PRODUCT OF OPERATORS AND NUMERICAL RANGE

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ABSTRACT. We show that a bounded linear operator $A \in B(H)$ is a multiple of a unitary operator if and only if AZ and ZA always have the same numerical radius or the same numerical range for all (rank one) $Z \in B(H)$. More generally, for any bounded linear operators $A, B \in B(H)$, we show that AZ and ZB always have the same numerical radius (resp., the same numerical range) for all (rank one) $Z \in B(H)$ if and only if $A = e^{it}B$ (resp., $A = B$) is a multiple of a unitary operator for some $t \in [0, 2\pi)$. We extend the result to other types of generalized numerical ranges including the k-numerical range and the higher-rank numerical range.

Keywords Numerical range; Numerical radius; k-numerical range; k-numerical radius; Higher-rank numerical range; Higher-rank numerical radius.
AMS Classification 47A12; 15A60.

1. Introduction

Let $B(H)$ be the algebra of bounded linear operators acting on a complex Hilbert space H. We identify $B(H)$ with M_n if H has dimension n. The *numerical range* of a bounded linear operator $A \in B(H)$ is defined by

$$W(A) = \{\langle Ax, x\rangle : x \in H, \|x\| = 1\} \subseteq \mathbb{C},$$

where $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the inner product and its associated norm in H, respectively. The *numerical radius* of A is

$$w(A) = \sup\{|z| : z \in W(A)\}.$$

It is known that $W(A)$ is a nonempty bounded convex subset of $\mathbb{C}$, and $w(A)$ is a norm on $B(H)$ satisfying $\|A\|/2 \le w(A) \le \|A\|$, where $\|A\|$ denotes the operator norm of A. The numerical range and numerical radius are useful in studying matrices and operators; for example, see [8, Chapter 22] or [7]. In particular, one can extract algebraic properties of an operator A by studying its numerical range and numerical radius. For instance, A is self-adjoint if and only if $W(A) \subseteq \mathbb{R}$; A is positive semi-definite if and only if $W(A) \subseteq [0, \infty)$; an invertible operator A is unitary if and only if $w(A) = w(A^{-1})$ ([14, Corollary]). In addition, there are interesting connection between the numerical range and radius of A and its spectrum $\sigma(A)$ and spectral radius $r(A)$. For example, $\sigma(A)$ is a subset of the closure of $W(A)$; $r(A) \le w(A)$ with equality holding if and only if $r(A) = \|A\|$.

In the study of spectrum and spectral radius, for any $A, B \in B(H)$, it is known that $\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}$ and $r(AB) = r(BA)$. However, $W(AB)$ may not be the same as $W(BA)$ in general. For example, for any nonzero vector $x, y \in H$, let $xy^* \in B(H)$ be the rank one operator such that $xy^*(z) = \langle z, y \rangle x$. Then for an orthonormal pair of vectors x and y, if $A = xy^*$ and $B = xx^*$, $W(AB) = \{0\}$ and $W(BA) = \{\mu \in \mathbb{C} : |\mu| \le 1/2\}$.

It is interesting to determine the pair of operators A and B such that $W(AB) = W(BA)$ and $w(AB) = w(BA)$. But one may not be able to get much information about the algebraic structure of the operators. For example, if $A = I_2 \oplus A_1$ and $B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \oplus B_1$ for some contractions A_1, B_1, then $W(AB) = W(BA) = \{\mu \in \mathbb{C} : |\mu| \le 1\}$. Not much can be said about A_1 and B_1.

It turns out that one can extract more interesting algebraic property of an operator $A \in B(H)$ by studying the following problem.

Problem 1. Determine $A \in B(H)$ such that $W(AZ) = W(ZA)$ or $w(AZ) = w(ZA)$ for all $Z \in B(H)$.

As we shall see, such operators must be a multiple of a unitary operator, and one needs only check $w(AZ) = w(ZA)$ for all rank one operators $Z \in B(H)$ to arrive at the conclusion.

We will also study the connection of two operators A and B by considering the following.

Problem 2. Characterize $A, B \in B(H)$ such that $W(AZ) = W(ZB)$ or $w(AZ) = w(ZB)$ for all $Z \in B(H)$.

It turns out that such a pair must be multiples of each other. We will also consider the problem for other types of generalized numerical ranges including the k-numerical range and the higher-rank numerical range. As one will see, we are able to get similar conclusion in these cases. In some cases, one can adapt the proofs on classical numerical range and numerical radius; in other cases, the proofs are more challenging.

2. Classical numerical range (radius) and k-numerical range (radius)

We begin with the following.

Lemma 2.1. *If $A \in B(H)$ has rank one, then $w(A) = (|\mathrm{tr}\, A| + \|A\|)/2$.*

Proof. By the fact that $W(A)$ is an elliptical disk with foci 0, $\mathrm{tr}\, A$, and length of minor axis $\sqrt{\|A\|^2 - |\mathrm{tr}\, A|^2}$, we get the conclusion. ∎

The main result of this section is the following theorem.

Theorem 2.2. *Let $A, B \in B(H)$. The following conditions are equivalent.*

(a) *$w(AZ) = w(ZB)$ for all $Z \in B(H)$.*

(b) *$w(Axy^*) = w(xy^*B)$ for all unit vectors $x, y \in H$.*

(c) $A = e^{it}B$ *is a multiple of a unitary operator for some* $t \in [0, 2\pi)$.

Proof. (c) $\Rightarrow$ (a) $\Rightarrow$ (b) are clear.

Suppose (b) holds. We may actually assume that (b) holds for any nonzero vectors $x, y \in H$. By Lemma 2.1, we have

$$|\langle Ax, y\rangle| + \|Ax\|\|y\| = 2w(Axy^*) = 2w(xy^*B) = |\langle Bx, y\rangle| + \|x\|\|B^*y\| \tag{1}$$

for all nonzero vectors $x, y \in H$. We want to show that $\|A\| = \|B\|$. Indeed, for any unit vector $x \in H$, replace y by Ax in (1), we have

$$2\|Ax\|^2 = |\langle x, B^*Ax\rangle| + \|x\|\|B^*Ax\| \leq 2\|B^*Ax\| \leq 2\|B^*\|\|Ax\|.$$

Thus, we deduce that $\|A\| \leq \|B^*\|$. On the other hand, for any unit vector $y \in H$, replace x by B^*y in (1), we have

$$2\|B^*y\|^2 = |\langle AB^*y, y\rangle| + \|AB^*y\|\|y\| \leq 2\|AB^*y\| \leq 2\|A\|\|B^*y\|.$$

Thus, we infer that $\|B^*\| \leq \|A\|$. Hence $\|A\| = \|B\|$ as desired. Now, we may assume that $\|A\| = \|B\| = 1$. Consider the following two cases separately:

(i) $\dim H = n < \infty$. Fix a unit vector $x_0 \in H$ with $\|Ax_0\| = 1$, from (1), we have

$$\begin{aligned} 2 &= |\langle Ax_0, Ax_0\rangle| + \|Ax_0\|\|Ax_0\| = |\langle Bx_0, Ax_0\rangle| + \|B^*Ax_0\| \\ &\leq \|Bx_0\|\|Ax_0\| + \|B^*Ax_0\| \leq 2. \end{aligned}$$

These equalities imply that $\|B^*Ax_0\| = 1$ and $Ax_0 = e^{i\theta}Bx_0$ for some real θ. Moreover, for any unit vector $y \in H$ with $y \perp Ax_0$, from (1), we have $\|B^*y\| = \|y\| = 1$. Since $BB^* \leq I_n$ and $(\{Ax_0\} \cup \{Ax_0\}^\perp) \subseteq \ker(I_n - BB^*)$, thus $\ker(I_n - BB^*) = H$ or $BB^* = I_n$. Hence B is unitary. On the other hand, for any unit vector $x \in H$, replace y by Bx in (1), we obtain $2 = |\langle Ax, Bx\rangle| + \|Ax\| \leq 2$. It forces that $\|Ax\| = 1$. Hence A is also unitary.

(ii) $\dim H = \infty$. Fix a unit vector $x \in H$, let $M_x = \operatorname{span}\{Ax, Bx\}$. For any unit vector $y \in M_x^\perp$, by (1), we have $\|Ax\| = \|B^*y\|$, it follows that

$$\lambda_x \equiv \langle A^*Ax, x\rangle = \|Ax\|^2 = \|B^*y\|^2 = \langle BB^*y, y\rangle. \tag{2}$$

For any unit vectors $x_1, x_2 \in H$, take a unit vector $y \in M_{x_1}^\perp \cap M_{x_2}^\perp$, (2) implies that $\lambda_{x_1} = \langle BB^*y, y\rangle = \lambda_{x_2}$. Thus $\lambda_x = c$, a constant, for all unit vector $x \in H$. Moreover, we also obtain $A^*A = cI = BB^*$ from (2). Since $\|A\| = \|B\| = 1$, then $c = 1$ or $A^*A = BB^* = I$.

Finally, in both cases (i) and (ii), by (1), we have

$$|\langle Ax, y\rangle| = |\langle Bx, y\rangle| \tag{3}$$

for all unit vectors $x, y \in H$. Replace y by Ax in (3), we have $1 = |\langle Bx, Ax\rangle| = |\langle A^*Bx, x\rangle|$ for all unit vector $x \in H$. Since the numerical range is convex, it forces that $W(A^*B) = \{e^{it}\}$ for some $t \in [0, 2\pi)$, that is, $A^*B = e^{it}I$. Note that $A^*A = BB^* = I$, therefore,

$A^* = A^*BB^* = e^{it}B^*$ and $B^*B = A^*A = B^*B = I$. Hence we conclude that B is unitary and $A = e^{-it}B$. ∎

By Theorem 2.2, we have the following corollaries immediately.

Corollary 2.3. *Let $A, B \in B(H)$. The following conditions are equivalent.*
(a) *$W(AZ) = W(ZB)$ for all $Z \in B(H)$.*
(b) *$W(Axy^*) = W(xy^*B)$ for all unit vectors $x, y \in H$.*
(c) *$A = B$ is a multiple of a unitary operator.*

For any bounded linear operator $A \in B(H)$, Corollary 1 in [14] asserts that A is unitary if and only if $w(A), w(A^{-1}) \le 1$. Moreover, [9, Theorem 3.9] shows that A is a nonzero multiple of a unitary operator if and only if $w(A^{-p}) = w(A)^{-p}$ for some positive integer p. The following corollaries are analogous to these results.

Corollary 2.4. *Let $A \in B(H)$. The following conditions are equivalent.*
(a) *$w(AZ) = w(ZA)$ for all $Z \in B(H)$.*
(b) *$w(Axy^*) = w(xy^*A)$ for all unit vectors $x, y \in H$.*
(c) *A is a multiple of a unitary operator.*

Corollary 2.5. *Let $A \in B(H)$. The following conditions are equivalent.*
(a) *$W(AZ) = W(ZA)$ for all $Z \in B(H)$.*
(b) *$W(Axy^*) = W(xy^*A)$ for all unit vectors $x, y \in H$.*
(c) *A is a multiple of a unitary operator.*

Recall that the k-numerical range and the k-numerical radius of $A \in B(H)$ are defined by

$$W_k(A) = \{\sum_{j=1}^{k} \langle Ax_j, x_j \rangle : \{x_1, \dots, x_k\} \text{ is an orthormal set in } H\},$$

and

$$w_k(A) = \sup\{|\mu| : \mu \in W_k(A)\}.$$

It is known that $W_k(A)$ is also a nonempty bounded convex subset of $\mathbb{C}$ (cf. [8, Problem 211]).

We can extend our results to the k-numerical range and k-numerical radius. If $k = \dim H$, then $W_k(A) = \{\operatorname{tr} A\}$, assuming convergence. So, we always assume that $k < \dim H$ in our consideration. The proof depends on the extension of Lemma 2.1.

Lemma 2.6. *If $A \in B(H)$ has rank one, then $W_k(A) = W(A)$ and $w_k(A) = w(A) = (|\operatorname{tr} A| + \|A\|)/2$.*

Proof. Note that $W_k(A) = \{\mathrm{tr}\,(X^*AX) : X^*X = I_k\}$. Suppose A has rank one. Then so is X^*AX and there is a unitary $U \in M_k$ such that $U^*X^*AXU = aE_{11} + bE_{12}$. Thus, $\mathrm{tr}\,(X^*AX) = a \in W(X^*AX) \subseteq W(A)$.

Suppose $a \in W(A)$. Then A is unitarily similar to $\tilde{A} = A_1 \oplus 0$ with $A_1 \in M_2$ with $(1,1)$ entry equal to a. Let $X = [e_1|e_3|\cdots|e_{k+1}]$, where Te_j equals the jth column of the operator matrix T, so that $X^*X = I_k$ and $a = \mathrm{tr}\,(X^*AX) \in W_k(A)$.

Hence, $W_k(A) = W(A)$ and $w_k(A) = w(A)$. ∎

Note that the above result is not true if A has rank larger then 1. For example, if $A = I - xx^*$ for some unit vector $x \in H$, then $W(A) = [0,1]$ and $W_k(A) = [k-1, k]$ for $k > 1$. We can see that $W(A) \cap W_k(A) = \emptyset$ for all $k > 1$.

By the preceding lemma, we have the following.

Theorem 2.7. *Let $A, B \in B(H)$, and $1 \le k < \dim H$. The following conditions are equivalent.*

(a) $w_k(AZ) = w_k(ZB)$ *for all* $Z \in B(H)$.

(b) $w_k(Axy^*) = w_k(xy^*B)$ *for all unit vectors* $x, y \in H$.

(c) $w(Axy^*) = w(xy^*B)$ *for all unit vectors* $x, y \in H$.

(d) $A = e^{it}B$ *is a multiple of a unitary operator for some* $t \in [0, 2\pi)$.

Proof. (c) ⇒ (d) follows from Theorem 2.2. (d) ⇒ (a) ⇒ (b) are clear, and (b) ⇒ (c) follows from Lemma 2.6. ∎

¿From Theorem 2.7, we immediately get the following corollaries.

Corollary 2.8. *Let $A, B \in B(H)$. The following conditions are equivalent.*

(a) $W_k(AZ) = W_k(ZB)$ *for all* $Z \in B(H)$.

(b) $W_k(Axy^*) = W_k(xy^*B)$ *for all unit vectors* $x, y \in H$.

(c) $A = B$ *is a multiple of a unitary operator.*

Corollary 2.9. *Let $A \in B(H)$ and $1 \le k < \dim H$. The following conditions are equivalent.*

(a) $w_k(AZ) = w_k(ZA)$ *for all* $Z \in B(H)$.

(b) $w_k(Axy^*) = w_k(xy^*A)$ *for all unit vectors* $x, y \in H$.

(c) A *is a multiple of a unitary operator.*

Corollary 2.10. *Let $A \in B(H)$ and $1 \le k < \dim H$. The following conditions are equivalent.*

(a) $W_k(AZ) = W_k(ZA)$ *for all* $Z \in B(H)$.

(b) $W_k(Axy^*) = W_k(xy^*A)$ *for all unit vectors* $x, y \in H$.

(c) A *is a multiple of a unitary operator.*

3. **Higher-rank numerical range and radius**

Let $1 \leq k < \dim H$. The rank-k numerical range $\Lambda_k(A)$ defined by

$$\Lambda_k(A) = \{\mu \in \mathbb{C} : X^*AX = \mu I_k \text{ for some } X^*X = I_k\}.$$

The study of the higher-rank numerical range was motivated by the investigation of quantum error correction, [1, 3, 4], and has attracted the attentions of many researchers, [5, 6, 10, 11, 12, 13, 15]. It has been shown that $\Lambda_k(A)$ possesses many nice properties as $W(A)$. For example, if $A \in M_n$ and $1 \leq k < n$, then $\Lambda_k(A)$ is always convex [15] and equals the set of complex number $\lambda \in \mathbb{C}$ for which $\mathrm{Re}\,(e^{-i\theta}\lambda) \leq \lambda_k(\mathrm{Re}\,(e^{-i\theta}A))$ for all real θ [10, Theorem 2.2]. For theoretical interest, researchers also consider the higher-rank numerical radius of A defined by

$$w_k^\Lambda(A) = \sup\{|\mu| : \mu \in \Lambda_k(A)\},$$

where we use the convention that $w_k^\Lambda(A) = -\infty$ if $\Lambda_k(A) = \emptyset$. It is known that if A has rank less than k, then $\Lambda_k(A) \subseteq \{0\}$. Thus, not much information on $A, B \in B(H)$ can be extract from the condition that $w_k^\Lambda(AZ) = w_k^\Lambda(ZB)$ for all $Z \in B(H)$ with rank less than k. Hence, we will focus on the condition that $w_k^\Lambda(AZ) = w_k^\Lambda(ZB)$ for all (rank k) operators $Z \in B(H)$.

Let us begin with the following lemmas.

Lemma 3.1. *Suppose that $A, B \in B(H)$ have rank at least $k < \dim H$. If $w_k^\Lambda(AZ) = w_k^\Lambda(ZB)$ for all rank k operators $Z \in B(H)$. Then* $\ker A \subseteq \ker B$ *and* $\ker B^* \subseteq \ker A^*$.

Proof. We first show that $\ker A \subseteq \ker B$. Assume otherwise, if $x \in H$ is a unit vector such that $Ax = 0$ and $Bx \neq 0$, let $u_1 = x/\|Bx\|$, then $\|Bu_1\| = 1$ and $Au_1 = 0$. Since $\operatorname{rank} B \geq k$, we may choose vectors $u_2, \ldots, u_k \in H$ such that $\{Bu_1, Bu_2, \ldots, Bu_k\}$ is an orthonormal subset of $\operatorname{ran} B$. Clearly, these vectors $u_1, u_2, \ldots, u_k$ are linearly independent. Let $Z = u_1(Bu_1)^* + u_2(Bu_2)^* + \cdots + u_k(Bu_k)^*$. Then Z is a rank k operator in $B(H)$. Note that $Au_1 = 0$ implies that $\operatorname{rank} AZ \leq k-1$, hence either $\Lambda_k(AZ) = \{0\}$ or $\Lambda_k(AZ) = \emptyset$. Let $M = \operatorname{span}\{u_1, u_2, \ldots, u_k\}$ and P_M be the (orthogonal) projection of H onto M. We check that $P_M(ZB)P_M = P_M$. Indeed, for any vector $y = \sum_{j=1}^k c_j u_j \in M$, since $\{Bu_1, Bu_2, \ldots, Bu_k\}$ is orthonormal, we have

$$(ZB)y = Z(\sum_{j=1}^k c_j Bu_j) = \sum_{i=1}^k \sum_{j=1}^k c_j \langle Bu_j, Bu_i\rangle u_i = \sum_{j=1}^k c_j u_j = y.$$

Therefore, $1 \in \Lambda_k(ZB)$ or $w_k^\Lambda(AZ) \leq 0 < 1 \leq w_k^\Lambda(ZB)$, this contradicts to our assumption. Hence we conclude that $Bx = 0$ and $\ker A \subseteq \ker B$.

To prove $\ker B^* \subseteq \ker A^*$, note that $w_k^\Lambda(AZ) = w_k^\Lambda(ZB)$ for all rank k operators $Z \in B(H)$, then $w_k^\Lambda(B^*Z) = w_k^\Lambda(ZA^*)$ for all rank k operators $Z \in B(H)$. From the above proof, we obtain $\ker B^* \subseteq \ker A^*$. ∎

Lemma 3.2. *Suppose that $A, B \in B(H)$ have rank at least $k < \dim H$. If $w_k^\Lambda(AZ) = w_k^\Lambda(ZB)$ for all rank k operators $Z \in B(H)$, then $\{0\} = \ker A = \ker B^*$.*

Proof. We first show that $\ker A = \{0\}$. Since $\operatorname{rank} B^* \geq k$, we may choose vectors $u_1, \ldots, u_k \in H$ such that $\{B^*u_1, B^*u_2, \ldots, B^*u_k\}$ is an orthonormal subset of $\operatorname{ran} B^*$. Assume otherwise, if $\ker A \neq \{0\}$, there exists a unit vector $v \in H$ such that $Av = 0$. By Lemma 3.1, we have $\ker A \subseteq \ker B = (\operatorname{ran} B^*)^\perp$, thus $v \perp B^*u_j$ for all $1 \leq j \leq k$. Let $Z = vu_1^* + (1/2)\big((B^*u_2)u_2^* + \cdots + (B^*u_k)u_k^*\big)$ be a rank k operator in $B(H)$. Note that $Av = 0$ implies $\operatorname{rank} AZ \leq k-1$, hence either $\Lambda_k(AZ) = \{0\}$ or $\Lambda_k(AZ) = \emptyset$. On the other hand, since $\{v, B^*u_1, B^*u_2, \ldots, B^*u_k\}$ is orthonormal, we have

$$\begin{aligned} ZB &= v(B^*u_1)^* + \frac{1}{2}\sum_{j=2}^{k}(B^*u_j)(B^*u_j)^* \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \oplus \frac{1}{2}I_{k-1} \oplus 0 \quad \text{on } H = M \oplus N \oplus (M \cup N)^\perp, \end{aligned}$$

where $M = \operatorname{span}\{B^*u_1, v\}$ and $N = \operatorname{span}\{B^*u_2, \ldots, B^*u_k\}$. Therefore, $1/2 \in \Lambda_k(ZB)$ or $w_k^\Lambda(AZ) \leq 0 \neq 1/2 = w_k^\Lambda(ZB)$, this contradicts to our assumption. Hence we conclude that $\ker A = \{0\}$.

To prove $\ker B^* = \{0\}$, since $w_k^\Lambda(AZ) = w_k^\Lambda(ZB)$ for all rank k operators $Z \in B(H)$, thus $w_k^\Lambda(B^*Z) = w_k^\Lambda(ZA^*)$ for all rank k operators $Z \in B(H)$. ¿From the above proof, $\ker B^* = \{0\}$. ■

For a positive integer r, let $\mathcal{F}_r$ be the set of operators in $B(H)$ with rank at most r.

Proposition 3.3. [11, Theorem 3.1] *Let $1 \leq r < k < \infty$. Suppose $A \in B(H)$ and $F \in \mathcal{F}_r$. Then $\Lambda_k(A) \subseteq \Lambda_{k-r}(A+F)$. Consequently,*

$$\Lambda_k(A) \subseteq \cap\{\Lambda_{k-r}(A+F) : F \in \mathcal{F}_r\}.$$

Corollary 3.4. *Let $Z = x_1y_1^* + \cdots + x_ky_k^*$ be an operator in $B(H)$. Then $\Lambda_k(Z) \subseteq W(x_1y_1^*)$.*

Proof. Let $F = -(x_2y_2^* + \cdots + x_ky_k^*) \in \mathcal{F}_{k-1}$. By Proposition 3.3, we have $\Lambda_k(Z) \subseteq \Lambda_{k-(k-1)}(Z+F) = W(x_1y_1^*)$. ■

Lemma 3.5. *Suppose $k < \dim H$, $x_0, y_0 \in H$ are unit vectors and $A \in B(H)$ has $\ker A = \{0\}$. Then there exist unit vectors $x_1, \ldots, x_{k-1} \in H$ and $c_1, \ldots, c_{k-1} > 0$ such that*

$$w_k^\Lambda\Big(A\big(x_0y_0^* + \sum_{j=1}^{k-1} c_jx_j(Ax_j)^*\big)\Big) = w\big((Ax_0)y_0^*\big).$$

Proof. Since $\ker A = \{0\}$, thus $\operatorname{rank} A = \dim H$. For $j = 1, \ldots, k-1$, we may choose a unit vector $x_j \in \{A^*y_0, A^*Ax_0, \ldots, A^*Ax_{j-1}\}^\perp$. Since $\ker A = \{0\}$, then $\|Ax_j\| > 0$ and $y_0 \perp Ax_j$ for $j = 1, \ldots, k-1$. Note that $0 = \langle x_j, A^*Ax_i\rangle = \langle Ax_j, Ax_i\rangle$ for $0 \leq i < j \leq k-1$, thus

$\{Ax_0, Ax_1, \ldots, Ax_{k-1}\}$ is linearly independent and orthogonal. Let $z_0 \in W\big((Ax_0)y_0^*\big)$ so that $|z_0| = w\big((Ax_0)y_0^*\big)$, and let $c_j = z_0/\|Ax_j\|^2$ for $j = 1, \ldots, k-1$. Let $F = \sum_{j=1}^{k-1} c_j x_j (Ax_j)^*$ and $Z = x_0 y_0^* + F$. Let $M = \operatorname{span}\{Ax_1, \ldots, Ax_{k-1}\}$, we can take a unit vector $u \in (\{y_0\} \cup M)^\perp$ such that $\operatorname{span}\{y_0, Ax_0\} \subseteq \operatorname{span}\{y_0, u\}$, since $\{y_0, Ax_0\} \perp M$ and $k < \dim H$. Then

$$\begin{aligned} AZ &= (Ax_0)y_0^* + \sum_{j=1}^{k-1} c_j (Ax_j)(Ax_j)^* \\ &= \begin{bmatrix} \langle Ax_0, y_0\rangle & 0 \\ \langle Ax_0, u\rangle & 0 \end{bmatrix} \oplus z_0 I_{k-1} \oplus 0 \quad \text{on} \quad H = \operatorname{span}\{y_0, u\} \oplus M \oplus (\{y_0, u\} \cup M)^\perp. \end{aligned}$$

Since

$$W((Ax_0)y_0^*) = W(\begin{bmatrix} \langle Ax_0, y_0\rangle & 0 \\ \langle Ax_0, u\rangle & 0 \end{bmatrix}),$$

we have $z_0 \in \Lambda_k(AZ)$ and $w\big((Ax_0)y_0^*\big) = |z_0| \le w_k^\Lambda(AZ)$. But Corollary 3.4 yields that $w_k^\Lambda(AZ) \le w\big((Ax_0)y_0^*\big)$, hence we conclude that $w_k^\Lambda(AZ) = w\big((Ax_0)y_0^*\big)$. ■

Theorem 3.6. *Suppose $A, B \in B(H)$ have rank at least $k < \dim H$. If $w_k^\Lambda(AZ) = w_k^\Lambda(ZB)$ for all rank k operators $Z \in B(H)$, then $w(Axy^*) = w(xy^*B)$ for all unit vectors $x, y \in H$.*

Proof. By Lemma 3.2, we have $\ker A = \ker B^* = \{0\}$. Let $x_0, y_0 \in H$ be unit vectors. By Lemma 3.5, there exist unit vectors $x_1, \ldots, x_{k-1} \in H$ and $c_1, \ldots, c_{k-1} > 0$ such that

$$w_k^\Lambda\Big(A\big(x_0 y_0^* + \sum_{j=1}^{k-1} c_j x_j (Ax_j)^*\big)\Big) = w\big((Ax_0)y_0^*\big).$$

Let $Z = x_0 y_0^* + \sum_{j=1}^{k-1} c_j x_j (Ax_j)^*$, then $ZB = x_0 (B^* y_0)^* + \sum_{j=1}^{k-1} c_j x_j (B^* A x_j)^*$. Corollary 3.4 yields that $w_k^\Lambda(ZB) \le w(x_0 (B^* y_0)^*)$. Therefore, we obtain

$$w(Ax_0 y_0^*) = w_k^\Lambda(AZ) = w_k^\Lambda(ZB) \le w(x_0 (B^* y_0)^*) = w(x_0 y_0^* B).$$

Conversely, since $\ker B^* = \{0\}$, by Lemma 3.5, there exist unit vectors $u_1, \ldots, u_{k-1} \in H$ and $s_1, \ldots, s_{k-1} > 0$ such that

$$w_k^\Lambda\Big(B^*\big(y_0 x_0^* + \sum_{j=1}^{k-1} s_j u_j (B^* u_j)^*\big)\Big) = w\big((B^* y_0) x_0^*\big).$$

Let $Z' = y_0 x_0^* + \sum_{j=1}^{k-1} s_j u_j (B^* u_j)^*$, then $Z' A^* = y_0 (Ax_0)^* + \sum_{j=1}^{k-1} s_j u_j (AB^* u_j)^*$. Corollary 3.4 yields that $w_k^\Lambda(Z' A^*) \le w(y_0 (Ax_0)^*)$. Therefore, we obtain

$$w(x_0 y_0^* B) = w((B^* y_0) x_0^*) = w_k^\Lambda(B^* Z') = w_k^\Lambda(Z' A^*) \le w(y_0 (Ax_0)^*) = w(Ax_0 y_0^*).$$

Hence $w(A(x_0 y_0^*)) = w((x_0 y_0^*)B)$ for all unit vectors $x_0, y_0 \in H$. ■

We are ready to show our main result of this section.

Corollary 3.7. *Suppose $A, B \in B(H)$ have rank at least $k < \dim H$. The following are equivalent.*

(a) $w_k^\Lambda(AZ) = w_k^\Lambda(ZB)$ *for all* $Z \in B(H)$.

(b) $w_k^\Lambda(AZ) = w_k^\Lambda(ZB)$ *for all rank* k *operators* $Z \in B(H)$.

(c) $w(Axy^*) = w(xy^*B)$ *for all unit vectors* $x, y \in H$.

(d) $A = e^{it}B$ *is a multiple of a unitary operator for some* $t \in [0, 2\pi)$.

Proof. (c) ⇒ (d) follows from Theorem 2.2. (d) ⇒ (a) ⇒ (b) are clear, and (b) ⇒ (c) follows from Theorem 3.6. ■

¿From the preceding result, we immediately get the following corollaries.

Corollary 3.8. *Suppose $A, B \in B(H)$ have rank at least $k < \dim H$. The following are equivalent.*

(a) $\Lambda_k(AZ) = \Lambda_k(ZB)$ *for all* $Z \in B(H)$.

(b) $\Lambda_k(AZ) = \Lambda_k(ZB)$ *for all rank* k *operators* $Z \in B(H)$.

(c) $A = B$ *is a multiple of a unitary operator.*

Corollary 3.9. *Suppose $A \in B(H)$ has rank at least $k < \dim H$. The following are equivalent.*

(a) $w_k^\Lambda(AZ) = w_k^\Lambda(ZA)$ *for all* $Z \in B(H)$.

(b) $w_k^\Lambda(AZ) = w_k^\Lambda(ZA)$ *for all rank* k *operators* $Z \in B(H)$.

(c) A *is a multiple of a unitary operator.*

Corollary 3.10. *Suppose $A \in B(H)$ has rank at least $k < \dim H$. The following are equivalent.*

(a) $\Lambda_k(AZ) = \Lambda_k(ZA)$ *for all* $Z \in B(H)$.

(b) $\Lambda_k(AZ) = \Lambda_k(ZA)$ *for all rank* k *operators* $Z \in B(H)$.

(c) A *is a multiple of a unitary operator.*

Acknowledgments

The authors would like to thank Ngai-Ching Wong for some stimulating discussion. The research of Gau, Tsai and Wang was partially supported by the Ministry of Science and Technology of the Republic of China under projects MOST 103-2115-M-008-006, MOST 103-2811-M-110-027 and MOST 103-2115-M-009-001, respectively. The research of Li was partially supported by NSF USA; he is an affiliate member of the Institute for Quantum Computing at University of Waterloo, an honorary professor of the University of Hong Kong, and the Shanghai University. This research began when he was visiting Taiwan in the summer of 2014 supported by the Ministry of Science and Technology of the Republic of China.

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