phi : \mathcal{N}_1(X) \to \mathcal{N}_1(Y)$ is surjective. To prove injectivity, assume that $\Phi(P) = \Phi(Q)$ for $P, Q \in \mathcal{N}_1(X)$. By Claim 3, there exists $\delta \in \mathbb{C}$ such that $\delta I = \Phi(P) - \Phi(Q) + \delta I = \Phi(P - Q)$. Note that $P^2 = 0$, $Q^2 = 0$ and $\Phi(P)^2 = 0$. We have $\{\delta^2\} = \sigma_\pi(2(\Phi(P - Q))^2) = \sigma_\pi(2(P - Q)^2) = -2\sigma_\pi(PQ + QP) = -2\sigma_\pi(\Phi(P)\Phi(Q) + \Phi(Q)\Phi(P)) = \{0\}$, and hence $\delta = 0$. Now Lemma 2.4 implies that $P = Q$. Hence Φ is injective, and thus $\Phi : \mathcal{N}_1(X) \to \mathcal{N}_1(Y)$ is bijective.

Next we prove that Φ satisfies the condition in Lemma 2.5. For any $P, Q \in \mathcal{N}_1(X)$ with $P+Q \in \mathcal{N}_1(X)$, Claims 2 and 3 imply that $\Phi(P)+\Phi(Q) = \Phi(P+Q) \in \mathcal{N}_1(Y)$. Conversely, for $P, Q \in \mathcal{N}_1(X)$, Claim 2 entails that $\Phi(P), \Phi(Q) \in \mathcal{N}_1(Y)$. Assume that $\Phi(P)+\Phi(Q) \in \mathcal{N}_1(Y)$. Then, Claim 2 implies again that there exists $R \in \mathcal{N}_1(X)$ such that $\Phi(R) = \Phi(P) + \Phi(Q)$. For any $x \in X$ and $f \in X^*$ with $f(x) = 0$, there exist $y \in Y$ and $g \in Y^*$ with $g(y) = 0$ such that $\Phi(x \otimes f) = y \otimes g$. It follows from Lemma 2.2 that $f(Px) = g(\Phi(P)y)$, $f(Qx) = g(\Phi(Q)y)$ and

$$f(Rx) = g(\Phi(R)y) = g(\Phi(P)y) + g(\Phi(Q)y),$$

so $f((P + Q)x) = f(Rx)$, and therefore, $P + Q = R \in \mathcal{N}_1(X)$. Now Lemma 2.5 implies that there exists a ring automorphism τ of $\mathbb{C}$ such that either

(i) there exists a τ-linear transformation $T : X \to Y$ satisfying $\Phi(N) = \lambda_N TNT^{-1}$ for all $N \in \mathcal{N}_1(X)$, where λ_N a scalar depending on N; or

(ii) there exists a τ-linear transformation $T : X^* \to Y$ satisfying $\Phi(N) = \lambda_N TN^*T^{-1}$ for all $N \in \mathcal{N}_1(X)$, where λ_N a scalar depending on N.

And if X is infinite dimensional, the transformation T is an invertible bounded linear or conjugate linear operator.

Assume that Φ is of the form (i). For any $x \in X$ and $f, g \in X^*$ with $f(x) = 0$ and $g(x) = 0$, there exist scalars $\lambda_{x,f}, \lambda_{x,g}$ and $\lambda_{x,f,g}$ such that $\Phi(x \otimes f) = \lambda_{x,f} Tx \otimes fT^{-1}$, $\Phi(x \otimes g) = \lambda_{x,g} Tx \otimes gT^{-1}$ and $\Phi(x \otimes (f+g)) = \lambda_{x,f,g} Tx \otimes (f+g)T^{-1}$. Claim 3 implies that $\Phi(x \otimes (f+g)) = \Phi(x \otimes f) + \Phi(x \otimes g)$, so it follows that $\lambda_{x,f} = \lambda_{x,g} = \lambda_{x,f,g} = \varepsilon$ is a nonzero constant. Claim 3 implies again that, for every $A \in \mathcal{N}_1(X)$ and any $\alpha \in \mathbb{C}$, we have $\Phi(\alpha A) = \alpha \Phi(A)$, so $\tau(\alpha) = \alpha$, and hence T is linear. So there exist $\varepsilon \in \mathbb{C}$ and an invertible operator $T \in \mathcal{B}(X, Y)$ such that $\Phi(N) = \varepsilon TNT^{-1}$ for every $N \in \mathcal{N}_1(X)$.

Take arbitrary $x, y \in X$ and $f, g \in X^*$ such that $f(x) = 1$, $g(y) = 1$, $g(x) = 0$ and $f(y) = 0$. Then it follows from $\{\varepsilon^2\} = \sigma_\pi(\Phi(x \otimes g)\Phi(y \otimes f) + \Phi(y \otimes f)\Phi(x \otimes g)) = \sigma_\pi(x \otimes f + y \otimes g) = \{1\}$ that $\varepsilon = \pm 1$, and hence $\Phi(N) = \varepsilon TNT^{-1}$ for every $N \in \mathcal{N}_1(X)$, where $\varepsilon \in \{-1, 1\}$.

If Φ has the form (ii), one can similarly obtain that (2) holds in Claim 4.

Claim 5. Φ has the form in Theorem 2.1.

Assume that the case (1) in Claim 4 holds. If $\varepsilon = -1$, consider $-\Phi$, then $-\Phi$ has the same property as Φ has. So assume that $\varepsilon = 1$. Let $A \in \mathcal{A}$ be arbitrary. For any $x \in X$ and $f \in X^*$ with $f(x) = 0$, we have

$$\begin{aligned} \{f(Ax)\} &= \sigma_\pi(Ax \otimes f + x \otimes fA) \\ &= \sigma_\pi(\Phi(A)\Phi(x \otimes f) + \Phi(x \otimes f)\Phi(A)) \\ &= \sigma_\pi(\Phi(A)Tx \otimes fT^{-1} + Tx \otimes fT^{-1}\Phi(A)) \\ &= \sigma_\pi(T^{-1}\Phi(A)Tx \otimes f + x \otimes fT^{-1}\Phi(A)T) \\ &= \{f(T^{-1}\Phi(A)Tx)\}, \end{aligned}$$

and consequently, there exists $\alpha_A \in \mathbb{C}$ such that $\Phi(A) = TAT^{-1} + \alpha_A I$. We claim that $\alpha_A = 0$ for every $A \in \mathcal{A}$. For any rank one idempotent P, we have $\{1\} = \sigma_\pi(P) = \sigma_\pi(\Phi(P)^2) = \{(\alpha_P + 1)^2\}$, so $\alpha_P = 0$ or $\alpha_P = -2$. Take an arbitrary rank one idempotent $Q \in \mathcal{A}$ such that $PQ = 0 = QP$. Then it can be easily checked that $\alpha_P = 0$ for every rank one idempotent P. If A is a scalar operator, written as $A = \xi I$. Take an arbitrary rank one idempotent P, then $\{2(\xi + \alpha_A)\} = \sigma_\pi(\Phi(A)\Phi(P) + \Phi(P)\Phi(A)) = \sigma_\pi(AP + PA) = \{2\xi\}$ implies that $\alpha_A = 0$. So now assume that A is not a scalar operator. Then there exists $x \in X$ such that Ax and x are linearly independent. Take $f \in X^*$ such that $f(x) = 1$ and $f(Ax) = 0$. If $f(A^2x) \neq 0$, Lemma 2.2 implies that

$$\{\pm\sqrt{f(A^2x)}\} = \sigma_\pi(Ax \otimes f + x \otimes fA) = \sigma_\pi((A + \alpha_A)x \otimes f + x \otimes f(A + \alpha_A)),$$

which, together with Lemma 2.2 again, implies that $f((A + \alpha_A I)x) = 0$, and therefore, $\alpha_A = 0$; if $f(A^2x) = 0$, applying again Lemma 2.2, one has

$$\{0\} = \sigma_\pi(Ax \otimes f + x \otimes fA) = \sigma_\pi(Ax \otimes f + x \otimes fA + 2\alpha_A x \otimes f),$$

so $\alpha_A = 0$. Thus $\Phi(A) = TAT^{-1}$ for every $A \in \mathcal{A}$.

If Φ has the form (2) in Claim 4, a similar discussion implies that $\Phi(A) = \varepsilon TA^*T^{-1}$ for every $A \in \mathcal{A}$, where $\varepsilon \in \{-1, 1\}$. In this case, X and Y are reflexive. This can be easily proved since Φ preserves rank one operators in both directions and $T : X^* \to Y$ is bijective. This completes the proof of the case $\dim X \geq 3$.

Now we deal with the case $\dim X \leq 2$. By Claim 1, it follows that $\dim X = \dim Y \leq 2$. If $\dim X = \dim Y = 1$, then, clearly, Φ is a positive or negative multiple of the identity map of $\mathbb{C}$. Next, suppose $\dim X = \dim Y = 2$. Then we can identify $\mathcal{A} = \mathcal{B} = M_2$.

Claim 6. $\Phi(I) = \xi I$ and for every $A \in M_2$, $\sigma_\pi(A) = \xi\sigma_\pi(\Phi(A))$, where $\xi \in \{-1, 1\}$.

Obviously, a matrix $A \in M_2$ satisfies $\sigma_\pi(AN + NA) = 0$ for all rank one nilpotent $N \in M_2$ if and only if A is a scalar matrix. Since Φ is surjective and preserves rank one nilpotents, we have $\Phi(\mathbb{C}I) = \mathbb{C}I$. Note that $\sigma_\pi(\Phi(I)^2) = \sigma_\pi(I) = \{1\}$, we see that $\Phi(I) = \xi I$ with $\xi \in \{1, -1\}$ and for every $A \in M_2$, $\sigma_\pi(A) = \xi\sigma_\pi(\Phi(A))$.

Claim 7. *Let M_2' denote the set of trace zero matrices in M_2. Then there is a scalar $\varepsilon \in \{-1, 1\}$ and a nonsingular matrix $S \in M_2$ such that*

$$(1)\ \Phi(A) = \varepsilon SAS^{-1} \text{ for all } A \in M_2', \quad \text{or} \quad (2)\ \Phi(A) = \varepsilon SA^tS^{-1} \text{ for all } A \in M_2'.$$

A matrix $A \in M_2$ is rank one nilpotent if and only if $\sigma_\pi(A) = 0$, and consequently, it follows from Claim 6 that $\Phi(A)$ is rank one nilpotent if and only if A is rank one nilpotent. Define $\Psi(A) = \Phi(A) - (\mathrm{tr}\Phi(A))I/2$ for every $A \in M_2$, where $\mathrm{tr}A$ denotes the trace of a matrix A. Thus, if $B \in M_2'$ is rank one nilpotent, then $\Psi(B) = \Phi(B)$ is rank one nilpotent, and $\Psi : M_2' \to M_2'$ preserves rank one nilpotent matrices.

Next we prove that $\Psi : M_2' \to M_2'$ is an invertible linear map. Let $A_1, A_2, A_3 \in M_2'$ be three linearly independent rank one nilpotent matrices. Then for any $B \in M_2'$,

$$\mathrm{tr}(A_jB) = \mathrm{tr}(\Phi(A_j)\Phi(B)) = \mathrm{tr}(\Phi(A_j)\Psi(B)).$$

Let $R(A) = (2a_{11}, a_{12}, a_{21})$ for $A = (a_{ij}) \in M_2'$ and $C(B) = (b_{11}, b_{21}, b_{12})^t$ for $B = (b_{ij}) \in M_2'$. Then $\mathrm{tr}(AB) = R(A)C(B)$. Suppose

$$T = \begin{pmatrix} R(A_1) \\ R(A_2) \\ R(A_3) \end{pmatrix} \qquad \text{and} \qquad S = \begin{pmatrix} R(\Phi(A_1)) \\ R(\Phi(A_2)) \\ R(\Phi(A_3)) \end{pmatrix}.$$

Then

$$TC(B) = SC(\Psi(B)) \qquad \text{for any } B \in M_2'.$$

We can choose three linearly independent $B_1, B_2, B_3 \in M_2'$ so that

$$T[C(B_1)|C(B_2)|C(B_3)] = S[C(\Psi(B_1))|C(\Psi(B_2))|C(\Psi(B_3))]$$

is an invertible matrix. Thus, T and S are invertible and

$$C(\Psi(B)) = S^{-1}TC(B) \qquad \text{for any } B \in M_2'.$$

It follows that $\Psi : M_2' \to M_2'$ is an invertible linear map preserving rank one nilpotent matrices. By Lemma 2.6, there is a nonzero scalar c and an invertible $S \in M_2$ such that one of the followings holds:

$$(1)\ \Psi(A) = cSAS^{-1} \text{ for all } A \in M_2', \qquad (2)\ \Psi(A) = cSA^tS^{-1} \text{ for all } A \in M_2'.$$

For any $A \in M_2'$ with $\operatorname{rank} A = 2$, denote by $\det A$ the determinant of A, it follows from Claim 6 that

$$\left\{\frac{\mathrm{tr}(\Phi(\mathrm{A}))}{2} + c\sqrt{-\det A}, \frac{\mathrm{tr}(\Phi(\mathrm{A}))}{2} - c\sqrt{-\det A}\right\} \supseteq \sigma_\pi(\Phi(A)) = \xi\sigma_\pi(A) = \left\{\pm\sqrt{-\det A}\right\},$$

which entails that $\mathrm{tr}(\Phi(\mathrm{A})) = 0$ and $c = \pm 1$. This completes the proof of Claim 7.

Claim 8. *The map Φ has the desired form.*

We may assume that (1) in Claim 7 holds. Otherwise, replace Φ by the map $A \mapsto \Phi(A^t)$. We may further assume that $S = I_2$. Otherwise, replace Φ by the map $X \mapsto S^{-1}\Phi(A)S$. So, $\Phi(A) = A$ if $A \in M_2'$. Note that $\Phi(I) = \pm I$ by Claim 6. If $\Phi(I) = -I$, compose Φ with the map $X \mapsto -RX^tR^t$ with $R = E_{12} - E_{21}$. Here $E_{ij} = (a_{kl})$ with $a_{ij} = 1$ and $a_{kl} = 0$ when $(k,l) \neq (i,j)$, $i,j = 1,2$. One sees that the modified map will fix I and $A \in M_2'$. As a result,

$$\sigma_\pi(I\Phi(A) + \Phi(A)I) = \sigma_\pi(IA + AI) \qquad \text{for all } A \in M_2.$$

Now, suppose $A = (a_{ij}) \in M_2$ with $a_{11} + a_{22} \neq 0$, and $\Phi(A) = B = (b_{ij})$. Let $N = E_{12}, E_{21} \in M_2'$, respectively, and use the fact that $\sigma_\pi(AN+NA) = \sigma_\pi(BN+NB)$, we see that $a_{12} = b_{12}$ and $a_{21} = b_{21}$. In particular, $\Phi(E_{11}) = a_1E_{11} + a_2E_{22}$, and $\Phi(E_{22}) = b_1E_{11} + b_2E_{22}$. Since $\{0\} = \sigma_\pi(E_{11}E_{22} + E_{22}E_{11}) = \sigma_\pi(\Phi(E_{11})\Phi(E_{22}) + \Phi(E_{22})\Phi(E_{11}))$, we see that $a_1b_1 = a_2b_2 = 0$. Now, $\sigma_\pi(\Phi(A)) = \sigma_\pi(A)$, and $\sigma_\pi(E_{jj}N + NE_{jj}) = \sigma_\pi(\Phi(E_{jj})\Phi(N) + \Phi(N)\Phi(E_{jj}))$ for $N = E_{11} - E_{22}$, we conclude that $\Phi(E_{ii}) = E_{ii}$ for $i = 1,2$. Hence Φ fixes E_{11}, E_{22}, I and all matrices in M_2'. Now, continue to consider $A = (a_{ij}) \in M_2$ and $B = \Phi(A)$. If $a_{12}a_{21} = 0$, then $B = \Phi(A) = A$ by the fact that $\sigma_\pi(AX + XA) = \sigma_\pi(\Phi(A)X + X\Phi(A))$ for $X = E_{11}, E_{12}, E_{21}, E_{22}$.

Next, suppose $a_{12}a_{21} = d \neq 0$. Then $\sigma(E_{11}A + AE_{11}) = \{a_{11} + \sqrt{a_{11}^2 + d}, a_{11} - \sqrt{a_{11}^2 + d}\}$ and $\sigma(E_{11}B + BE_{11}) = \{b_{11} + \sqrt{b_{11}^2 + d}, b_{11} - \sqrt{b_{11}^2 + d}\}$. If $a_{11} + r\sqrt{d} \in \sigma_\pi(E_{11}A + AE_{11})$ with $r \in \{1,-1\}$, then $a_{11} + r\sqrt{a_{11}^2 + d} = b_{11} + \hat{r}\sqrt{b_{11}^2 + d}$ with $\hat{r} \in \{1,-1\}$. Thus,

$$(a_{11} - b_{11})^2 = a_{11}^2 + b_{11}^2 + 2d + 2r\hat{r}\sqrt{(a_{11}^2 + d)(b_{11}^2 + d)}.$$

It follows that

$$0 = (a_{11}^2 + d)(b_{11}^2 + d) - (d + a_{11}b_{11})^2 = d(a_{11} - b_{11})^2.$$

Since $d \neq 0$, we see that $a_{11} = b_{11}$. Similarly, we can show that $a_{22} = b_{22}$. Thus $A = B$ and completes the proof. $\square$

3. Jordan products of self-adjoint operators on Hilbert spaces

Let H and K be two complex Hilbert spaces, and $\mathcal{S}(H)$ and $\mathcal{S}(K)$ be the real linear spaces of all self-adjoint operators in $\mathcal{B}(H)$ and $\mathcal{B}(K)$, respectively. Then $\mathcal{S}(H)$ and $\mathcal{S}(K)$ are Jordan algebras. In this section, a standard real Jordan algebra on H is a subalgebra of $\mathcal{S}(H)$ which contains all finite rank self-adjoint operators and the identity operator. In this section we will characterize maps preserving peripheral spectrum of Jordan products of self-adjoint operators.

Observe that for any $x \in H$ and nonzero $A \in \mathcal{S}(H)$,

$$\sigma_\pi(Ax \otimes x + x \otimes xA) = \begin{cases} \{\langle Ax, x\rangle + \|Ax\|\|x\|\}, & \text{if} \quad \langle Ax, x\rangle > 0, \\ \{\langle Ax, x\rangle - \|Ax\|\|x\|\}, & \text{if} \quad \langle Ax, x\rangle < 0, \\ \{\pm\|Ax\|\|x\|\}, & \text{if} \quad \langle Ax, x\rangle = 0. \end{cases} \tag{3.1}$$

In fact, if there exists a nonzero $\alpha \in \mathbb{R}$ such that $Ax = \alpha x$, clearly Eq. (3.1) holds. Now assume that Ax and x are linearly independent. Then there exist nonzero $\gamma \in \mathbb{R}$ and $z \in H$ such that

$$(Ax \otimes x + x \otimes xA)z = \gamma z.$$

A direct computation implies that $\gamma = \langle Ax, x\rangle \pm \|Ax\|\|x\|$, and therefore Eq. (3.1) follows.

Now we state the main result in this section.

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be standard real Jordan algebras on complex Hilbert spaces H and K, respectively. Suppose that $\Phi : \mathcal{A} \to \mathcal{B}$ is a map of which range contains all rank one projections and trace zero rank two self-adjoint operators. Then Φ satisfies that*

$$\sigma_\pi(\Phi(A)\Phi(B) + \Phi(B)\Phi(A)) = \sigma_\pi(AB + BA) \qquad \text{for all } A, B \in \mathcal{A}$$

if and only if there exist $\varepsilon \in \{1, -1\}$ *and a unitary operator* $U : H \to K$ *such that*

$$\Phi(A) = \varepsilon UAU^* \qquad \text{for all } A \in \mathcal{A}$$

or

$$\Phi(A) = \varepsilon UA^tU^* \qquad \text{for all } A \in \mathcal{A},$$

where A^t *is the transpose of* A *for an arbitrarily but fixed orthonormal basis of* H.

To prove Theorem 3.1, we need several lemmas. In the sequel, we assume always that Φ *satisfies the conditions in Theorem 3.1.*

Lemma 3.2. *For* $A \in \mathcal{S}(H)$, *then* $\|Ax \otimes x + x \otimes xA\| = 2$ *for all unit vectors* $x \in H$ *implies that* $A = \varepsilon I$ *with* $\varepsilon \in \{1, -1\}$.

Proof. Note that, for every $A \in \mathcal{S}(H)$, we have always $\|A\|$ or $-\|A\| \in \sigma(A)$ (see, for example, [12, Property 6.1.7]). Let $A \in \mathcal{S}(H)$. For any unit vector $x \in H$, it follows from Eq. (3.1) that

$$2 = \|Ax \otimes x + x \otimes xA\| = |\langle Ax, x\rangle| + \|Ax\| \leq 2\|Ax\|, \tag{3.2}$$

and hence $\|Ax\| \geq 1$ for any unit vector $x \in H$, and $\|A\| \geq 1$. On the other hand, for all unit vectors $x \in H$, $2|\langle Ax, x\rangle| \leq |\langle Ax, x\rangle| + \|Ax\| = 2$ implies that $|\langle Ax, x\rangle| \leq 1$, therefore $\|A\| \leq 1$, thus $\|A\| = 1$ and $\|Ax\| = 1$ for all unit vectors $x \in H$. Now Eq. (3.2) implies that $A = \varepsilon I$ with $\varepsilon \in \{1, -1\}$. □

Lemma 3.3. $\Phi(I) = \varepsilon I$ *with* $\varepsilon \in \{-1, 1\}$.

Proof. For all $A, B \in \mathcal{A}$, since $\sigma_\pi(AB + BA) = \sigma_\pi(\Phi(A)\Phi(B) + \Phi(B)\Phi(A))$, it follows that $r(A) = r(\Phi(A))$ for every $A \in \mathcal{A}$. Let $\Phi(I) = B$. For any unit vector $y \in H$, there exists $A \in \mathcal{A}$ such that $\Phi(A) = y \otimes y$, thus $\sigma_\pi(2A) = \sigma_\pi(By \otimes y + y \otimes yB)$, which implies that $\|By \otimes y + y \otimes yB\| = 2$ for all unit vectors $y \in H$. Now Lemma 3.2 implies that $B \in \{I, -I\}$. □

If $\Phi(I) = -I$, considering $-\Phi$, then $-\Phi$ satisfies the conditions in Theorem 3.1, so we may as well assume $\Phi(I) = I$ in the following, and hence $\sigma_\pi(A) = \sigma_\pi(\Phi(A))$ for every $A \in \mathcal{A}$.

Lemma 3.4. Φ preserves rank one projections in both directions.

Proof. For any unit vector $x \in H$, let $A = x \otimes x$ and $\Phi(A) = B$. We will prove that B is a rank one projection.

Claim 1. $\dim\ker(B - I) = 1$.

Note that $\sigma_\pi(B) = \sigma_\pi(A) = \{1\}$. Then $1 \in \sigma(B) \subseteq (-1, 1]$. It follows that either (i) $\dim\ker(B - I) \geq 1$ or (ii) $B - I$ is injective but not surjective.

Assume that (ii) occurs. Since $1 \in \sigma_\pi(B)$, we have $\|B\| = 1$ and $B \leq I$. So, according to some space decomposition of H, B has an operator matrix of the form

$$\begin{pmatrix} a & 0 & b & 0 & 0 \\ 0 & a & 0 & c & 0 \\ b & 0 & * & * & * \\ 0 & c & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix},$$

where $a > 1/2$ and $b, c \geq 0$. To see this, one can first choose three orthonormal vectors x_1, x_2, x_3 such that $1 - d < \langle Bx_j, x_j\rangle < 1$ for some sufficiently small $d \in (0, 1/4)$. Suppose the compression $\hat{B}$ of B onto the span of $\{x_1, x_2, x_3\}$ has eigenvalues $\mu_1 \geq \mu_2 \geq \mu_3$. Then

$$\mu_2 \geq (\mu_2 + \mu_3)/2 \geq [(3 - 3d) - 1]/2 > 1/2.$$

Let $\mu_2 = a$. Then $\hat{B}$ is unitarily similar to

$$\begin{pmatrix} a & 0 & * \\ 0 & a & * \\ * & * & * \end{pmatrix}.$$

Thus, there exists a space decomposition such that B has an operator matrix of the form

$$\begin{pmatrix} aI_2 & B_{12} \\ B_{12}^* & * \end{pmatrix}.$$

Clearly, there are unitary U, V such that $UB_{12}V^*$ has operator matrix of the form

$$\begin{pmatrix} b & 0 & 0 \\ 0 & c & 0 \end{pmatrix},$$

where $b, c \geq 0$. So B has the desired operator matrix form. Under the same space decomposition, take $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 0$, then $r(BS + SB) \geq 2a > 1$, and there exists $R \in \mathcal{A}$ such that $\Phi(R) = S$, it follows that $\sigma_\pi(S) = \{1, -1\}$. So $\|R\| = 1$ and $\|Ru\| \leq 1$ for all unit vectors $u \in H$. But $\sigma_\pi(AR + RA) = \sigma_\pi(Rx \otimes x + x \otimes xR)$ is either a singleton or $\{\pm\|Rx\|\}$ with $\|Rx\| \leq 1$. This contradicts the fact $r(AR + RA) = r(BS + SB) \geq 2a > 1$.

So $\dim\ker(B - I) \geq 1$. Assume that $\dim\ker(B - I) = n \geq 2$. According to the space decomposition $H = \ker(B - I) \oplus \ker(B - I)^\perp$, B has an operator matrix $I_n \oplus T$. Under the same space decomposition, take $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 0$. Similar to the previous discussion, one gets a contradiction again. So $\dim\ker(B - I) = 1$.

Claim 2. There exists a unit vector $y \in H$ such that $B = y \otimes y$.

If it is not true, then, by Claim 1, there exists a unit vector $y \in \ker(B - I)$ and a nonzero $B_2 \in \mathcal{B}$ with $B_2 y = 0$ such that $B = y \otimes y + B_2$. So there exists a unit vector $z \in [y]^\perp$ such that $B_2 z \neq 0$. Let $C_1 = y \otimes y$ and $C_2 = z \otimes z$. Then $\sigma_\pi(BC_1 + C_1 B) = \{2\}$, $C_1C_2 + C_2C_1 = 0$, and $BC_2 + C_2 B \neq 0$. Since the range of Φ contains all rank one projections, there exist D_1 and D_2 in $\mathcal{A}$ such that $\Phi(D_1) = C_1$ and $\Phi(D_2) = C_2$. Then $\sigma_\pi(D_1) = \sigma_\pi(D_2) = \{1\}$, $\sigma_\pi(AD_1 + D_1 A) = \{2\}$, $D_1D_2 + D_2D_1 = 0$, and $AD_2 + D_2 A \neq 0$.

Since $\{2\} = \sigma_\pi(AD_1 + D_1 A) = \sigma_\pi(D_1 x \otimes x + x \otimes xD_1)$, it follows from Eq. (3.1) that $|\langle D_1 x, x\rangle| + \|D_1 x\| = 2$, which, together with $\|D_1\| = 1$, implies that $D_1 x = x$. So, according to the space decomposition $H = [x] \oplus [x]^\perp$, $D_1 = [1] \oplus Z$ with $\sigma(Z) \subseteq (-1, 1]$. If D_2 has an operator matrix $\begin{pmatrix} v_{11} & V_{12} \\ V_{12}^* & V_{22} \end{pmatrix}$ accordingly, then

$$0 = D_1D_2 + D_2D_1 = \begin{pmatrix} 2v_{11} & V_{12} + V_{12}Z \\ ZV_{12}^* + V_{12}^* & ZV_{22} + V_{22}Z \end{pmatrix}.$$

Since $I + Z$ is invertible, we see that $V_{12} = 0$. Clearly, $v_{11} = 0$. So, $D_2 = 0 \oplus V_{22}$. But then it contradicts the fact that $AD_2 + D_2 A \neq 0$. So Claim 2 holds and Φ preserves rank one projections.

Conversely, assume that $\Phi(A)$ is a rank one orthogonal projection, then a similar discussion implies that A is a rank one projection. This completes the proof. $\square$

The following lemma was proved in [15, Lemma 3.3].

Lemma 3.5. *Let H be a complex Hilbert space and $A, B \in \mathcal{B}(H)$ be self-adjoint operators. If*

$$|\langle Ax, x\rangle| + \|Ax\|\|x\| = |\langle Bx, x\rangle| + \|Bx\|\|x\|$$

holds for all $x \in H$, then $A = \pm B$.

Proof of Theorem 3.1. By Lemmas 3.4, Φ preserves rank one projections in both directions. So, for every unit vector $x \in H$, there exists a unit vector $y_x \in H$ such that $\Phi(x\otimes x) = y_x \otimes y_x$. For any unit vector $x, x' \in H$, let $A = x \otimes x$ and $B = x' \otimes x'$. Since $\sigma_\pi(\Phi(A)\Phi(B) + \Phi(B)\Phi(A)) = \sigma_\pi(AB + BA)$, by Eq. (3.1), $\langle x, x'\rangle = 0$ if and only if $\langle y_x, y_{x'}\rangle = 0$, and when $\langle x, x'\rangle \neq 0$,

$$|\langle y_x, y_{x'}\rangle|^2 + |\langle y_x, y_{x'}\rangle| = |\langle x, x'\rangle|^2 + |\langle x, x'\rangle|.$$

It follows that

$$|\langle y_x, y_{x'}\rangle| = |\langle x, x'\rangle|, \qquad \text{for all } x, x' \in H.$$

Let $[x]$ denote one dimensional subspace of H spanned by x, and $\mathbb{P}H$ (or $\mathbb{P}K$) denote the set of all one dimensional subspaces of H (or K). Thus, Φ induces a bijective transformation $\varphi : \mathbb{P}H \to \mathbb{P}K$ such that $\varphi([x]) = [y_x]$ for every $x \in H$. Wigner's Theorem states that every bijective transformation on the set of all one-dimensional subspaces of a complex Hilbert space which preserves the angle between every pair of such subspace can be induced by a unitary or conjugate unitary operator on H. So there exists a unitary or conjugate unitary operator $U : H \to K$ such that $y_x = \alpha_x Ux$ for every $x \in H$, where $\alpha_x \in \mathbb{C}$ with $|\alpha_x| = 1$. So $\Phi(x \otimes x) = Ux \otimes Ux$ for every $x \in H$.

Assume first that U is unitary. Let $A \in \mathcal{A}$ be arbitrary. For any unit vector $x \in H$,

$$\sigma_\pi(\Phi(A)Ux \otimes Ux + Ux \otimes Ux\Phi(A)) = \sigma_\pi(Ax \otimes x + x \otimes xA).$$

Applying Eq. (3.1), for any unit vector $x \in H$, one has

$$|\langle U^*\Phi(A)Ux, x\rangle| + \|U^*\Phi(A)Ux\| = |\langle Ax, x\rangle| + \|Ax\|,$$

and hence Lemma 3.5 implies that $U^*\Phi(A)U = \pm A$. That is, $\Phi(A) = \pm UAU^*$ for every $A \in \mathcal{A}$. We claim that either $\Phi(A) = UAU^*$ for every $A \in \mathcal{A}$ or $\Phi(A) = -UAU^*$ for every $A \in \mathcal{A}$. Otherwise, take $A, B \in \mathcal{A}$ with $\sigma_\pi(AB + BA)$ being a singleton different from $\{0\}$ such that $\Phi(A) = UAU^*$ and $\Phi(B) = -UBU^*$. Then $\sigma_\pi(AB + BA) = \sigma_\pi(\Phi(A)\Phi(B) + \Phi(B)\Phi(A)) = -\sigma_\pi(AB + BA)$, which implies that $AB + BA = 0$, a contradiction. So $\Phi(A) = \varepsilon UAU^*$ for every $A \in \mathcal{A}$, where $\varepsilon \in \{1, -1\}$.

Now assume that U is conjugate unitary. Take an orthonormal basis $\{e_i\}_{i\in\Lambda}$ of H and define J by $J(\sum_{i\in\Lambda} \xi_i x_i) = \sum_{i\in\Lambda} \bar{\xi}_i e_i$. Then $J : H \to H$ is conjugate unitary. Let $V = UJ$. Then a similar discussion as above implies that $\Phi(A) = \varepsilon V A^t V^*$ for all $A \in \mathcal{A}$, where $\varepsilon \in \{1, -1\}$ and A^t is the transpose of A for an arbitrarily but fixed orthonormal basis. The proof is complete. □

4. The product $AB^* + B^*A$ on Hilbert spaces

Let $\mathcal{A}$ and $\mathcal{B}$ be (not necessarily unital or closed) standard operator algebras on complex Hilbert spaces H and K, respectively. In this section, we will characterize maps preserving the peripheral spectrum of the product $AB^* + B^*A$. The following is our main result.

Theorem 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on complex Hilbert spaces H and K, respectively. Assume that $\Phi : \mathcal{A} \to \mathcal{B}$ is a map of which range contains all operators with rank at most two. Then Φ satisfies that*

$$\sigma_\pi(\Phi(A)\Phi(B)^* + \Phi(B)^*\Phi(A)) = \sigma_\pi(AB^* + B^*A) \qquad \textit{for all } A, B \in \mathcal{A}$$

if and only if there exist a scalar γ with $|\gamma| = 1$ and a unitary operator $U \in \mathcal{B}(H, K)$ such that either

$$\Phi(A) = \gamma UAU^* \qquad \text{for every } A \in \mathcal{A},$$

or

$$\Phi(A) = \gamma UA^tU^* \qquad \text{for every } A \in \mathcal{A},$$

where A^t is the transpose of A for an arbitrarily but fixed orthonormal basis of H. If $\dim H \leq 2$, no assumption on the range of Φ is needed.

To prove Theorem 4.1, we need the following analog of Lemma 2.3.

Lemma 4.2. *Assume that $\dim H \geq 3$. Let $A \in \mathcal{A}$ be nonzero. Then the following statements are equivalent:*

(1) *A is rank one nilpotent.*
(2) *For any $B \in \mathcal{A}$, $\sigma_\pi(AB^* + B^*A)$ is a singleton.*
(3) *For any $B \in \mathcal{A}$ with $\operatorname{rank} B \leq 2$, $\sigma_\pi(AB^* + B^*A)$ is a singleton.*

Now we are in a position to present the

Proof of Theorem 4.1. We will finish the proof of the theorem by considering two cases $\dim H \geq 3$ and $\dim H \leq 2$. First assume that $\dim H \geq 3$. Similar to discussions in the proof of Theorem 2.1, it follows that there must be $\dim K \geq 3$. A similar discussion just as Claim 2-Claim 4 in the proof of Theorem 2.1 implies that there exists a nonzero constant λ and either

(i) there exists an invertible operator $T \in \mathcal{B}(H, K)$ such that $\Phi(N) = \lambda TNT^{-1}$ for every $N \in \mathcal{N}_1(H)$; or

(ii) there exists a conjugate linear bounded bijective operator $T : H \to K$ such that $\Phi(N) = \lambda TN^*T^{-1}$ for any $N \in \mathcal{N}_1(H)$.

Assume that (i) holds. For any orthogonal vectors $x, y \in H$, it follows from

$$\begin{aligned} & |\lambda|^2\sigma_\pi(Tx \otimes yT^{-1}(Ty \otimes xT^{-1})^* + (Ty \otimes xT^{-1})^*Tx \otimes yT^{-1}) \\ = \ & \sigma_\pi(x \otimes y(y \otimes x)^* + (y \otimes x)^*x \otimes y) \\ = \ & \{0\} \end{aligned}$$

that

$$\langle (T^*T)^{-1}x, y\rangle = 0 \qquad \text{or} \qquad \langle T^*Tx, y\rangle = 0,$$

and therefore $T^*T = \alpha I$ with $\alpha > 0$. So we can choose T is unitary.

For any orthogonal unit vectors $x, y \in H$, it follows from

$$\begin{aligned} \{|\lambda|^2\} &= \sigma_\pi(Tx \otimes yT^{-1}(Tx \otimes yT^{-1})^* + (Tx \otimes yT^{-1})^*Tx \otimes yT^{-1}) \\ &= \sigma_\pi(x \otimes y(x \otimes y)^* + (x \otimes y)^*x \otimes y) \\ &= \{1\} \end{aligned}$$

that $|\lambda| = 1$. Considering $\bar{\lambda}\Phi$, then $\bar{\lambda}\Phi$ has the same property as Φ has, one can assume that $\Phi(N) = TNT^*$ for every $N \in \mathcal{N}_1(H)$. A similar discussion of Claim 5 in the proof of Theorem 2.1 implies that, for every $A \in \mathcal{A}$, there exists a scalar α_A such that $\Phi(A) = TAT^{-1} + \alpha_A I$. For any rank one projection $P \in \mathcal{A}$, take a rank one projection $Q \in \mathcal{A}$ such that $PQ = QP = 0$. Then $\sigma_\pi(\alpha_P\overline{\alpha_Q}I + \alpha_P Q + \overline{\alpha_Q}P) = \{0\}$ implies that $\alpha_P = 0$. Now the rest of the proof follows a similar discussion of Claim 5 in the proof of Theorem 2.1. So $\Phi(A) = TAT^*$ for every $A \in \mathcal{A}$.

Now assume the case (ii) occurs. Similar to the above proof, there exists a complex unit λ and a conjugate unitary operator $U : H \to K$ such that $\Phi(A) = \lambda UA^*U^*$ for every $A \in \mathcal{A}$. Let

J be just as assumption in the proof of Theorem 3.1 and $V = UJ$. Then $V : H \to K$ is unitary and $JA^*J = A^t$, where A^t is the transpose of A for an arbitrarily but fixed orthonormal basis. Thus $\Phi(A) = \lambda V A^t V^*$ for every $A \in \mathcal{A}$.

In the case $\dim H \leq 2$, a similar discussion just as the corresponding part in Theorem 2.1 implies that there exist a complex unit γ and a nonsingular matrix $S \in M_2$ such that $\Phi(A) = \gamma SAS^{-1}$ for every $A \in M_2$ or $\Phi(A) = \gamma SA^tS^{-1}$ for every $A \in M_2$. It can be easily checked that S can be chosen as unitary. □

Acknowledgement

Thanks are due to the referee for his/her careful reading of the manuscript, and some helpful comments.

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