PSEUDOSPECTRA OF SPECIAL OPERATORS AND PSEUDOSECTRUM PRESERVERS

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ABSTRACT. Denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space H. Let $A \in \mathcal{B}(H)$, and denote by $\sigma(A)$ the spectrum of A. For $\varepsilon > 0$, define the ε-pseudospectrum $\sigma_\varepsilon(A)$ of A as

$$\sigma_\varepsilon(A) = \{z \in \sigma(A+E) : E \in \mathcal{B}(H),\ \|E\| < \varepsilon\}.$$

In this paper, the pseudospectra of several special classes of operators are characterized. As an application, complete descriptions are given of the maps of $B(H)$ leaving invariant the pseudospectra of $A \bullet B$ for different kind of binary operations $\bullet$ on operators such as the difference $A - B$, the operator product AB, and the Jordan product $AB + BA$.

1. Introduction

Denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space H. If H has dimension $n < \infty$ we identify $\mathcal{B}(H)$ as the algebra M_n of $n \times n$ complex matrices. Let $A \in \mathcal{B}(H)$ and

$$\sigma(A) = \{z \in \mathbb{C} : zI - A \text{ is not invertible in } \mathcal{B}(H)\}$$

the spectrum of A. The spectrum of an operator provides a lot of useful information about it. For instance, $\lim_{m\to\infty} \sum_{k=0}^m A^k$ exists if $\sigma(A)$ lies inside a disk centered at the origin with radius $r < 1$; a system of differential equations governed by $\frac{\mathrm{d}}{\mathrm{dt}}x = Ax$ for a given matrix A always has an equilibrium solution if $\sigma(A) \subseteq \{z : (z + z^*) < 0\}$. So, there is a lot of interest in finding efficient ways to determine or estimate $\sigma(A)$. Moreover, due to numerical and measuring errors, and also due to the fact that $\sigma(A)$ is very sensitive to perturbation, researchers propose the study of the ε-pseudospectrum of A for a given $\varepsilon > 0$ defined by

$$\sigma_\varepsilon(A) = \cup\{\sigma(A+E) : E \in \mathcal{B}(H), \|E\| < \varepsilon\}. \tag{1.1}$$

Here $\|E\|$ is the (spectral) norm of E. Evidently, for $\varepsilon \in (0, \infty)$, the ε-pseudospectra of A form a family of strictly nested closed sets, which grow to fill the whole complex plane as

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$\varepsilon \to \infty$. It follows from the upper-semicontinuity of the spectrum that the intersection of all the pseudospectra is the spectrum,

$$\cap_{\varepsilon>0}\sigma_\varepsilon(A) = \sigma(A).$$

The operator $(zI - A)^{-1}$ is the resolvent of an operator A at the point $z \in \mathbb{C}$. One may also define the ε-pseudospectrum as follows:

$$\sigma_\varepsilon(A) = \{z \in \mathbb{C} : \|(zI - A)^{-1}\| > \varepsilon^{-1}\}. \tag{1.2}$$

Here we use the convention that $\|(zI - A)^{-1}\| = \infty$ if $z \in \sigma(A)$.

Two other equivalent definitions of the ε-pseudospectrum are respectively

$$\sigma_\varepsilon(A) = \{z \in \mathbb{C} : \|(zI - A)x\| < \varepsilon \text{ for some unit vector } x \in H\}, \tag{1.3}$$

and

$$\sigma_\varepsilon(A) = \{z \in \mathbb{C} : s_{\min}(zI - A) < \varepsilon\}, \tag{1.4}$$

where $s_{\min}$ denotes the minimal singular value in the matrix case or the smallest s-number for an operator [6]. Clearly, the pseudospectrum is invariant under unitary similarities. Denote by A^t the transpose of A relative to an arbitrary but fixed orthonormal basis of H. Then $\sigma_\varepsilon(A) = \sigma_\varepsilon(A^t)$ because $\sigma(X) = \sigma(X^t)$ for any $X \in \mathcal{B}(H)$.

Let us recall other properties of the pseudospectrum (see [12]), which will be frequently used in our proof. Let $\varepsilon > 0$ be arbitrary and $D(a, \varepsilon) = \{\mu \in \mathbb{C} : |\mu - a| < \varepsilon\}$, where $a \in \mathbb{C}$.

Property 1.1. *Let $\varepsilon > 0$ and let $A \in \mathcal{B}(H)$.*

(1) $\sigma(A) + D(0, \varepsilon) \subseteq \sigma_\varepsilon(A)$.

(2) *If A is normal, then $\sigma_\varepsilon(A) = \sigma(A) + D(0, \varepsilon)$.*

(3) *For any $c \in \mathbb{C}$, $\sigma_\varepsilon(A + cI) = c + \sigma_\varepsilon(A)$.*

(4) *For any nonzero $c \in \mathbb{C}$, $\sigma_\varepsilon(cA) = c\sigma_{\frac{\varepsilon}{|c|}}(A)$.*

In this paper, we give complete descriptions of some special classes of operators in terms of the pseudospectrum in Section 2. For example, we prove that an operator is a multiple of a self-adjoint operator if and only if its pseudospectrum lies in the set $\{z \in \mathbb{C} : |\mathrm{Im}(z)| < \varepsilon\}$. In particular, an operator $A = \alpha I$ for some scalar α if and only if $\sigma_\varepsilon(A) = D(\alpha, \varepsilon)$; an operator A is a nontrivial projection if and only if $\sigma_\varepsilon(A) = D(0, \varepsilon) \cup D(1, \varepsilon)$. Furthermore, we show that if an operator A satisfies $A^2 = 0$, then $\sigma_\varepsilon(A) = D(0, \sqrt{\varepsilon^2 + \|A\|\varepsilon})$. In particular, if A is of rank one, then A is nilpotent if and only if $\sigma_\varepsilon(A) = D(0, \sqrt{\varepsilon^2 + \|A\|\varepsilon})$; we also prove that the map $(\varepsilon, A) \mapsto \sigma_\varepsilon(A)$ is continuous. In Sections 3–5, we characterize the maps Φ on operators such that $\sigma_\varepsilon(A \bullet B) = \sigma_\varepsilon(\Phi(A) \bullet \Phi(B))$, where $A \bullet B$ is one of the binary operations: $A - B, AB, AB + BA$. We note that the study of similar problems on matrix algebras was done in [4]. In this paper, we develop additional tools to lift the results to the infinite dimensional case, and treat the Jordan product $AB + BA$. Linear preservers of pseudospectrum have also been studied in a recent paper by Kumar and Kulkarni [8]. In our study, we show that maps preserving different classes of pseudospectrum preservers are automatically linear.

To conclude this section, let us fix some notations. Let H be a complex Hilbert space. Denote by $\mathcal{B}(H)$ the C^*-algebra of all bounded linear operators on H and by $\mathcal{B}_s(H)$ the set of all self-adjoint operators in $\mathcal{B}(H)$. For $A \in \mathcal{B}(H)$, $\ker A$ and $\text{rng}A$ denote the kernel and range of A, respectively. For a closed subspace M of H, $A|_M$ denotes a restriction of A to M, and $M^\perp$ denotes the orthogonal complement of M in H, and P_M the orthogonal projection from H onto M. For $x \in H$, $[x]$ denotes the linear space spanned by x. For any nonzero $x, f \in H$, denote by $x \otimes f$ rank one operator $z \mapsto \langle z, f\rangle x$, and all rank one operators in $\mathcal{B}(H)$ can be written into this form. Denote by I the identity operator in $\mathcal{B}(H)$. Let $\text{Tr}A$ denote the trace of a finite rank operator A.

2. Some properties of the pseudospectrum

In this section, we give complete descriptions of the pseudospectra of self-adjoint operators. We also classify the pseudospectra of several classes of operators such as projections and square zero operators. Moreover, we obtain a result concerning the continuity of the pseudospectrum.

Lemma 2.1. *Let $\varepsilon > 0$ and $A \in \mathcal{B}(H)$. Assume that $\alpha \in \sigma_p(A)$ (the set of all point spectra of A). If $\ker(\alpha I - A)$ is not a reduced subspace of A, then there exists $r > \varepsilon$ such that $\overline{D(\alpha, r)} \subset \sigma_\varepsilon(A)$.*

Proof. Let $\alpha \in \sigma_p(A)$, then $\ker(\alpha I - A)$ is an invariant subspace of A. According to the space decomposition $H = \ker(\alpha I - A) \oplus \ker(\alpha I - A)^\perp$, A has an operator matrix representation

$$A = \begin{pmatrix} \alpha I & B \\ 0 & C \end{pmatrix}.$$

Since $\ker(\alpha I - A)$ is not a reduced subspace of A, we have $B \neq 0$. Therefore there exist orthogonal unit vectors $u \in \ker(A - \alpha I)^\perp$ and $v \in \ker(A - \alpha I)$ such that $\langle Bu, v\rangle \neq 0$. Let $P = u \otimes u + v \otimes v$. For each $z \in \overline{D(0, \varepsilon)}$, take $\alpha_z = \alpha + z$. Then $P(\alpha_z I - A)^*(\alpha_z I - A)P$ is unitarily similar to $B_z \oplus 0$, where

$$B_z = \begin{pmatrix} |z|^2 & -\overline{z}\langle Bu, v\rangle \\ -z\langle v, Bu\rangle & |\langle Bu, v\rangle|^2 + \|(\alpha_z I - C)u\|^2 \end{pmatrix}.$$

Thus the minimal eigenvalue λ_z of $P(\alpha_z I - A)^*(\alpha_z I - A)P$ is smaller than $|z|^2$, and hence there exists a unit vector $x_z \in [u, v]$ such that $\|(\alpha_z I - A)Px_z\| = \sqrt{\lambda_z} < |z| \leq \varepsilon$. Since the map $z \to \lambda_z$ is continuous on $\overline{D(0, \varepsilon)}$, there exists $d < \varepsilon$ such that $\|(\alpha_z I - A)Px_z\| \leq d$ for all $z \in \overline{D(0, \varepsilon)}$. Let $r = \dfrac{3\varepsilon - d}{2} > \varepsilon$. For every $z \in \overline{D(0, r)}$, consider the following cases:

Case 1 $|z| \leq \varepsilon$. From the above discussion, we have a unit vector $x_z \in [u, v]$ such that $\|(\alpha_z I - A)Px_z\| \leq d < \varepsilon$. Hence, $\alpha + z \in \sigma_\varepsilon(A)$.

Case 2 $|z| > \varepsilon$. Let $z = |z|e^{it}$ with $t \in \mathbb{R}$. Let $\varepsilon_t = \varepsilon e^{it}$. From the above discussion, we have a unit vector $x_{\varepsilon_t} \in [u, v]$ such that $\|(\alpha_{\varepsilon_t} I - A)Px_{\varepsilon_t}\| \le d < \varepsilon$. We have

$$\begin{aligned} & \|((\alpha + z)I - A)x_{\varepsilon_t}\| = \|(\alpha + \varepsilon e^{it} + (|z| - \varepsilon)e^{it})I - A)x_{\varepsilon_t}\| \\ \le \quad & \|(\alpha_{\varepsilon_t} I - A)Px_{\varepsilon_t}\| + |z| - \varepsilon \le d + r - \varepsilon = \frac{\varepsilon + d}{2} < \varepsilon \end{aligned}$$

Hence, $\alpha + z \in \sigma_\varepsilon(A)$.

Thus, $\overline{D(\alpha, r)} \subset \sigma_\varepsilon(A)$ and the proof is complete. □

Theorem 2.2. *Let $\varepsilon > 0$, $A \in \mathcal{B}(H)$, and $t \in \mathbb{R}$. Then $e^{it}A$ is self-adjoint if and only if $\sigma_\varepsilon(A) \subset \{z \in \mathbb{C} : |\text{Im } e^{it}z| < \varepsilon\}$.*

Proof. Since $\sigma_\varepsilon(e^{it}A) = e^{it}\sigma_\varepsilon(A)$, it suffices to prove the case when $t = 0$.

Let $A \in \mathcal{B}(H)$ be any self-adjoint operator. Since $\sigma_\varepsilon(A) = \cup_{\alpha \in \sigma(A)} D(\alpha, \varepsilon)$, it follows that for every $z \in \sigma_\varepsilon(A)$, $|\text{Im } z| < \varepsilon$.

To prove the sufficiency, assume that $\sigma_\varepsilon(A) \subset \{z \in \mathbb{C} : |\text{Im } z| < \varepsilon\}$. For every $\alpha \in \sigma(A)$, since $D(\alpha, \varepsilon) \subset \sigma_\varepsilon(A) \subset \{z \in \mathbb{C} : |\text{Im } z| < \varepsilon\}$, we have $\alpha \in \mathbb{R}$.

First we consider the case when $\sigma(A) = \sigma_p(A)$.

Since $\sigma(A) = \sigma_p(A)$, for every $\alpha \in \sigma(A)$, $\ker(\alpha I - A)$ is a non-zero invariant subspace of A. Assume that there exists $\alpha \in \sigma(A)$ such that $\ker(\alpha I - A)$ is not a reduced subspace of A, then, by Lemma 2.1, there exists $r > \varepsilon$ such that $\alpha + ir \in \sigma_\varepsilon(A)$, a contradiction. So $\ker(\alpha I - A)$ reduces A for every $\alpha \in \sigma(A)$.

Clearly, for distinct $\alpha,\ \beta \in \sigma(A)$, $\ker(\alpha I - A)$ and $\ker(\beta I - A)$ are orthogonal and H is spanned by $\{\ker(\alpha I - A) : \alpha \in \sigma(A)\}$. Since every x can be expressed as $x = \sum_\alpha x_\alpha$, where $x_\alpha \in \ker(\alpha I - A)$, $\langle Ax, x\rangle$ is real. Hence, A is self-adjoint.

For the general case, we can use Berberian's construction [2] to get a unital isometric $*$-representation $\phi : \mathcal{B}(H) \to \mathcal{B}(K)$ such that for every $A \in \mathcal{B}(H)$, $\sigma_a(A) = \sigma_a(\phi(A)) = \sigma_p(\phi(A))$, where $\sigma_a(A)$ denotes the set of approximate point spectra of A. Hence, by (1.2), $\sigma_\varepsilon(A) = \sigma_\varepsilon(\phi(A))$ for every $A \in \mathcal{B}(H)$. We have

$$\begin{aligned} & \sigma_\varepsilon(A) \subset \{z \in \mathbb{C} : |\text{Im } z| < \varepsilon\} \\ \Rightarrow \quad & \sigma_\varepsilon(\phi(A)) \subset \{z \in \mathbb{C} : |\text{Im } z| < \varepsilon\} \\ \Rightarrow \quad & \phi(A) \text{ is self-adjoint.} \end{aligned}$$

Since $\phi(A^*) = \phi(A)^* = \phi(A)$, and ϕ is linear isometry, we have $A = A^*$, that is, A is self-adjoint. This completes the proof. □

Corollary 2.3. *Let $\varepsilon > 0$ and $A \in \mathcal{B}(H)$. Then the following statements hold.*

(1) *$A = aI$ if and only if $\sigma_\varepsilon(A) = D(a, \varepsilon)$, where $a \in \mathbb{C}$.*

(2) *Let $a \in \mathbb{C}$ be nonzero. There exists a nontrivial projection $P \in \mathcal{B}(H)$ such that $A = aP$ if and only if $\sigma_\varepsilon(A) = D(0, \varepsilon) \cup D(a, \varepsilon)$.*

Proof. (1) The sufficiency is clear. Conversely, assume that $\sigma_\varepsilon(A) = D(a, \varepsilon)$. Let $a = |a|e^{it}$, where $t \in \mathbb{R}$, then $\sigma_\varepsilon(e^{-it}A) = D(|a|, \varepsilon)$. By Theorem 2.2, $e^{-it}A$ is self-adjoint. Because $D(\lambda, \varepsilon) \subseteq \sigma_\varepsilon(e^{-it}A)$ for every $\lambda \in \sigma(e^{-it}A)$, we see that $\sigma(e^{-it}A) = \{|a|\}$ and the result follows.

The proof of (2) is similar. $\square$

Proposition 2.4. *Let $\varepsilon > 0$. If $X \in \mathcal{B}(H)$ satisfies $X^2 = 0$, then*

$$\sigma_\varepsilon(X) = D(0, \sqrt{\varepsilon^2 + \|X\|\varepsilon}).$$

Proof. The case for $X = 0$ is clear. Now assume that $X^2 = 0$ and $X \neq 0$. Then $\ker X \neq H$. According to the decomposition $H = \ker X \oplus \ker X^\perp$, X has an operator matrix representation

$$X = \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix},$$

where $N \in \mathcal{B}(\ker X^\perp, \ker X)$ is injective. Let $z \in \mathbb{C}$ and write $X_z = zI - X$. For $\lambda > 0$, we have $\lambda \in \sigma(X_z X_z^*)$ if and only if $\frac{1}{\lambda}(\lambda - |z|^2)^2 \in \sigma(NN^*)$. By direct computation,

$$\min \sigma(X_z X_z^*) = \frac{2|z|^2 + \|N\|^2 - \|N\|\sqrt{\|N\|^2 + 4|z|^2}}{2}.$$

Hence, if $s_{\min}(X_z) < \varepsilon$, then

$$\sqrt{\frac{2|z|^2 + \|N\|^2 - \|N\|\sqrt{\|N\|^2 + 4|z|^2}}{2}} < \varepsilon.$$

As a result, $|z| < \sqrt{\varepsilon^2 + \|N\|\varepsilon}$. Since $\|X\| = \|N\|$, we have $\sigma_\varepsilon(X) = D(0, \sqrt{\varepsilon^2 + \|X\|\varepsilon})$. $\square$

The following result has been proved in [4, Proposition 2.7].

Lemma 2.5. *Let $\varepsilon > 0$. Denote by $r_\varepsilon(A) = \sup\{|z| : z \in \sigma_\varepsilon(A)\}$. If $x, f \in H$, then*

$$r_\varepsilon(x \otimes f) = \frac{1}{2}\left(\sqrt{|\langle x, f\rangle|^2 + 4\varepsilon^2 + 4\varepsilon\|x\|\|f\|} + |\langle x, f\rangle|\right),$$

which is attained at a point in $\sigma_\varepsilon(x \otimes f)$ in the direction $\langle x, f\rangle$.

The next result follows immediately from Proposition 2.4 and Lemma 2.5.

Corollary 2.6. *Let $\varepsilon > 0$ and $x, f \in H$ be arbitrary. Then $\langle x, f\rangle = 0$ if and only if*

$$\sigma_\varepsilon(x \otimes f) = D(0, \sqrt{\varepsilon^2 + \|x\|\|f\|\varepsilon}).$$

The next result shows that $\sigma_\varepsilon(A)$ varies continuously with respect to A and ε. It is well known that the spectrum is upper semi-continuous, but, in general, it may not be continuous (for example, see [1] and [7]).

Proposition 2.7. *The map* $(\varepsilon, A) \mapsto \sigma_\varepsilon(A)$, *which sends a positive number* ε *and* $A \in \mathcal{B}(H)$ *to the bounded set* $\sigma_\varepsilon(A)$ *in* $\mathbb{C}$, *is continuous using the metric*

$$d((\varepsilon_1, A_1),\ (\varepsilon_2, A_2)) = \|A_1 - A_2\| + |\varepsilon_1 - \varepsilon_2|$$

in the domain and the metric

$$d(\Lambda, \Delta) = \max\left\{\sup_{s\in\Lambda}\inf_{t\in\Delta}|s-t|, \sup_{t\in\Delta}\inf_{s\in\Lambda}|s-t|\right\}$$

in the co-domain, where Λ *and* Δ *are two sets in* $\mathbb{C}$.

Proof. Given (ε, A) and $0 < \delta < \varepsilon/2$, clearly $\sigma_{\varepsilon-\delta}(A) \subset \sigma_\varepsilon(A) \subset \sigma_{\varepsilon+\delta}(A)$. Take an arbitrary $T \in \mathcal{B}(H)$ and $\epsilon' > 0$ such that

$$\|A - T\| + |\varepsilon - \varepsilon'| < \delta\,.$$

We have

$$\varepsilon - \varepsilon' < \delta \Rightarrow \varepsilon' > \varepsilon - \delta > \delta\,.$$

Then, for any $z \in \sigma_{\varepsilon'}(T)$, there exists a unit vector $x \in X$ such that $\|(zI - T)x\| < \varepsilon'$, and hence,

$$\|(zI - A)x\| \le \|(zI - T)x\| + \|(A - T)x\| < \varepsilon' + \delta - |\varepsilon - \varepsilon'| < \varepsilon + \delta,$$

which implies that $z \in \sigma_{\varepsilon+\delta}(A)$. So $\sigma_{\varepsilon'}(T) \subseteq \sigma_{\varepsilon+\delta}(A)$; similarly it can be proved that $\sigma_{\varepsilon-\delta}(A) \subseteq \sigma_{\varepsilon'}(T)$. Thus $d(\sigma_\varepsilon(A), \sigma_{\varepsilon'}(T)) \le d(\sigma_{\varepsilon-\delta}(A), \sigma_{\varepsilon+\delta}(A)) \le 2\delta \to 0$ when $\delta \to 0$. Hence the pseudospectrum is jointly continuous in $(\varepsilon,\ A)$. □

As pointed out by the referee, the above proposition is a folklore result in pseudospectrum. However, it is not easy to locate a proper reference. Note that our proof also works if we replace H by an arbitrary Banach space, and define the ε-spectrum on the bounded operators of the Banach space accordingly.

3. PSEUDOSPECTRUM PRESERVERS OF THE DIFFERENCE OF OPERATORS

In this section, we characterize pseudospectrum preservers of the difference of any pair of operators.

Theorem 3.1. *Let* $\varepsilon > 0$. *A surjective map* $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ *satisfies*

$$\sigma_\varepsilon(\Phi(A) - \Phi(B)) = \sigma_\varepsilon(A - B), \qquad \text{for all } A, B \in \mathcal{B}(H)$$

if and only if there exist $S \in \mathcal{B}(H)$ *and a unitary operator* $U \in \mathcal{B}(H)$ *such that* Φ *has the form*

$$A \mapsto UAU^* + S \qquad \text{or} \qquad A \mapsto UA^tU^* + S,$$

where A^t *denotes the transpose of* A *relative to an arbitrary but fixed orthonormal basis of* H. *Furthermore, when* H *is finite dimensional, the surjectivity assumption on* Φ *can be removed.*

Proof. The case when H is finite dimensional has been proven in [4]. Therefore, we may assume that dim H is infinite.

Let $\Psi(A) = \Phi(A) - \Phi(0)$ for every $A \in \mathcal{B}(H)$, then $\Psi(0) = 0$, $\sigma_\varepsilon(\Psi(A)) = \sigma_\varepsilon(A)$ for all $A \in \mathcal{B}(H)$ and $\sigma_\varepsilon(\Psi(A) - \Psi(B)) = \sigma_\varepsilon(A - B)$ for all $A, B \in \mathcal{B}(H)$.

Since $\sigma_\varepsilon(\Psi(A)) = \sigma_\varepsilon(A)$, by Theorem 2.2, $A \in \mathcal{B}_s(H)$ if and only if $\Psi(A) \in \mathcal{B}_s(H)$. So $\Psi : \mathcal{B}_s(H) \to \mathcal{B}_s(H)$ is bijective. Similarly, $\Psi : i\mathcal{B}_s(H) \to i\mathcal{B}_s(H)$ is bijective.

For all self-adjoint operators $A, B \in \mathcal{B}_s(H)$, it follows from

$$\sigma(A - B) + D(0, \varepsilon) = \sigma_\varepsilon(A - B) = \sigma_\varepsilon(\Psi(A) - \Psi(B)) = \sigma(\Psi(A) - \Psi(B)) + D(0, \varepsilon)$$

that $\sigma(A - B) = \sigma(\Psi(A) - \Psi(B))$. Therefore, $\|A - B\| = \|\Psi(A) - \Psi(B)\|$. So Ψ is real linear on $\mathcal{B}_s(H)$ and $\sigma(A) = \sigma(\Psi(A))$. It follows from [9, Theorem 2] that there exists a unitary operator U such that $\Psi(A) = UAU^*$ for every $A \in \mathcal{B}_s(H)$ or $\Psi(A) = UA^tU^*$ for every $A \in \mathcal{B}_s(H)$, where A^t denotes the transpose of A relative to an arbitrary but fixed orthonormal basis of H.

Considering the transformation $A \mapsto U^*\Psi(A)U$ or $A \mapsto (U^*\Psi(A)U)^t$, then this transformation satisfies the condition in the theorem, so we might as well assume that $\Psi(A) = A$ for every $A \in \mathcal{B}_s(H)$.

Claim 1. For any $A \in \mathcal{B}_s(H)$, $\Psi(iA) = iA$.

Since $\Psi : i\mathcal{B}_s(H) \to i\mathcal{B}_s(H)$ is bijective, $\Psi(iA) = iB$ for some $B \in \mathcal{B}_s(H)$. By Property 1.1 (4), we have

$$\begin{aligned} & \sigma_\varepsilon(A - iA) = \sigma_\varepsilon(\Phi(A) - \Phi(iA)) = \sigma_\varepsilon(A - iB) \\ \Rightarrow\quad & (1 + i)\sigma_\varepsilon(A - iA) = (1 + i)\sigma_\varepsilon(A - iB) \\ \Rightarrow\quad & \sigma_{\sqrt{2}\varepsilon}((1 + i)(1 - i)A) = \sigma_{\sqrt{2}\varepsilon}((1 + i)(A - iB)) \\ \Rightarrow\quad & \sigma_{\sqrt{2}\varepsilon}(2A) = \sigma_{\sqrt{2}\varepsilon}((1 + i)(A - iB))\,. \end{aligned}$$

By Theorem 2.2, $(1 + i)(A - iB)$ is self-adjoint. Therefore, we have

$$\begin{aligned} & (1 + i)(A - iB) = (1 - i)(A + iB) \\ \Rightarrow\quad & (A + B) + i(A - B) = (A + B) + i(B - A) \\ \Rightarrow\quad & A = B\,. \end{aligned}$$

This proves Claim 1.

Claim 2. For any $A \in \mathcal{B}(H)$, $\Psi(A) = A$.

Let $A = B + iC$ with $B,\ C \in \mathcal{B}_s(H)$. Suppose $\Psi(A) = T + iS$ for some $T, S \in \mathcal{B}_s(H)$. Since

$$\sigma_\varepsilon(T + iS - B) = \sigma_\varepsilon(\Psi(A) - \Psi(B)) = \sigma_\varepsilon(A - B) = \sigma_\varepsilon(iC).$$

Theorem 2.2 implies that $T+iS-B$ is a skew-Hermitian and consequently, $T=B$. Similarly, by Claim 1, one can show that $S=C$. So $\Psi(B+iC)=B+iC$ for all self-adjoint operators B,C. This completes the proof. □

From Theorem 3.1, one readily obtains the following result.

Corollary 3.2. *Let $\varepsilon>0$. A surjective map $\Phi:\mathcal{B}(H)\to\mathcal{B}(H)$ satisfies*

$$\sigma_\varepsilon(\Phi(A)+\Phi(B))=\sigma_\varepsilon(A+B) \quad \text{for all } A,B\in\mathcal{B}(H)$$

if and only if there is a unitary operator $U\in\mathcal{B}(H)$ such that Φ has the form

$$A\mapsto UAU^* \qquad \text{or} \qquad A\mapsto UA^tU^*,$$

where A^t denotes the transpose of A relative to an arbitrary but fixed orthonormal basis of H. Furthermore, when H is finite dimensional, the surjectivity assumption on Φ can be removed.

4. Pseudospectrum preservers of the product of operators

In this section, we characterize pseudospectrum preservers of the product of operators. Note that the preservers of spectrum has been treated in [10].

Theorem 4.1. *Let $\varepsilon>0$. Then a surjective map $\Phi:\mathcal{B}(H)\to\mathcal{B}(H)$ satisfies*

$$\sigma_\varepsilon(\Phi(A)\Phi(B))=\sigma_\varepsilon(AB) \qquad \text{for all } A,B\in\mathcal{B}(H)$$

if and only if there exist a unitary operator $U\in\mathcal{B}(H)$ and $\mu\in\{1,-1\}$ such that

$$\Phi(A)=\mu UAU^* \qquad \text{for every } A\in\mathcal{B}(H).$$

Furthermore, when H is finite dimensional, the surjectivity assumption on Φ can be removed.

Proof. The case when H is finite dimensional has been proven in [4]. Therefore, we may assume that dim H is infinite.

We establish the proof of the theorem by proving several claims.

Claim 1. There exist a unitary operator $U\in\mathcal{B}(H)$ and a functional $h:\mathcal{B}(H)\to\mathbb{C}\setminus\{0\}$ such that $\Phi(x\otimes f)=h(x\otimes f)Ux\otimes Uf$ for all $x,f\in H$, and $h(x\otimes x)=\pm1$ for every unit vector $x\in H$.

For $A,B\in\mathcal{B}(H)$, by Corollary 2.3, $AB=0$ if and only if $\Phi(A)\Phi(B)=0$. By [3], there exist a bounded invertible linear or conjugate linear operator T on H and a functional $h:\mathcal{B}(H)\to\mathbb{C}\setminus\{0\}$ such that $\Phi(x\otimes f)=h(x\otimes f)Tx\otimes fT^{-1}$ for any $x,f\in H$. Assume first that T is linear. It follows from Corollary 2.3 that, for any unit vector $x\in H$, $\Phi(x\otimes x)^2=h(x\otimes x)^2Tx\otimes xT^{-1}$ is a rank one projection, so $h(x\otimes x)=\pm1$ and T can be chosen as unitary. Now assume that T is conjugate linear. Similar to the previous discussion, T can be chosen as conjugate unitary. Assume that T is conjugate unitary. For any unit vector $x \in H$, take $\alpha = a + ib$ with nonzero $a, b \in \mathbb{R}$, then

$$\sigma_\varepsilon(\alpha^2 x \otimes x) = \sigma_\varepsilon((\alpha x \otimes x)^2) = \sigma_\varepsilon(\Phi(\alpha x \otimes x)^2) = \sigma_\varepsilon(|h(\alpha x \otimes x)|^2 |\alpha|^2 Tx \otimes xT^{-1})$$

implies that $(a^2 - b^2) + 2abi = \alpha^2 = |h(\alpha x \otimes x)|^2 |\alpha|^2 \in \mathbb{R}$, a contradiction. So T can be chosen as unitary and Claim 1 holds.

Claim 2. Either $h(x, x) \equiv 1$ for any unit vector $x \in H$, or $h(x, x) \equiv -1$ for any unit vector $x \in H$.

By Claim 1, we only need to prove that $h(x \otimes x) = h(f \otimes f)$ for all unit vectors $x, f \in H$. Let $x, f \in H$ be any unit vector. Assume first that $\langle x, f\rangle \neq 0$. It follows from Corollary 2.3 that

$$\begin{aligned} D(0, \varepsilon) \cup D(\langle x, f\rangle, \varepsilon) &= \sigma_\varepsilon((f \otimes f)(x \otimes f)) \\ &= \sigma_\varepsilon(\Phi(f \otimes f)\Phi(x \otimes f)) \\ &= D(0, \varepsilon) \cup D(h(f \otimes f)h(x \otimes f)\langle x, f\rangle, \varepsilon), \end{aligned}$$

so

$$h(f \otimes f)h(x \otimes f) = 1. \tag{4.1}$$

Similarly, it follows respectively from $\sigma_\varepsilon((x \otimes x)(f \otimes x)) = \sigma_\varepsilon(\Phi(x \otimes x)\Phi(f \otimes x))$ and $\sigma_\varepsilon((x \otimes f)(f \otimes x)) = \sigma_\varepsilon(\Phi(x \otimes f)\Phi(f \otimes x))$ that

$$h(x \otimes x)h(f \otimes x) = 1 \quad \text{and} \quad h(x \otimes f)h(f \otimes x) = 1. \tag{4.2}$$

Combining (4.1) with (4.2), one has $h(x \otimes x)h(f \otimes f) = 1$, and hence Claim 1 implies that $h(x \otimes x) = h(f \otimes f)$.

Now assume that $\langle x, f\rangle = 0$. Take a unit vector $u \in H$ such that $\langle u, f\rangle \neq 0$ and $\langle x, u\rangle \neq 0$. Then, similarly we get that $h(x \otimes x) = h(u \otimes u)$ and $h(f \otimes f) = h(u \otimes u)$, so $h(x \otimes x) = h(f \otimes f)$ and Claim 2 follows.

Claim 3. Either $h(x \otimes f) \equiv 1$ for all unit vectors $x, f \in H$; or $h(x \otimes f) \equiv -1$ for all unit vectors $x, f \in H$.

Let $x, f \in H$ be arbitrary unit vectors. By Claim 2, we might as well assume that $h(x \otimes x) = 1$ for all unit vector $x \in H$. If $\langle x, f\rangle \neq 0$, then (4.1) implies that $h(x \otimes f) = 1$. Now assume that $\langle x, f\rangle = 0$. Take a unit vector $u \in H$ such that $\langle u, x\rangle \neq 0$ and $\langle u, f\rangle \neq 0$. Then $\sigma_\varepsilon(\Phi(x \otimes f)\Phi(u \otimes x)) = \sigma_\varepsilon((x \otimes f)(u \otimes x))$ implies that $h(x \otimes f)h(u \otimes x) = 1$, and therefore $h(x \otimes f) = 1$, which completes the proof of Claim 3.

By replacing Φ by $-\Phi$ if necessary, we may assume that $h(x \otimes f) = 1$ for all unit vectors $x, f \in H$. Then $\Phi(x \otimes f) = Ux \otimes Uf$ for all unit vectors $x, f \in H$. Let $A \in \mathcal{B}(H)$ be arbitrary. For any unit vectors $x, f \in H$, we have

$$\sigma_\varepsilon(\Phi(A)Ux \otimes Uf) = \sigma_\varepsilon(Ax \otimes f). \tag{4.3}$$

If $Ax = 0$, then (4.3) implies that $\Phi(A)Ux \otimes Uf = 0$ for all $f \in H$. Hence, $\Phi(A) = 0$.

If $Ax \neq 0$, take $f = \frac{Ax}{\|Ax\|}$ in (4.3), then we have

$$\begin{aligned} & \sigma_\varepsilon\left(\Phi(A)Ux \otimes U\frac{Ax}{\|Ax\|}\right) = \sigma_\varepsilon\left(Ax \otimes \frac{Ax}{\|Ax\|}\right) \\ \Rightarrow \quad & \sigma_{\|Ax\|\varepsilon}(\Phi(A)Ux \otimes UAx) = \sigma_{\|Ax\|\varepsilon}(Ax \otimes Ax)\,. \end{aligned}$$

It follows from Corollary 2.3 that $\Phi(A)Ux \otimes UAx = \|Ax\|^2 P$ for a rank one projection P. Therefore, we have

$$\begin{aligned} \Phi(A)Ux \otimes UAx &= \|Ax\|^2\left(\frac{UAx}{\|UAx\|}\right) \otimes \left(\frac{UAx}{\|UAx\|}\right) \\ = \quad & \|UAx\|^2\left(\frac{UAx}{\|UAx\|}\right) \otimes \left(\frac{UAx}{\|UAx\|}\right) = UAx \otimes UAx. \end{aligned}$$

Hence, for all unit vector x, we have $\Phi(A)Ux \otimes UAx = UAx \otimes UAx$, and consequently, $(\Phi(A)U - UA)x = 0$ for all unit vector x. Therefore, $\Phi(A) = UAU^*$ for every $A \in \mathcal{B}(H)$. The proof is complete. □

5. Pseudospectrum preservers of Jordan product

In abstract algebra, a Jordan algebra is an (not necessarily associative) algebra over a field whose multiplication satisfies the following axioms:

- $xy = yx$ (commutative law)
- $(xy)(xx) = x(y(xx))$ (Jordan identity).

Jordan algebras were first introduced by Pascual Jordan (1933) to formalize the notion of an algebra of observables in quantum mechanics. The real algebra of all self-adjoint operators on a Hilbert space is a Jordan algebra. In this section we first characterize maps preserving pseudospectrum of Jordan products on real Jordan algebras, furthermore, we will also consider pseudospectrum preservers of Jordan products on $\mathcal{B}(H)$. Denote by $\mathcal{B}_s(H)$ the set of all self-adjoint operators on a Hilbert space H, and by $\mathcal{P}(H)$ the set of all projections on H.

Theorem 5.1. *Let $\varepsilon > 0$. Then a surjective map $\Phi : \mathcal{B}_s(H) \to \mathcal{B}_s(H)$ satisfies*

$$\sigma_\varepsilon(AB + BA) = \sigma_\varepsilon(\Phi(A)\Phi(B) + \Phi(B)\Phi(A)) \qquad \text{for all } A, B \in \mathcal{B}_s(H)$$

if and only if there exists a unitary operator $U \in \mathcal{B}(H)$ such that Φ has the form

$$A \mapsto \mu UAU^* \qquad \text{or} \qquad A \mapsto \mu UA^tU^*,$$

where $\mu \in \{1, -1\}$ and A^t denotes the transpose of A relative to an arbitrary but fixed orthonormal basis of H. Furthermore, when H is finite dimensional, the surjectivity assumption on Φ can be removed.

Proof. In the finite dimensional case, note that $\sigma_\varepsilon(A) = \cup\{D(\lambda,\varepsilon) : \lambda \in \sigma(A)\}$ for every $A \in \mathcal{B}_s(H)$. Thus, Φ leaves invariant the pseudospectrum of $AB+BA$ if and only if Φ leaves invariant the spectrum of $AB+BA$. The conclusion then follows from Theorem 3.2 in [5]. Therefore, we may assume that dim H is infinite.

The proof of the theorem will be completed after proving several claims.

Claim 1. $\Phi(I) = I$ or $\Phi(I) = -I$.

It follows from $\sigma_\varepsilon(2\Phi(I)^2) = D(2,\varepsilon)$ and Corollary 2.3 that $\Phi(I)^2 = I$, and hence, $\sigma(\Phi(I)) \subseteq \{-1,1\}$. Assume that $\sigma(\Phi(I)) = \{-1,1\}$. Since $\Phi(I)^2 = I$ and $\Phi(I)$ is self-adjoint, there exists a nontrivial projection $P \in \mathcal{B}(H)$ such that $\Phi(I) = 2P - I$. Take unit vectors $x, f \in H$ such that $Px = x$ and $Pf = 0$, and let $Y = x\otimes f + f\otimes x$, then there is $X \in \mathcal{B}_s(H)$ such that $\Phi(X) = Y$. On the one hand, $\sigma_\varepsilon(2X^2) = \sigma_\varepsilon(2Y^2) = D(0,\varepsilon)\cup D(2,\varepsilon)$ implies that X^2 is a nontrivial projection; on the other hand, $\sigma_\varepsilon(2X) = \sigma_\varepsilon(\Phi(I)Y + Y\Phi(I)) = D(0,\varepsilon)$ implies that $X = 0$, a contradiction. So $\sigma(\Phi(I)) = \{1\}$ or $\{-1\}$, and hence, $\Phi(I) = \pm I$. This completes the proof of Claim 1.

If $\Phi(I) = -I$, consider $-\Phi$, then $-\Phi$ satisfies the condition of the theorem. So we may assume that $\Phi(I) = I$.

Claim 2. There exists a unitary or conjugate unitary operator U on H such that $\Phi(P) = UPU^*$ for every $P \in \mathcal{P}(H)$.

The equality $\sigma_\varepsilon(2\Phi(\frac{1}{2}I)) = \sigma_\varepsilon(\Phi(\frac{1}{2}I)\Phi(I) + \Phi(I)\Phi(\frac{1}{2}I)) = \sigma_\varepsilon(I) = D(1,\varepsilon)$, together with Corollary 2.3 (1), implies that $\Phi(\frac{1}{2}I) = \frac{1}{2}I$. For any $A \in \mathcal{B}_s(H)$, it follows that

$$\sigma_\varepsilon(\Phi(A)) = \sigma_\varepsilon(\Phi(A)\Phi(\frac{1}{2}I) + \Phi(\frac{1}{2}I)\Phi(A)) = \sigma_\varepsilon(A).$$

Thus $P \in \mathcal{B}(H)$ is a projection if and only if $\Phi(P)$ is a projection. We claim that $\Phi : \mathcal{P}(H) \to \mathcal{P}(H)$ is bijective. Clearly, we only need to prove that Φ is injective. Assume that $P, Q \in \mathcal{P}(H)$ such that $\Phi(P) = \Phi(Q)$. For any unit vector $x \in H$, we have

$$\begin{aligned}\sigma_\varepsilon(Px\otimes x + x\otimes xP) &= \sigma_\varepsilon(\Phi(P)\Phi(x\otimes x) + \Phi(x\otimes x)\Phi(P))\\ &= \sigma_\varepsilon(\Phi(Q)\Phi(x\otimes x) + \Phi(x\otimes x)\Phi(Q))\\ &= \sigma_\varepsilon(Qx\otimes x + x\otimes xQ).\end{aligned}$$

Note that $Px\otimes x + x\otimes xP$ is self-adjoint, the above equality implies that $\sigma(Px\otimes x + x\otimes xP) = \sigma(Qx\otimes x + x\otimes xQ)$, and hence $\langle Px, x\rangle = \langle Qx, x\rangle$ for every unit vector x, so $P = Q$. That is, Φ is injective. Now it can be easily checked that $PQ = 0$ if and only if $\Phi(P)\Phi(Q) = 0$ for all $P, Q \in \mathcal{P}(H)$. Thus, $\Phi : \mathcal{P}(H) \to \mathcal{P}(H)$ is a bijective map which preserves the orthogonality in both directions. So, by [11], there exists a unitary or conjugate unitary operator U on H such that $\Phi(P) = UPU^*$ for every $P \in \mathcal{P}(H)$.

Claim 3. The result in the theorem holds.

Let $A \in \mathcal{B}_s(H)$ be arbitrary. For any unit vector $x \in H$, we have

$$\sigma_\varepsilon(x \otimes xA + Ax \otimes x) = \sigma_\varepsilon(Ux \otimes xU^*\Phi(A) + \Phi(A)Ux \otimes xU^*).$$

First assume that U is unitary. Then

$$\begin{aligned} 2\langle Ax, x\rangle &= \mathrm{Tr}(x \otimes xA + Ax \otimes x) \\ &= \mathrm{Tr}(Ux \otimes xU^*\Phi(A) + \Phi(A)Ux \otimes xU^*) \\ &= 2\langle U^*\Phi(A)Ux, x\rangle, \end{aligned}$$

where $\mathrm{Tr}(A)$ denotes the trace of A. So $\Phi(A) = UAU^*$ for every $A \in \mathcal{B}_s(H)$.

Now assume that U is conjugate unitary. Take an orthonormal basis $\{e_i\}_{i\in\Lambda}$ of H and define J by $J(\sum_{i\in\Lambda} \xi_i x_i) = \sum_{i\in\Lambda} \bar{\xi}_i e_i$. Then $J : H \to H$ is conjugate unitary and $JA^*J = A^t$, where A^t is the transpose of A for an arbitrarily but fixed orthonormal basis of H. Let $U = VJ$. Then V is unitary, and $\Phi(A) = VJAJV^* = VA^tV^*$ for every $A \in \mathcal{B}_s(H)$. □

Next we consider the pseudospectrum preservers of Jordan product on $\mathcal{B}(H)$.

Theorem 5.2. *Let $\varepsilon > 0$. Then a surjective map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ satisfies*

$$\sigma_\varepsilon(AB + BA) = \sigma_\varepsilon(\Phi(A)\Phi(B) + \Phi(B)\Phi(A)) \qquad \textit{for all } A, B \in \mathcal{B}(H)$$

if and only if there exist $\mu \in \{1, -1\}$ and a unitary operator $U \in \mathcal{B}(H)$ such that Φ has the form

$$A \mapsto \mu UAU^* \qquad \textit{or} \qquad A \mapsto \mu UA^tU^*,$$

where A^t denotes the transpose of A relative to an arbitrary but fixed orthonormal basis of H. Furthermore, when H is finite dimensional, the surjectivity assumption on Φ can be removed.

Proof. First consider the case when the dimension of H is infinite. We will divide the proof into several claims.

Claim 1. $\Phi(I) = I$ or $\Phi(I) = -I$.

It follows from Claim 1 of Theorem 5.1 that we only need to prove $\Phi(I)$ is self-adjoint. Since $\Phi(I)^2 = I$, [13, Theorem 1.1] implies that $\Phi(I)$ is unitarily similar to

$$\begin{pmatrix} I_{H_1} & A \\ 0 & -I_{H_1} \end{pmatrix} \oplus I_{H_2} \oplus I_{H_3}$$

on $H = (H_1 \oplus H_1) \oplus H_2 \oplus H_3$, where $A \in \mathcal{B}(H_1)$ is positive, that is, $\langle Ax, x\rangle \geq 0$ for all nonzero vector $x \in H_1$. Assume that $A \neq 0$. Since $\|A\|$ equals to the numerical radius of A, take orthogonal unit vectors $x_1, x_2 \in H_1$ such that $a = \langle Ax_1, x_1\rangle$ is close to $\|A\|$,

$[x_1, Ax_1] \subseteq [x_1, x_2]$ and $b = \langle Ax_1, x_2 \rangle \geq 0$. Then, according to the space decomposition $H = [x_1] \oplus [x_2] \oplus (H_1 \ominus [x_1, x_2])$, A has the form

$$\begin{pmatrix} a & b & 0 \\ b & * & * \\ 0 & * & * \end{pmatrix}.$$

Let $M = [x_1, x_2]$, then the restriction of $\Phi(I)$ to $M \oplus M$ equals to

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & b & * \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Thus, with the decomposition $H = ((M \oplus M) \oplus (H_1 \ominus M) \oplus (H_1 \ominus M)) \oplus H_2 \oplus H_3$, $\Phi(I)$ has the form

$$\begin{pmatrix} 1 & 0 & a & b & 0 & 0 \\ 0 & 1 & b & * & 0 & * \\ 0 & 0 & -1 & 0 & 0 & * \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & * & I & * \\ 0 & 0 & 0 & 0 & 0 & -I \end{pmatrix} \oplus I_{H_2} \oplus I_{H_3}.$$

With the corresponding space decomposition, let

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = Y_1 \oplus 0_{H_2} \oplus 0_{H_3}.$$

Since Φ is surjective, there exists $X \in \mathcal{B}(H)$ such that $\Phi(X) = Y$. Now it follows from $\sigma_\varepsilon(2X^2) = \sigma_\varepsilon(2Y^2) = D(0, \varepsilon)$ and Corollary 2.3 that $X^2 = 0$. So Proposition 2.4 ensures that

$$\sigma_\varepsilon(2X) = D(0, \sqrt{\varepsilon^2 + 2\|X\|\varepsilon}).$$

On the other hand, $\sigma_\varepsilon(2X) = \sigma_\varepsilon(\Phi(I)Y + Y\Phi(I)) = \sigma_\varepsilon(Z) \cup D(0, \varepsilon)$ with

$$Z = \begin{pmatrix} a & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $\sigma_\varepsilon(Z) = \sigma_\varepsilon(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix})$ and $D(0,\varepsilon) \subseteq \sigma_\varepsilon(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix})$, we have

$$\sigma_\varepsilon(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}) = \sigma_\varepsilon(2X) = D(0, \sqrt{\varepsilon^2 + 2\|X\|\varepsilon}),$$

which, together with Lemma 2.5, implies that $a = 0$, a contradiction. So $\Phi(I)$ is self-adjoint. This completes the proof of Claim 1.

If $\Phi(I) = -I$, consider $-\Phi$, then $-\Phi$ satisfies the condition of the theorem. So we may assume that $\Phi(I) = I$. Similar to the discussion of Claim 2 in Theorem 5.1, we have

$$\sigma_\varepsilon(\Phi(A)) = \sigma_\varepsilon(A) \text{ for every } A \in \mathcal{B}(H) \tag{5.3}$$

It follows from Theorem 2.2 and (5.3) that $A \in \mathcal{B}_s(H)$ that if and only if $\Phi(A) \in \mathcal{B}_s(H)$. Therefore, $\Phi : \mathcal{B}_s(H) \to \mathcal{B}_s(H)$ is surjective. By Theorem 5.1, there exists a unitary operator $U \in \mathcal{B}(H)$ such that $\Phi(A) = UAU^*$ for every $A \in \mathcal{B}_s(H)$ or $\Phi(A) = UA^tU^*$ for every $A \in \mathcal{B}_s(H)$.

Since the map $A \mapsto A^t$ and the map $A \mapsto U^*\Phi(A)U$ preserve the pseudospectrum of Jordan products, we may as well assume that $\Phi(A) = A$ for every $A \in \mathcal{B}_s(H)$.

Claim 2. $\Phi(A) = A$ for every $A \in \mathcal{B}(H)$.

Let $A \in \mathcal{B}(H)$ be arbitrary and $x \in H$ be an arbitrary non-zero vector. For any $\lambda > 0$, we have $\Phi(\lambda x \otimes x) = \lambda x \otimes x$. Now it follows from Proposition 1.1 (4) that

$$\begin{aligned} & \lambda\sigma_{\frac{\varepsilon}{\lambda}}(Ax \otimes x + x \otimes xA) = \sigma_\varepsilon(A\lambda x \otimes x + \lambda x \otimes xA) \\ = \quad & \sigma_\varepsilon(\Phi(A)\lambda x \otimes x + \lambda x \otimes x\Phi(A)) = \lambda\sigma_{\frac{\varepsilon}{\lambda}}(\Phi(A)x \otimes x + x \otimes x\Phi(A)). \end{aligned}$$

Notice that

$$\sigma(Ax \otimes x + x \otimes xA) = \cap_{\lambda>0}\sigma_{\frac{\varepsilon}{\lambda}}(Ax \otimes x + x \otimes xA).$$

Thus $\sigma(Ax \otimes x + x \otimes xA) = \sigma(\Phi(A)x \otimes x + x \otimes x\Phi(A))$. Hence, $\langle Ax, x\rangle = \langle \Phi(A)x, x\rangle$ for any nonzero vector $x \in H$. As a result, $\Phi(A) = A$.

Next consider the case when H is finite dimensional. Using the proof in Claim 1 and the first part of Claim 2 in Theorem 5.1, we can show that A is self-adjoint if and only if $\Phi(A)$ is self-adjoint. Applying Theorem 5.1 to the restriction of Φ on self-adjoint matrices, we see that Φ has the standard form on Hermitian matrices. Compose Φ with a suitable unitary similarity or a conjugate unitary similarity, we may assume that $\Phi(X) = X$ for every Hermitian matrix X. In particular, $\Phi(\lambda x \otimes x) = \lambda x \otimes x$ for any $\lambda > 0$ and unit vector x. Now, we can use the proof of Claim 2 in the infinite dimensional case to finish the proof. $\square$

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