OFF-DIAGONAL SUBMATRICES OF A HERMITIAN MATRIX

CHI-KWONG LI AND YIU-TUNG POON

ABSTRACT. A necessary and sufficient condition is given to an $p \times q$ complex matrix X to be an off-diagonal block of an $n \times n$ Hermitian matrix C with prescribed eigenvalues (in terms of the eigenvalues of C and singular values of X). The proof depends on some recent breakthroughs in the study of spectral inequalities on the sum of Hermitian matrices by Klyachko and Fulton. Some interesting geometrical properties of the set S of all such matrices are derived from the main result. These results improve earlier ones that only give partial information for the set S.

1. INTRODUCTION

Let $\mathcal{H}_n$ be the real linear space of $n \times n$ complex Hermitian matrices, and let $\mathcal{U}_n$ be the group of $n \times n$ unitary matrices. For $c = (c_1, \cdots, c_n) \in \mathbb{R}^n$, with $c_1 \geq \cdots \geq c_n$, let $\mathcal{U}(c)$ be the set of matrices in $\mathcal{H}_n$ with eigenvalues $c_1, \cdots, c_n$. Equivalently, $\mathcal{U}(c)$ is the *unitary* (*similarity*) *orbit* of the diagonal matrix diag $(c_1, \cdots, c_n)$. In this paper, a necessary and sufficient condition is given for an $p \times q$ complex matrix X to be an off-diagonal block of a matrix $C \in \mathcal{U}(c)$ in terms of the singular values of X. The set of all such matrices X can be viewed as the projection of $\mathcal{U}(c)$.

Let p and q be positive integers such that $p + q \leq n$. Consider the set

$$S_{p,q}(c) = \{A_{13} \in \mathbb{C}^{p \times q} : \text{ there exists } A = (A_{ij})_{1 \leq i,j \leq 3} \in \mathcal{U}(c)\}.$$

Clearly, in the block matrix $A = (A_{ij})_{1 \leq i,j \leq 3}$ above, $A_{11} \in \mathcal{H}_p$ and $A_{33} \in \mathcal{H}_q$. In statistics and other applications, there is considerable interest (see [1, 2, 4, 15] and their references) in studying inequalities relating the singular values of matrices in $S_{p,n-p}(c)$ and c. In [15], the authors proved that if $p \leq n/2$ and $X \in S_{p,n-p}(c)$ has singular values $s_1 \geq \cdots \geq s_p$, then

$$\frac{1}{2} \sum_{j=1}^{m} (c_j - c_{n-j+1}) \geq \sum_{j=1}^{m} s_j, \qquad m = 1, \ldots, p. \tag{1}$$

Using this result and the theory on Schur convex function, one can (see [15]) refine and shorten the proofs of many existing inequalities. Evidently, the above result on $S_{p,n-p}(c)$ is equivalent to

$$S_{p,n-p}(c) \subseteq \left\{ X \in \mathbb{C}^{p \times (n-p)} : \begin{array}{l} X \text{ has singular values} \\ s_1 \geq \cdots \geq s_p \text{ satisfying (1)} \end{array} \right\}. \tag{2}$$

In [2] the authors studied some convexity features of $\mathcal{U}(c)$, and showed that the equality in (2) holds if and only if $S_{p,n-p}(c)$ is convex, and this happens if and

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only if $(c_1, \ldots, c_p)$ and $(c_{n-p+1}, \ldots, c_n)$ are arithmetic progressions with the same common difference.

In this paper, we give a complete description of $S_{p,q}(c)$ for general (p, q). In particular, it follows from our result that the set $S_{p,q}(c)$ is always star-shaped with the zero matrix as a star-center; see Corollary 2.4. Some examples and remarks related to our results are also given.

There are two key ingredients in our proofs. First, we need the recent major breakthrough in the study of spectral inequalities on the sum of Hermitian matrices by Klyachko [11] and its generalization by Fulton [8]; see also [7] for an excellent survey. Second, we need some matrix techniques to related the results on sum of Hermitian matrices to off-diagonal block of a Hermitian matrix.

Although our discussion is on complex Hermitian matrices, all our results are valid (with the same proofs) for real symmetric and Hermitian matrices over real quaternions.

In our discussion, the sets of eigenvalues or singular values always mean the multi-sets of eigenvalues or singular values counting multiplicities.

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2. Main Result

As mentioned in the introduction, there are two important ingredients in our proofs. We first present a result of Fulton [8] concerning the sum of Hermitian matrices. This result is a generalization of a remarkable result of Klyachko[11] on spectral inequalities on the sum of Hermitian matrices.

Let $I = (i_1, \ldots, i_m)$, be an increasing subsequences of $[n] = (1, \ldots, n)$. Define

$$\lambda(I) = (i_m - m, \ldots, i_1 - 1).$$

Suppose $J = (j_1, \ldots, j_m)$ and $K = (k_1, \ldots, k_m)$ are also increasing subsequences of $[n]$. We say that $(I, J, K) \in LR_m^n$ if the Littlewood-Richardson coefficient $c_{\lambda(I)\lambda(J)}^{\lambda(K)}$ of the three partitions $\lambda(I), \lambda(J)$, and $\lambda(K)$ is equal to one; see [6]. Our proofs depend on the following result in [8]; see Theorem 1, the discussion after the statement of Theorem 2, and the proofs in Section 2 of the paper.

Lemma 2.1. *There exist Hermitian matrices A, B, C with eigenvalues*

$$a_1 \geq \cdots \geq a_n, \quad b_1 \geq \cdots \geq b_n, \quad \text{and} \quad c_1 \geq \cdots \geq c_n, \tag{3}$$

respectively, such that

$$A + B \geq C, \quad \text{i.e., } A + B - C \text{ is positive semidefinite}$$

if and only if for each $m \in \{1, \ldots, n\}$ and $(I, J, K) \in LR_m^n$,

$$\sum_{i \in I} a_i + \sum_{j \in J} b_j \geq \sum_{k \in K} c_k. \tag{4}$$

Moreover, if there is an $m \in \{1, \ldots, n\}$ such that the equality in (4) holds for some $(I, J, K) \in LR_m^n$, then there exist $A_1, B_1, C_1 \in \mathcal{H}_m$ with sets of eigenvalues (counting multiplicities) $\{a_i : i \in I\}$, $\{b_j : j \in J\}$, $\{c_k : k \in K\}$, respectively, so that $A_1 + B_1 = C_1$, and in the case of $n > m$, there exist $A_2, B_2, C_2 \in \mathcal{H}_{n-m}$ with sets of eigenvalues (counting multiplicities) $\{a_i : i \in [n] \setminus I\}$, $\{b_j : j \in [n] \setminus J\}$, $\{c_k : k \in [n] \setminus K\}$, respectively, so that $A_2 + B_2 \geq C_2$.

Next, we establish a lemma relating Hermitian matrix sum to the off-diagonal block of a Hermitian matrix.

Lemma 2.2. *Suppose* $a_1 \geq \cdots \geq a_n$, $b_1 \geq \cdots \geq b_n$, $c_1 \geq \cdots \geq c_n$, *and* $t \in [0,1]$ *are such that* $\sum_{j=1}^n c_n = t\left\{\sum_{j=1}^n (a_j + b_j)\right\}$ *and for each* $m \in \{1,\ldots,n\}$ *and* $(I,J,K) \in LR_m^n$,

$$t\left\{\sum_{i\in I} a_i + \sum_{j\in J} b_j\right\} \geq \sum_{k\in K} c_k.$$

Then there exist $A, B, C \in \mathcal{H}_n$ *with eigenvalues* $a_1 \geq \cdots \geq a_n$, $b_1 \geq \cdots \geq b_n$, $c_1 \geq \cdots \geq c_n$, *respectively, such that*

$$C = t(A+B)\,.$$

Moreover, there exists a unitary matrix $U \in \mathcal{U}_{2n}$ *such that*

$$U^* \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} U = \begin{pmatrix} * & C/2 \\ C/2 & * \end{pmatrix}.$$

Proof. The first assertion follows from Lemma 2.1. To prove the second assertion, let $\theta \in [0, \pi/4]$ be such that $\sin 2\theta = t$, and let

$$U = \begin{pmatrix} \cos\theta I_n & \sin\theta I_n \\ -\sin\theta I_n & \cos\theta I_n \end{pmatrix}.$$

Then

$$U^* \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} U = \begin{pmatrix} * & (A+B)\sin 2\theta/2 \\ (A+B)\sin 2\theta/2 & * \end{pmatrix}.$$

□

We are ready to give the characterization of

$$S_{p,q}(c) = \{A_{13} \in \mathbb{C}^{p\times q} : \text{ there exists } A = (A_{ij})_{1\leq i,j\leq 3} \in \mathcal{U}(c)\},$$

where p and q are positive integers such that $p+q \leq n$. Since $X \in S_{p,q}(c)$ if and only if $X^* \in S_{q,p}(c)$, we only need to consider the case when $p \leq q$.

Theorem 2.3. *Let* $c_1 \geq \cdots \geq c_n$. *Suppose* p *and* q *are positive integers such that* $p \leq q$ *and* $p+q \leq n$, *and* $X \in \mathbb{C}^{p\times q}$ *has singular values* $s_1 \geq \cdots \geq s_p$. *The following conditions are equivalent:*

(a) $X \in S_{p,q}(c)$.

(b) *For each* $m \in \{1,\ldots,p\}$ *and* $(I,J,K) \in LR_m^p$,

$$\sum_{i\in I} c_i - \sum_{j\in J} c_{n-j+1} \geq 2\sum_{k\in K} s_k.$$

(c) *There exist* $A, B, S \in \mathcal{H}_p$ *with eigenvalues* $c_1 \geq \cdots \geq c_p$, $c_{n-p+1} \geq \cdots \geq c_n$, *and* $s_1 \geq \cdots \geq s_p$, *respectively, such that* $A - B \geq 2S$.

Proof. The equivalence of (b) and (c) follows from Lemma 2.1.

(a) ⇒ (b): Suppose (a) holds, i.e., there is a matrix $A = (A_{ij})_{1\leq i,j\leq 3} \in \mathcal{U}(c)$, such that $A_{13} = X$. Then $\tilde{A} = -(-I_p \oplus I_{n-p})A(-I_p \oplus I_{n-p})$ has eigenvalues $-c_n \geq \cdots \geq -c_1$. Suppose the matrix

$$A + \tilde{A} = 2\begin{pmatrix} 0 & A_{12} & A_{13} \\ A_{12}^* & 0 & 0 \\ A_{13}^* & 0 & 0 \end{pmatrix}$$

has eigenvalues $2\tilde{s}_1 \geq \cdots \geq 2\tilde{s}_p \geq \overbrace{0 \geq \cdots \geq 0}^{n-2p} \geq -2\tilde{s}_p \geq \cdots \geq -2\tilde{s}_1$. By Lemma 2.1, condition (b) holds with s_j replaced by $\tilde{s}_j$ for $j = 1, \ldots, p$. Since $2\begin{pmatrix} 0 & A_{13} \\ A_{13}^* & 0 \end{pmatrix}$ is a principal submatrix of $A + \tilde{A}$, and has eigenvalues

$$2s_1 \geq \cdots \geq 2s_p \geq \overbrace{0 \geq \cdots \geq 0}^{q-p} \geq -2s_p \geq \cdots \geq -2s_1,$$

we have $\tilde{s}_j \geq s_j$ for $j = 1, \ldots, p$, by the result in [5] (see also [10, Theorem 4.3.15]). Thus, condition (b) holds.

(b) $\Rightarrow$ (a): Suppose (b) holds. It suffices to construct a matrix

$$A = \begin{pmatrix} A_{11} & 0 & S \\ 0 & A_{22} & 0 \\ S^* & 0 & A_{33} \end{pmatrix} \in \mathcal{U}(c) \tag{5}$$

such that $S \in \mathbb{C}^{p \times q}$ has singular values $s_1 \geq \cdots \geq s_p$. Then there exist $V_1 \in \mathcal{U}_p$ and $V_2 \in \mathcal{U}_q$ such that $V_1^* S V_2 = X$ is the $(1,3)$ block of the matrix

$$(V_1 \oplus I_{n-p-q} \oplus V_2)^* A (V_1 \oplus I_{n-p-q} \oplus V_2) \in \mathcal{U}(c).$$

To construct the matrix in (5), we first consider the case when $n = 2p$. We prove the result by induction on p. If $p = 1$, then $t(c_1 - c_2) = 2s_1$ for some $t \in [0,1]$. The result follows from Lemma 2.2. Suppose $p > 1$, and the implication holds for lower dimensions. Let $t \in [0,1]$ be such that for each $m \in \{1, \ldots, p\}$ and $(I, J, K) \in LR_m^p$,

$$t\left\{\sum_{i \in I} c_i - \sum_{j \in J} c_{n-j+1}\right\} \geq 2\sum_{k \in K} s_k,$$

and the equality occurs at $(I, J, K) \in LR_{m'}^p$ for some $m' \in \{1, \ldots, p\}$. If $m' = p$, condition (a) follows from Lemma 2.2. If $m' < p$, then by Lemma 2.1, there exist $A_1, B_1, S_1 \in \mathcal{H}_{m'}$ with sets of eigenvalues $\{c_i : i \in I\}$, $\{-c_{n-j+1} : j \in J\}$, and $\{s_k : k \in K\}$ respectively, satisfying $t(A_1 + B_1) = 2S_1$, and there exist $A_2, B_2, S_2 \in \mathcal{H}_{p-m'}$ with sets of eigenvalues $\{c_i : i \in [p] \setminus I\}$, $\{-c_{n-j+1} : j \in [p] \setminus J\}$, and $\{s_k : k \in [p] \setminus K\}$ respectively, satisfying $t(A_2 + B_2) \geq 2S_2$. Consequently, by Lemma 2.2 there exists $U_1 \in \mathcal{U}_{2m'}$ such that

$$U_1^* \begin{pmatrix} A_1 & 0 \\ 0 & -B_1 \end{pmatrix} U_1 = \begin{pmatrix} * & S_1 \\ S_1^* & * \end{pmatrix}.$$

Moreover, by Lemma 2.1 again, the eigenvalues of A_2, B_2, S_2 satisfy the inductive hypothesis of the implication (b) $\Rightarrow$ (a). So, there exists $U_2 \in \mathcal{U}_{2(p-m')}$ such that

$$U_2^* \begin{pmatrix} A_2 & 0 \\ 0 & -B_2 \end{pmatrix} U_2 = \begin{pmatrix} * & S_2 \\ S_2^* & * \end{pmatrix}.$$

Let $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$ and $S = S_1 \oplus S_2$. By a block permutation similarity, there exists $U_3 \in \mathcal{U}_{2p}$ such that

$$U_3^* \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} U_3 = \begin{pmatrix} P & S \\ S^* & Q \end{pmatrix},$$

which has eigenvalues $c_1, \ldots, c_n$.

Now, suppose $n > 2p$. Let

$$D = \operatorname{diag}(c_1, \ldots, c_p, c_{n-p+1}, \ldots, c_n).$$

By the above construction, we can get a unitary matrix $U \in \mathcal{U}_{2p}$ such that

$$U^*DU = \begin{pmatrix} P & S \\ S^* & Q \end{pmatrix},$$

where S is $p \times p$ with singular values $s_1 \geq \cdots \geq s_p$. If $D_0 = \operatorname{diag}(c_{p+1}, \ldots, c_{n-p})$, then $U^*DU \oplus D_0$ is permutationally similar to

$$\begin{pmatrix} P & 0 & S \\ 0 & D_0 & 0 \\ S^* & 0 & Q \end{pmatrix},$$

which is the desired matrix satisfying (5) after we redefine the block sizes. □

By the result in [2], the set $S_{p,q}(c)$ may not be convex; $S_{p,q}(c)$ is convex if and only if for $\ell = \min\{p, q\}$, the sequences $(c_1, \ldots, c_\ell)$ and $(c_{n-\ell+1}, \ldots, c_n)$ are in arithmetic progression with the same common difference. In general, we have the following.

Corollary 2.4. *Let $c_1 \geq \cdots \geq c_n$ and p, q be positive integers such that $p + q \leq n$. Suppose $X \in S_{p,q}(c)$ has singular values $s_1 \geq \cdots \geq s_\ell$ where $\ell = \min\{p, q\}$. If $Y \in \mathbb{C}^{p\times q}$ has singular values $t_1, \cdots, t_\ell$ satisfying $t_i \leq s_i$ for all $i = 1, \cdots, \ell$, then $Y \in S_{p,q}(c)$. Equivalently, $RXS \in S_{p,q}(c)$ for all contractions R, S of sizes p and q respectively. In particular, $tX \in S_{p,q}(c)$ for all $0 \leq t \leq 1$, i.e., $S_{p,q}(c)$ is star-shaped with the zero matrix as a star-center.*

Proof. Since $X \in S_{p,q}(c)$ if and only if $X^* \in S_{q,p}(c)$, we consider only the case $\ell = p$. If $X \in S_{p,q}(c)$ has singular values $s_1 \geq \cdots \geq s_p$, then Theorem 2.3 (b) holds. Since $t_i \leq s_i$ for all $i = 1, \cdots, p$, (b) also holds with s_i replaced by t_i. Hence, $Y \in S_{p,q}(c)$. Note that a matrix $Y \in \mathbb{C}^{p\times q}$ has singular values $t_i \leq s_i$, for all $i = 1, \cdots, p$ if and only if $Y = RXS$ for some contractions R, S of sizes p and q respectively. Therefore, the equivalent statement follows. The last assertion is clear. □

3. Remarks and Examples

It is interesting that by Corollary 2.4, if $X \in S_{p,q}(c)$, one can conclude without any additional information about c that $RXS \in S_{p,q}(c)$ for any contractions R and S of sizes p and q respectively; in other words, if $A = (A_{ij})_{i,j=1}^3 \in \mathcal{U}(A)$ so that $A_{13} \in \mathbb{C}^{p\times q}$, then there is $\tilde{A} = (\tilde{A}_{ij})_{i,j=1}^3 \in \mathcal{U}(A)$ with $\tilde{A}_{ij} = RA_{13}S$. Note that in general the matrix $T = R^* \oplus I_{n-p-q} \oplus S$ is not unitary, and therefore one cannot simply let $\tilde{A}$ to be T^*AT although its $(1,3)$ block has the form $RA_{13}S$. Actually, in computation, even constructing $\tilde{A} \in \mathcal{U}(c)$ such that $\tilde{A}_{13} = A_{13}/2$ for a given $A_{13} \in S_{p,q}(c)$ is a non-trivial problem.

In the proof of the implication (a) $\Rightarrow$ (b) in Theorem 2.3, one actually has

$$\sum_{i\in I} c_i - \sum_{j\in J} c_{n-j+1} \geq 2 \sum_{k\in K} s_k$$

for any $(I, J, K) \in LR_m^n$, whereas our necessary and sufficient conditions only require those inequalities associated with $(I, J, K) \in LR_m^p$. Thus many inequalities associated with $(I, J, K) \in LR_m^n$ are redundant.

In [15], using the inequalities (1), which is a subcollection of those in Theorem 2.3 (b), and the theory of Schur convex functions, the authors refine and shorten the proofs of results on matrix inequalities relating the eigenvalues of $C \in \mathcal{H}_n$ and the singular values of an off-diagonal submatrix of C. Using the whole set of inequalities in Theorem 2.3 (b), one should be able to get more information. For example, for any $1 \le i_1 < \cdots < i_\ell \le p$, we have

$$\sum_{r=1}^{m}(c_{i_r} - c_{n-r+1}) \ge 2\sum_{r=1}^{m} s_{i_r}, \qquad m = 1, \ldots, \ell.$$

Thus, for any increasing convex function $f : \mathbb{R} \to \mathbb{R}$, we have

$$\sum_{r=1}^{\ell} f(c_{i_r} - c_{n-r+1}) \ge \sum_{r=1}^{\ell} f(2s_{i_r}).$$

In our main result and the supporting lemmas, we may replace LR_m^n (or LR_m^p) by other sets. For example, we can use the set $\mathcal{L}R_m^n$: the set of (I, J, K) sequence triples such that the Littlewood-Richardson coefficient corresponding to the partitions $\lambda(I), \lambda(J), \lambda(K)$ is positive. As shown in [7, Theorem 18], these sequence triples (I, J, K) are the same as the Horn-Thompson consistent sequences, which can be generated systematically as follows; see [3, 7, 9] and their references.

Let $I = (i_1, \ldots, i_m), J = (j_1, \ldots, j_m), K = (k_1, \ldots, k_m) \in [n]$.

1. For $m \ge 1$, $\sum_{\ell=1}^{m} k_\ell + m(m+1)/2 = \sum_{\ell=1}^{m}(i_\ell + j_\ell)$.
2. If $m > 1$, then for any consistent triple (U, V, W), where $U = (u_1, \ldots, u_{m'})$, $V = (v_1, \ldots, v_{m'})$ and $W = (w_1, \ldots, w_{m'})$ with $m' \in \{1, \ldots, m-1\}$, we have
$$\sum_{\ell=1}^{m'} k_{w_\ell} + m'(m'+1)/2 \ge \sum_{\ell=1}^{m'}(i_{u_\ell} + j_{v_\ell}).$$

Although $\mathcal{L}R_m^n$ contain some redundant inequalities, it is relatively easy to generate them because one needs not verify that the Littlewood-Richardson coefficient of $\lambda(I), \lambda(J)$, and $\lambda(K)$ is one. Here we give a short explanation why LR_m^n can be replaced by $\mathcal{L}R_m^n$.

It is known [14] (see also [12, 13]) that $a_1 \ge \cdots \ge a_n$, $b_1 \ge \cdots \ge b_n$, and $c_1 \ge \cdots \ge c_n$ satisfy (4) for each $m \in \{1, \ldots, n\}$ and $(I, J, K) \in LR_m^n$ if and only if they satisfy (4) for each $m \in \{1, \ldots, n\}$ and $(I, J, K) \in \mathcal{L}R_m^n$. By the minimality of LR_m^n, if equality holds for some consistent triples $(I, J, K) \in \mathcal{L}R_m^n$ then we can find a triple $(I', J', K') \in LR_{m'}^n$ for which the equality holds. In fact, if equality holds in (4) for some $(I, J, K) \in \mathcal{L}R_m^n$ with $m \in \{1, \ldots, n\}$, but

$$\sum_{i \in I'} a_i + \sum_{j \in J'} b_j > \sum_{k \in K'} c_k$$

for all $(I', J', K') \in LR_{m'}^n$ with $m' \in \{1, \ldots, n\}$, then there exists $d > 0$ such that

$$\sum_{i \in I'}(a_i - d) + \sum_{j \in J'}(b_j - d) > \sum_{k \in K'} c_k$$

for all $(I', J', K') \in LR_{m'}^n$ with $m' \in \{1, \ldots, n\}$, which implies (by the minimality of LR_m^n),

$$\sum_{k \in K} c_k = \sum_{i \in J} a_i + \sum_{j \in J} b_j \le \sum_{i \in I}(a_i - d) + \sum_{j \in J}(b_j - d),$$

a contradiction.

In general, condition (b) in theorem 2.3 is difficult to check. If $c_1, \ldots, c_n$ have some special properties, the situations are more pleasant.

Examples: Let p and q be positive integers such that $p \leq q$ and $p + q \leq n$.
(i) If $(c_1, \ldots, c_p)$ and $(c_{n-p+1}, \ldots, c_n)$ are arithmetic progressions with the same common difference, then condition (b) reduces to

$$\sum_{j=1}^{r}(c_j - c_{n-j+1}) \geq \sum_{j=1}^{r} 2s_j, \qquad r = 1, \ldots, p.$$

(ii) If $c_1 = \cdots = c_p$ or $c_{n-p+1} = \cdots = c_n$, then (b) reduces to

$$c_j - c_{n-j+1} \geq 2s_j \quad \text{for each } j \in \{1, \ldots, p\}. \tag{6}$$

For the verification of (i), see [2]. To verify (ii), we focus on the case when $c_{n-p+1} = \cdots = c_n$. First, note that (6) are just the subcollection of inequalities in Theorem 2.3 (b) with $|I| = |J| = |K| = 1$ and $J = \{1\}$. Now, suppose (6) holds. Then for each $j \in \{1, \ldots, p\}$, one can construct a Hermitian matrix of the form

$$C_j = \begin{pmatrix} r_j & s_j \\ s_j & t_j \end{pmatrix}$$

such that $r_j t_j - s_j^2 = c_j c_n$ and $r_j + s_j = c_j + c_n$, and thus C_j has eigenvalues c_j, c_n. Now, a permutation similarity of the matrix

$$C_1 \oplus \cdots \oplus C_p \oplus \operatorname{diag}(c_{p+1}, \ldots, c_{n-p})$$

will yield $A = (A_{ij})_{i,j=1}^{3} \in \mathcal{U}(c)$ such that $A_{13} \in \mathbb{C}^{p \times q}$ has singular values $s_1, \ldots, s_p$. By the implication $(a) \Rightarrow (b)$ in Theorem 2.3, we see that all the inequalities in (b) hold.

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Department of Mathematics, College of William and Mary, P.O. Box 8795, Williamsburg, Virginia 23187-8795, USA
E-mail address: ckli@math.wm.edu

Department of Mathematics, Iowa State University, Ames, Iowa 50011, USA
E-mail address: ytpoon@iastate.edu