ELLIPTICAL RANGE THEOREMS FOR GENERALIZED NUMERICAL RANGES OF QUADRATIC OPERATORS

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Abstract. The classical numerical range of a quadratic operator is an elliptical disk. This result is extended to different kinds of generalized numerical ranges. In particular, it is shown that for a given quadratic operator, the rank-k numerical range, the essential numerical range, and the q-numerical range are elliptical disks; the c-numerical range is a sum of elliptical disks, and the Davis-Wielandt shell is an ellipsoid with or without interior.

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1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators acting on the Hilbert space $\mathcal{H}$. We identify $\mathcal{B}(\mathcal{H})$ with M_n if $\mathcal{H}$ has dimension n. The *numerical range* of $A \in \mathcal{B}(\mathcal{H})$ is defined by

$$W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H}, \langle x, x\rangle = 1\};$$

see [9, 10]. The numerical range is useful in studying matrices and operators. One of the basic properties of the numerical range is that $W(A)$ is always convex; for example, see [9]. In particular, if $A \in M_2$ has eigenvalues a_1 and a_2, then $W(A)$ is an elliptical disk with a_1, a_2 as foci and $\sqrt{\operatorname{tr}(A^*A) - |a_1|^2 - |a_2|^2}$ as the length of minor axis; for example, see [11]. This is known as the elliptical range theorem from which one can deduce the convexity of the numerical range of a general operator.

Motivated by theoretical study and applications, there has been many generalizations of the numerical range such as the k-numerical range, the q-numerical range, the c-numerical range, the essential numerical range, and the Davis-Wielandt shell; for example, see [2, 7, 8, 9, 10, 12, 16, 22] and their references. Recently, researchers have studied the higher rank numerical range in connection to quantum error correction; see [4, 5, 6, 13, 15] and Section 2. Each of these generalizations encodes certain specific information of the operator that leads to interesting applications. To advance the study of these generalized numerical ranges, it is useful to have concrete descriptions of the numerical ranges of certain operators. In most cases, it is relatively easy to solve the problem for self-adjoint or normal operators. The task is more challenging for general operators.

A non-scalar operator $A \in \mathcal{B}(\mathcal{H})$ is a *quadratic operator* if there is $a, b \in \mathbb{C}$ such that $(A - aI)(A - bI) = 0$. This class of operators include idempotent operators and square-zero operators. The following result on quadratic operators is known; e.g., see [21].

Theorem 1.1. *Let $A \in \mathcal{B}(\mathcal{H})$ be a non-scalar quadratic operator satisfying $(A-aI)(B-bI) = 0$ with $a, b \in \mathbb{C}$. Then $\mathcal{H}$ has a decomposition $\mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ such that A has an operator matrix of the form*

$$\begin{bmatrix} aI_r & P \\ 0 & bI_r \end{bmatrix} \oplus \gamma I_s,$$

where $\gamma \in \{a, b\}$, $\dim \mathcal{H}_1 = r$, $\dim \mathcal{H}_2 = s$, and $P : \mathcal{H}_1 \to \mathcal{H}_1$ is a positive semidefinite operator, i.e., $\langle Px, x\rangle \geq 0$ for all $x \in \mathcal{H}_1$, with the additional condition that $\langle Px, x\rangle \neq 0$ for all nonzero $x \in \mathcal{H}_1$ if $a = b$. The numerical range $W(A)$ is an elliptical disk with foci a, b and minor axis of length $\|P\|$ with all or none of the boundary points depending on whether there is a unit vector $x \in \mathcal{H}_1$ such that $\|Px\| = \|P\|$.

Note that in the above discussion of P, we have identified the subspaces $\mathcal{H}_1 \oplus 0 \oplus 0$ and $0 \oplus \mathcal{H}_1 \oplus 0$ with $\mathcal{H}_1$.

The shapes of different kinds of generalized numerical ranges of quadratic operators were studied by researchers. For example, the k-numerical range of a quadratic operator was described as the union of (infinitely many) circular disks in [3]; the essential numerical range of a quadratic operator was described in terms of the essential norm of a related operator, and some partial results on the c-numerical range of a quadratic operator were obtained in [19].

In this paper, we give explicit descriptions of different kinds of generalized numerical ranges of quadratic operators including the rank-k numerical range, the c-numerical range, the q-numerical range, the essential numerical range, and the Davis-Wielandt shell; see the definitions in Sections 2–4. In particular, we show that these generalized numerical ranges of quadratic operators are elliptical disks, the sum of elliptical disks, or ellipsoids with or without the interior. Our results cover and improve those of other researchers. One can readily use our results to construct the above generalized numerical ranges analytically or numerically.

For $S \subseteq \mathbb{C}$, we will use $\mathbf{int}(S)$, $\mathbf{cl}\,(S)$ and $\mathbf{conv}(S)$ to denote the interior, the closure and the convex hull of S, respectively. For $A \in \mathcal{B}(\mathcal{H})$, let $N(A)$ denote the null space of A. Let $\mathcal{V}$ be a closed subspace of $\mathcal{H}$ and Q the embedding of $\mathcal{V}$ into $\mathcal{H}$. Then $B = Q^*AQ$ is the *compression* of A onto $\mathcal{V}$. More generally, A has a compression B if A has an operator matrix $\begin{bmatrix} B & * \\ * & * \end{bmatrix}$ with respect to an orthonormal basis; alternatively, there is a closed subspace $\mathcal{V}$ of $\mathcal{H}$ and $X : \mathcal{V} \to \mathcal{H}$ such that $X^*X = I_{\mathcal{V}}$ and $X^*AX = B$. Note that, in this case, $X(\mathcal{V})$ is closed and X^*AX is the compression of A on $X(\mathcal{V})$.

2. Rank-k numerical ranges and essential numerical ranges

For a positive integer k, define the *rank-k numerical range* of $A \in \mathcal{B}(\mathcal{H})$ by

$$\Lambda_k(A) = \{\lambda \in \mathbb{C} : PAP = \lambda P \text{ for some rank-}k \text{ orthogonal projection } P\}.$$

This generalized numerical range is motivated by the study of quantum error correction; see [4, 5, 6].

To describe some basic results of $\Lambda_k(A)$, we need the following notation. Let $H \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator. If $\dim \mathcal{H} = n$, denote by $\lambda_1(H) \geq \cdots \geq \lambda_n(H)$ the eigenvalues of H. If $\mathcal{H}$ is infinite dimensional, define

$$\lambda_m(H) = \sup\{\lambda_m(X^*HX) : X^*X = I_m\}.$$

It is known (see [18]) and not hard to verify that $\lambda_m(H)$ of an infinite dimensional operator H can be determined as follows. Let

$$\sigma_e(A) = \cap\{\sigma(A+F) : F \in \mathcal{B}(\mathcal{H}) \text{ has finite rank}\}$$

be the *essential spectrum* of $A \in \mathcal{B}(\mathcal{H})$, and let

$$\lambda_\infty(H) = \sup \sigma_e(H),$$

which also equals the supremum of the set

$$\sigma(H) \setminus \{\mu \in \mathbb{C} : H - \mu I \text{ has a non-trivial finite dimensional null space}\}.$$

Then $\mathcal{S} = \sigma(H) \cap (\lambda_\infty(H), \infty)$ has only isolated points, and we can arrange the elements in descending order, say, $\lambda_1 \geq \lambda_2 \geq \cdots$ counting multiplicities, i.e., each element repeats according to the dimension of its eigenspace. If $\mathcal{S}$ is infinite, then $\lambda_j(H) = \lambda_j$ for each positive integer j. If $\mathcal{S}$ has m elements, then $\lambda_j(H) = \lambda_j$ for $j = 1, \ldots, m$, and $\lambda_j(H) = \lambda_\infty(H)$ for $j > m$.

Let

$$\Omega_k(A) = \bigcap_{\xi \in [0, 2\pi)} \left\{ \mu \in \mathbb{C} : \mathrm{Re}\,(e^{i\xi}\mu) \leq \lambda_k \left(\mathrm{Re}\,(e^{i\xi} A) \right) \right\},$$

where $\mathrm{Re}\,(B) = (B + B^*)/2$ is the real part of B. It was shown in [14] that

$$\mathbf{int}(\Omega_k(A)) \subseteq \Lambda_k(A) \subseteq \Omega_k(A) = \mathbf{cl}\,(\Lambda_k(A)).$$

In particular, $\Lambda_k(A) = \Omega_k(A)$ if $A \in M_n$; see also [15].

The rank-k numerical range of a quadratic operator can be an empty set, a singleton, a line segment or an elliptical disk with all or none of its boundary. The following theorem gives the precise description of the set using Theorem 1.1.

Theorem 2.1. *Suppose $A \in \mathcal{B}(\mathcal{H})$ is a quadratic operator with operator matrix in the form described in Theorem 1.1 and k is a positive integer not larger than* $\dim \mathcal{H}$.

(a) *If $r + s < k$, then $\Lambda_k(A) = \emptyset$.*
(b) *If $r < k \leq r + s$, then $\Lambda_k(A) = \{\gamma\}$.*
(c) *Suppose $k \leq r$. Then $\Lambda_k(A) = \mathcal{E}$ or $\Lambda_k(A) = \mathbf{int}(\mathcal{E})$, where $\mathcal{E}$ is the closed elliptical disk with foci a, b and minor axis of length $\lambda_k(P)$; the equality $\Lambda_k(A) = \mathcal{E}$ holds if and only if $P : \mathcal{H}_1 \to \mathcal{H}_1$ has a compression* $\mathrm{diag}\,(p_1, \ldots, p_k)$ *with $p_1 \geq \cdots \geq p_k = \lambda_k(P)$.*

Remark 2.2. In (c) of Theorem 2.1, it is not hard to show that another equivalent condition for $\Lambda_k(A) = \mathcal{E}$ is that

$$P : \mathcal{H}_1 \to \mathcal{H}_1 \text{ has a compression } \mathrm{diag}\,(\lambda_1(P), \ldots, \lambda_k(P)),$$

and therefore $\Lambda_\ell(A)$ is an elliptical disk with foci a, b and minor axis of length $\lambda_\ell(P)$ for any $\ell \in \{1, \ldots, k\}$. Also if $\lambda_k(P) = 0$, then $\mathcal{E}$ becomes the line segment joining a and b, i.e., $\mathcal{E} = \mathbf{conv}\{a, b\}$. In this case, $\Lambda_k(A)$ equals $\mathbf{conv}\{a, b\}$.

The following corollary is immediate.

Corollary 2.3. *Suppose $A \in \mathcal{B}(\mathcal{H})$ satisfies $(A - aI)^2 = 0$. Then $\Lambda_k(A)$ is an empty set, a singleton $\{a\}$, an open circular disk or a closed circular disk centered at a.*

We need two lemmas to prove Theorem 2.1. First of all, by the discussion after the definition of $\lambda_m(H)$ for a self-adjoint operator $H \in \mathcal{B}(\mathcal{H})$, we have the following observation.

Lemma 2.4. *Suppose P is a positive semidefinite operator in $\mathcal{B}(\mathcal{H}_1)$ with $\dim(\mathcal{H}_1) \geq k$. For any $\varepsilon > 0$, there exist $p_1, \dots, p_k \in [0, \infty)$ with $\lambda_j(P) - \varepsilon < p_j \leq \lambda_j(P)$ for $j = 1, \dots, k$, such that P has a compression of the form $\operatorname{diag}(p_1, \dots, p_k)$.*

Proof. We prove by induction on k. For $k = 1$, the result follows from definition. Suppose we have a $(k-1)$-dimensional subspace $\mathcal{V}_1$ and $X_1 : \mathcal{V}_1 \to \mathcal{H}_1$ such that $X_1^* X_1 = I_{\mathcal{V}_1}$ and $\lambda_j(P) - \varepsilon < \lambda_j\left(X_1^* P X_1\right) \leq \lambda_j(P)$ for $j = 1, \dots, k-1$. Choose a k-dimensional subspace $\mathcal{V}_2$ and $X_2 : \mathcal{V}_2 \to \mathcal{H}_1$ such that $X_2^* X_2 = I_{\mathcal{V}_2}$ and $\lambda_k(P) - \varepsilon < \lambda_k\left(X_2^* P X_2\right) \leq \lambda_k(P)$. Let $\mathcal{V} = X_1(\mathcal{V}_1) + X_2(\mathcal{V}_2)$ and $\hat{P}$ the compression of P on $\mathcal{V}$. Then

$$\lambda_j(P) - \varepsilon < \lambda_j(\hat{P}) \leq \lambda_j(P) \quad \text{for} \quad j = 1, \dots, k.$$

Therefore, the result is satisfied by taking the compression of $\hat{P}$ to the k-dimensional subspace spanned by the eigenvectors of $\hat{P}$ corresponding to $\lambda_j(\hat{P})$,$1 \leq j \leq k$. $\square$

Lemma 2.5. *Let $A \in \mathcal{B}(\mathcal{H})$ be a quadratic operator having the form described in Theorem 1.1 with the additional assumption that $r = \infty$. Suppose $\mathcal{V}_1$ is a k-dimensional subspace of $\mathcal{H}$. Then there is a $(4k+\ell)$-dimensional subspace $\mathcal{V}_2$ of $\mathcal{H}$ containing $\mathcal{V}_1$ with $\ell = \min\{s, k\}$ such that the compression of A on $\mathcal{V}_2$ has the form*

$$\begin{bmatrix} aI_{2k} & P' \\ 0 & bI_{2k} \end{bmatrix} \oplus \gamma I_\ell,$$

where $P' = \operatorname{diag}(p_1, \dots, p_{2k})$ is a compression of P, with $p_1 \geq \cdots \geq p_{2k}$ and $p_i \leq \lambda_i(P)$ for $1 \leq i \leq 2k$.

Proof. Suppose A has the form described in Theorem 1.1, with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\dim \mathcal{H}_1 = r = \infty$. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be k-dimensional subspaces of $\mathcal{H}_1$ such that $\mathcal{K}_1 \oplus 0 \oplus 0$ and $0 \oplus \mathcal{K}_2 \oplus 0$ contain the orthogonal projections of $\mathcal{V}_1$ on $\mathcal{H}_1 \oplus 0 \oplus 0$ and $0 \oplus \mathcal{H}_1 \oplus 0$, respectively. Also let $\mathcal{K}_3$ be a ℓ-dimensional subspace of $\mathcal{H}_2$, with $\ell = \min\{s, k\}$, such that $0 \oplus 0 \oplus \mathcal{K}_3$ contains the orthogonal projection of $\mathcal{V}_1$ on $0 \oplus 0 \oplus \mathcal{H}_2$. Clearly, $\mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3$ contains $\mathcal{V}_1$. Take a $2k$-dimensional subspace $\mathcal{K}$ of $\mathcal{H}_1$ containing $\mathcal{K}_1 + \mathcal{K}_2$ and $\mathcal{V}_2 = \mathcal{K} \oplus \mathcal{K} \oplus \mathcal{K}_3$. Then $\mathcal{V}_2$ also contains $\mathcal{V}_1$. Let $S : \mathcal{V}_2 \hookrightarrow \mathcal{H}$ be the imbedding of $\mathcal{V}_2$ into $\mathcal{H}$. Then S^*AS has operator matrix of the form

$$\begin{bmatrix} aI_{2k} & X^*PX \\ 0 & bI_{2k} \end{bmatrix} \oplus \gamma I_\ell,$$

where X is the imbedding of $\mathcal{K}$ into $\mathcal{H}_1$. Furthermore, we can find a unitary operator U such that

$$U^*X^*PXU = \operatorname{diag}(p_1, \dots, p_{2k}) \qquad \text{with} \qquad p_1 \geq \cdots \geq p_{2k}.$$

Let $T = S(U \oplus U \oplus I_\ell)$. Then $T^*T = I_{4k+\ell}$ and T^*AT has the asserted form. $\square$

Proof of Theorem 2.1. We first consider the finite dimensional case. Let $n = \dim \mathcal{H} = 2r + s$. Assume that P has eigenvalues $s_1 \geq \cdots \geq s_r \geq 0$. Then A is unitarily similar to

$$A_1 \oplus A_2 \oplus \cdots \oplus A_r \oplus \gamma I_s,$$

where

$$A_j = \begin{bmatrix} a & s_j \\ 0 & b \end{bmatrix}, \quad j = 1, 2, \dots, r.$$

We note that if $a = b$, then $s_j \neq 0$. Therefore, A_j is never a scalar matrix and $\Omega_2(A_j) = \emptyset$. Let $\mathcal{E}(a, b, \ell)$ denote the closed elliptical disk with foci at a and b and minor axis of length ℓ. It follows that $\mathcal{E}(a, b, \ell_1) \subseteq \mathcal{E}(a, b, \ell_2)$ for $\ell_1 < \ell_2$. It is known that $\Lambda_1(A_j) = \mathcal{E}(a, b, s_j)$; e.g., see [11]. For $\xi \in \mathbb{R}$, we have

$$\lambda_1\left(\operatorname{Re}(e^{i\xi}A_j)\right) = \tfrac{1}{2}\left[\ \operatorname{Re}(e^{i\xi}(a+b)) + \sqrt{(\operatorname{Re}(e^{i\xi}(a-b)))^2 + s_j^2}\ \right],$$

$$\lambda_2\left(\operatorname{Re}(e^{i\xi}A_j)\right) = \tfrac{1}{2}\left[\ \operatorname{Re}(e^{i\xi}(a+b)) - \sqrt{(\operatorname{Re}(e^{i\xi}(a-b)))^2 + s_j^2}\ \right].$$

Hence,

$$\lambda_1\left(\operatorname{Re}(e^{i\xi}A_1)\right) \geq \cdots \geq \lambda_1\left(\operatorname{Re}(e^{i\xi}A_r)\right)$$
$$\geq \operatorname{Re}(e^{i\xi}\gamma) \geq \lambda_2\left(\operatorname{Re}(e^{i\xi}A_r)\right) \geq \cdots \geq \lambda_2\left(\operatorname{Re}(e^{i\xi}A_1)\right).$$

Then $\lambda_k\left(\mathrm{Re}\,(e^{i\xi}A)\right)$ equals

$$\begin{cases} \lambda_1\left(\mathrm{Re}\,(e^{i\xi}A_k)\right) & \text{if } k \le r, \\ \mathrm{Re}\,(e^{i\xi}\gamma) & \text{if } r < k \le r+s, \\ \lambda_2\left(\mathrm{Re}\,(e^{i\xi}A_{n-k+1})\right) & \text{if } r+s < k \le n. \end{cases}$$

Recall that $\mu \in \Omega_k(A)$ if and only if $\mathrm{Re}\,(e^{i\xi}\mu) \le \lambda_k\left(\mathrm{Re}\,(e^{i\xi}A)\right)$ for all $\xi \in [0, 2\pi)$. We have

$$\Lambda_k(A) = \Omega_k(A) = \begin{cases} \Omega_1(A_k) & \text{if } k \le r, \\ \{\gamma\} & \text{if } r < k \le r+s, \\ \Omega_2(A_{n-k+1}) = \emptyset & \text{if } r+s < k \le n\,. \end{cases}$$

Then the assertion holds when $k > r$. If $k \le r$, then

$$\Lambda_k(A) = \Omega_k(A) = \Omega_1(A_k) = \Lambda_1(A_k) = \mathcal{E}(a, b, s_k).$$

Thus, the result holds for the finite dimensional case.

Next, suppose $\mathcal{H}$ is an infinite dimensional Hilbert space. If $r < k$, then $\Omega_k(A) = \{\gamma\}$ and hence $\Lambda_k(A) = \{\gamma\}$.

Suppose $r \ge k$ is finite or $\lambda_k(P) = 0$. Then $P : \mathcal{H}_1 \to \mathcal{H}_1$ has a compression $\mathrm{diag}\,(\lambda_1(P), \dots, \lambda_k(P))$. Let

$$\tilde{A} = A_1 \oplus \cdots \oplus A_k \in M_{2k}$$

with $A_j = \begin{bmatrix} a & \lambda_j(P) \\ 0 & b \end{bmatrix}$ for $j = 1, \dots, k$. Notice that $\tilde{A}$ is a compression of A and

$$\lambda_k\left(\mathrm{Re}\,\left(e^{i\xi}A\right)\right) = \lambda_k\left(\mathrm{Re}\,\left(e^{i\xi}\tilde{A}\right)\right) \qquad \text{for all } \xi \in [0, 2\pi).$$

Hence,

$$\Lambda_k(\tilde{A}) \subseteq \Lambda_k(A) \subseteq \Omega_k(A) = \Omega_k(\tilde{A}) = \Lambda_k(\tilde{A}).$$

Thus, $\Lambda_k(A) = \Lambda_k(\tilde{A})$ so that the result holds by the finite dimensional result.

Suppose r is infinite and $\lambda_k(P) > 0$. We prove that (c) holds with $\mathcal{E} = \mathcal{E}(a, b, \lambda_k(P))$. Let μ be an interior point of $\mathcal{E}$. Then there exists $\varepsilon > 0$ such that $\mu \in \mathcal{E}(a, b, \lambda_k(P) - \varepsilon)$. By Lemma 2.4, there exist a k-dimensional subspace $\mathcal{V}$ of $\mathcal{H}$ and $X : \mathcal{V} \to \mathcal{H}_1$ satisfying $X^*X = I_k$ and

$$\lambda_k(X^*PX) > \lambda_k(P) - \varepsilon.$$

Let $Z = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \oplus I_s$. Then we have $Z^*AZ = \begin{bmatrix} aI_k & X^*PX \\ 0 & bI_k \end{bmatrix} \oplus \gamma I_s$ and

$$\mu \in \mathcal{E}(a, b, \lambda_k(P) - \varepsilon) \subseteq \Lambda_k(Z^*AZ) \subseteq \Lambda_k(A).$$

Conversely, suppose $\mu \in \Lambda_k(A)$. Then there exist a k-dimensional subspace $\mathcal{V}_1$ of $\mathcal{H}$ and $X : \mathcal{V}_1 \to \mathcal{H}$ such that $X^*X = I_{\mathcal{V}_1}$ and $X^*AX = \mu I_{\mathcal{V}_1}$. By Lemma 2.5, there is a $(4k + \ell)$-dimensional subspace $\mathcal{V}_2$ containing $\mathcal{V}_1$ such that the compression of A on $\mathcal{V}_2$ has operator matrix

$$A' = \begin{bmatrix} aI_{2k} & P' \\ 0 & bI_{2k} \end{bmatrix} \oplus \gamma I_\ell \in M_{4k+\ell},$$

where $P' = \operatorname{diag}(p_1, \ldots, p_{2k})$ is a $2k$-dimensional compression of P, with $p_1 \geq \cdots \geq p_{2k}$ and $p_i \leq \lambda_i(P)$ for $1 \leq i \leq 2k$. By the result in the finite dimensional case, we have

$$\mu \in \Lambda_k(A') = \mathcal{E}(a, b, \lambda_k(P')) \subseteq \mathcal{E}(a, b, \lambda_k(P)).$$

So, we have shown that

$$\mathbf{int}(\mathcal{E}(a, b, \lambda_k(P))) \subseteq \Lambda_k(A) \subseteq \mathcal{E}(a, b, \lambda_k(P)).$$

Also, it follows from the above argument that if $\Lambda_k(A)$ contains a boundary point of $\mathcal{E}(a, b, \lambda_k(P))$, then $\lambda_k(P) = \lambda_k(P') = p_k$. In this case, P has a k-dimensional compression $\operatorname{diag}(p_1, \ldots, p_k)$ with $p_k = \lambda_k(P)$ and $\Lambda_k(A) = \mathcal{E}(a, b, \lambda_k(P))$. Conversely, it is clear that if P has a k-dimensional compression of the above diagonal from, $\Lambda_k(A)$ contains all the boundary point of $\mathcal{E}(a, b, \lambda_k(P))$. The proof is complete. □

For an infinite dimensional operator A, one can extend the definition of rank-k numerical range to $\Lambda_\infty(A)$ defined as the set of scalars $\lambda \in \mathbb{C}$ such that $PAP = \lambda P$ for an infinite rank orthogonal projection P on $\mathcal{H}$, see [14, 17]. Evidently, $\Lambda_\infty(A)$ consists of those $\lambda \in \mathbb{C}$ for which there exists an infinite orthonormal set $\{x_i \in \mathcal{H} : i \geq 1\}$ such that $\langle Ax_i, x_j \rangle = \delta_{ij}\lambda$ for all $i, j \geq 1$. It is shown in [14] that

$$\Lambda_\infty(A) = \bigcap_{k \geq 1} \Lambda_k(A) = \bigcap \{W(A + F) : F \in \mathcal{B}(\mathcal{H}) \text{ has a finite rank}\}.$$

Recall that $\lambda_\infty(H)$ is the supremum of the set

$$\sigma(H) \setminus \{\mu \in \mathbb{C} : H - \mu I \text{ has a non-trivial finite dimensional null space}\}.$$

One can extend the definition of $\Omega_k(A)$ to

$$\Omega_\infty(A) = \bigcap_{k\geq 1} \Omega_k(A).$$

By Theorem 5.1 in [14] (see also [1, Theorem 4]),

$$\Omega_\infty(A) = \bigcap \{\mathbf{cl}\,(W(A+F)) : F \in \mathcal{B}(\mathcal{H}) \text{ has a finite rank}\}$$

is the essential numerical range $W_e(A)$ of A; $\Omega_\infty(A) = \mathbf{cl}\,(\Lambda_\infty(A))$ if and only if $\Lambda_\infty(A)$ is non-empty.

By Theorem 2.1, we have the following corollary, which gives a complete description of $\Lambda_\infty(A)$ and the essential numerical range of a quadratic operator A. It turns out that each of them can be a singleton, a line segment or an elliptical disk. As a result, we also get the description of the essential numerical range of $A \in \mathcal{B}(\mathcal{H})$ obtained in [19, Theorem 2.2 and Corollary 2.3].

Corollary 2.6. *Suppose $A \in \mathcal{B}(\mathcal{H})$ is an infinite dimensional quadratic operator with operator matrix in the form in Theorem 1.1.*

(a) *If $r < \infty$, then $\Lambda_\infty(A) = \{\gamma\}$.*

(d) *Suppose $r = \infty$, and $\mathcal{E}$ is the closed elliptical disk with foci a, b and minor axis of length $\lambda_\infty(P)$. Then $\Lambda_\infty(A) = \mathcal{E}$ or $\lambda_\infty(A) = \mathbf{int}(\mathcal{E})$; the equality $\Lambda_\infty(A) = \mathcal{E}$ holds if and only if $\sigma(P) \cap (\lambda_\infty(P), \infty)$ is infinite or $P - \lambda_\infty(P)I$ has an infinite dimensional null space.*

Consequently, $W_e(A) = \Omega_\infty(A) = \mathbf{cl}\,(\Lambda_\infty(A))$ is a singleton, a line segment or a closed elliptical disk.

3. Davis-Wielandt shells and q-numerical ranges

The *Davis-Wielandt shell* of A is defined by

$$DW(A) = \{(\langle Ax, x\rangle, \langle Ax, Ax\rangle) : x \in \mathcal{H}, \langle x, x\rangle = 1\};$$

see [7, 8, 22]. Evidently, the projection of the set $DW(A)$ on the first co-ordinate is $W(A)$. So, $DW(A)$ captures more information about the operator A than $W(A)$. For example, in the finite dimensional case, normality of operators can be completely determined by the geometrical shape of their Davis-Wielandt shells, namely, $A \in M_n$ is normal if and only if $DW(A)$ is a polyhedron in $\mathbb{C} \times \mathbb{R}$ identified with $\mathbb{R}^3$.

Suppose $A \in \mathcal{B}(\mathcal{H})$. It is known that if $\dim \mathcal{H} \geq 3$ then $DW(A)$ is always convex. If $A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \in M_2$, then one of the following holds.

(1) $c = 0$ and $DW(A) = \mathbf{conv}\{(a, |a|^2), (b, |b|^2)\}$, which will be a singleton if $a = b$;
(2) $c \neq 0$ and $DW(A)$ is an ellipsoid centered at $(a+b, |a|^2+|b|^2+|c|^2)/2$, which is a sphere if $a = b$.

Suppose $\dim \mathcal{H} \geq 3$. The Davis-Wielandt shell of a quadratic operator can be a line segment, an ellipsoid with interior, or just the interior of an ellipsoid. The following theorem gives a precise description of the set.

Theorem 3.1. *Suppose $\dim \mathcal{H} \geq 3$ and $A \in \mathcal{B}(\mathcal{H})$ is a quadratic operator with operator matrix in the form in Theorem 1.1. Let $A_0 = \begin{bmatrix} a & \|P\| \\ 0 & b \end{bmatrix}$ and $\mathcal{E}$ be the closed ellipsoid $\mathbf{conv}DW(A_0)$. Then $DW(A) = \mathcal{E}$ or $\mathbf{int}(\mathcal{E})$. The equality $DW(A) = \mathcal{E}$ holds if and only if there is a unit vector $x \in \mathcal{H}_1$ such that $\|Px\| = \|P\|$.*

We start with the following lemma.

Lemma 3.2. *Suppose $C = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$ and $D = \begin{bmatrix} a & d \\ 0 & b \end{bmatrix}$ with $c \geq d$. Then $\mathbf{conv}DW(D) \subseteq \mathbf{conv}DW(C)$.*

Proof. Suppose $U \in M_2$ is unitary and $U^*DU = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ so that $(e, |e|^2 + |g|^2) \in DW(D)$. Since $W(D) \subseteq W(C)$ and $\operatorname{tr} D = \operatorname{tr} C$, there is a unitary $V \in M_2$ such that $V^*CV = \begin{bmatrix} e & f_1 \\ g_1 & h \end{bmatrix}$. Since $X \in M_2$ and $X^t \in M_2$ are always unitarily similar, we may assume that $|f_1| \geq |g_1|$. Note that

$$eh - f_1 g_1 = \det(C) = \det(D) = eh - gf.$$

So, $f_1 g_1 = fg$. Also,

$$|f_1|^2 + |g_1|^2 - |f|^2 - |g|^2 = \operatorname{tr}(C^*C) - \operatorname{tr}(D^*D) = |c|^2 - |d|^2 \geq 0$$

and hence $|f_1|^2 + |g_1|^2 \geq |f|^2 + |g|^2$. Then both $|f|$ and $|g|$ must lie between the interval $[|g_1|, |f_1|]$. It follows that the point $(e, |e|^2 + |g|^2)$ is a convex combination of the two points in $DW(C)$, namely, $(e, |e|^2 + |g_1|^2)$ and $(e, |e|^2 + |f_1|^2)$. Thus, $DW(D) \subseteq \mathbf{conv}DW(C)$. $\square$

Proof of Theorem 3.1. Suppose $r > 0$ is finite. Then A is unitarily similar to

$$A_1 \oplus \cdots \oplus A_r \oplus \gamma I_s$$

with $A_j = \begin{bmatrix} a & \lambda_j(P) \\ 0 & b \end{bmatrix}$ for $j = 1, \ldots, r$ and $\|P\| = \lambda_1(P) \geq \cdots \geq \lambda_r(P) \geq 0$. Note that for any two operators X and Y we have

$$DW(X \oplus Y) = \mathbf{conv}(DW(X) \cup DW(Y)).$$

By Lemma 3.2, $DW(A_j) \subseteq \mathbf{conv} DW(A_1) = \mathcal{E}$ for all $j = 2, \ldots, r$. Moreover, we have $DW(\gamma I_s) \subseteq \mathbf{conv} DW(A_1) = \mathcal{E}$. Thus, $DW(A) = \mathcal{E}$. Clearly, there exists a unit vector $x \in \mathcal{H}_1$ such that $\|Px\| = \|P\|$.

Suppose $r = \infty$. Without loss of generality, we may assume that $\gamma = b$. Decompose $\mathcal{H}$ into $\hat{\mathcal{H}}_1 \oplus \hat{\mathcal{H}}_2$ such that A has an operator matrix

$$\begin{bmatrix} aI_r & Q \\ 0 & bI_{r+s} \end{bmatrix} \quad \text{with} \quad Q = \begin{bmatrix} P & 0 \end{bmatrix}.$$

Suppose $(\mu, \nu) = (\langle Ax, x \rangle, \|Ax\|^2) \in DW(A)$. Write $x = u_1 + v_1$ and $Ax = u_2 + v_2$ for some $u_1, u_2 \in \hat{\mathcal{H}}_1$ and $v_1, v_2 \in \hat{\mathcal{H}}_2$. Let S be the subspace spanned by $\{u_1, u_2, v_1, v_2\}$. Since $Au_j = au_j$ and $A^* v_j = \bar{b} v_j$ for $j = 1, 2$, the compression of A onto S has the form

$$\tilde{A} = \begin{bmatrix} aI_p & Q' \\ 0 & bI_q \end{bmatrix} \in M_{p+q}$$

so that $\langle Ax, x \rangle = \langle \tilde{A}x, x \rangle$ and $\|Ax\| = \|\tilde{A}x\|$, where p and q are the dimension of subspace spanned by $\{u_1, u_2\}$ and $\{v_1, v_2\}$ respectively. Notice that $\|Q'\| \leq \|Q\| = \|P\|$. Hence, $(\mu, \nu) \in DW(\tilde{A}) \subseteq \mathbf{conv} DW(A_0)$.

On the other hand, for any $\varepsilon > 0$, there are unit vectors $x \in \hat{\mathcal{H}}_2$ and $y \in \hat{\mathcal{H}}_1$ such that $Qx = qy$ with $q > \|P\| - \varepsilon$. Using an orthonormal basis with $y \oplus 0, 0 \oplus x \in \hat{\mathcal{H}}_1 \oplus \hat{\mathcal{H}}_2$ as the first two vectors, we see that the operator matrix of A has the form

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad \text{with} \quad A_{11} = \begin{bmatrix} a & q \\ 0 & b \end{bmatrix}.$$

Thus, $DW(A_{11}) \subseteq DW(A)$. By convexity, $\mathbf{conv} DW(A_{11}) \subseteq DW(A)$. Letting $\varepsilon \to 0$, we see that $DW(A)$ contains the interior of $DW(A_0)$. It is easy to determine the boundary behavior of $DW(A)$. □

For $q \in [0,1]$, the *q-numerical range* of A is

$$W_q(A) = \{\langle Ax, y\rangle : x, y \in \mathcal{H}, \langle x, x\rangle = \langle y, y\rangle = 1, \langle x, y\rangle = q\}.$$

There is a close connection between $W_q(A)$ and $DW(A)$, namely,

$$W_q(A) = \left\{q\mu + \sqrt{1-q^2}\nu : (\mu, |\mu|^2 + |\nu|^2) \in DW(A)\right\};$$

see [12, 20]. By Theorem 3.1, we have the following description of $W_q(A)$ for a quadratic operator $A \in \mathcal{B}(\mathcal{H})$. In particular, $W_q(A)$ will always be an open or closed elliptical disk, which may degenerate to a line segment or a point.

Corollary 3.3. *Use the notation in Theorem 3.1. For any $q \in [0,1]$, either $W_q(A) = W_q(A_0)$ or $W_q(A) = \mathbf{int}\,(W_q(A_0))$; the equality $W_q(A) = W_q(A_0)$ holds if and only if there is a unit vector $x \in \mathcal{H}_1$ such that $\|Px\| = \|P\|$.*

4. c-numerical ranges

For $c = (c_1, \dots, c_k)$ with $c_1 \geq \cdots \geq c_k$ and $k \leq \dim \mathcal{H}$, the *c-numerical range* of A is

$$W_c(A) = \left\{\sum_{j=1}^{k} c_j \langle Ax_j, x_j\rangle : \{x_1, \dots, x_k\} \subseteq \mathcal{H} \text{ is an orthonormal set}\right\}.$$

If $(c_1, \dots, c_k) = (1, \dots, 1)$, then $W_c(A)$ reduces to the *k-numerical range*; see [9].

Suppose $A = \begin{bmatrix} a & d \\ 0 & b \end{bmatrix} \in M_2$ and $c = (c_1, c_2)$. Then

$$W_c(A) = (c_1 - c_2)W(A) + c_2 \mathrm{tr}\, A = W((c_1 - c_2)A + (c_2 \mathrm{tr}\, A)I_2)$$

is the elliptical disk with foci $ac_1 + bc_2$ and $ac_2 + bc_1$, and minor axis of length $|(c_1 - c_2)d|$.

For a self-adjoint operator $H \in B(\mathcal{H})$, we have

$$\mathbf{cl}\,(W_c(H)) = [m_c(H), M_c(H)],$$

where

$$m_c(H) = \inf\left\{-\sum_{j=1}^{\ell} c_j \lambda_j(-H) + \sum_{j=1}^{k-\ell} c_{k-j+1}\lambda_j(H) : 0 \leq \ell \leq k\right\}$$

and

$$M_c(H) = \sup \left\{ \sum_{j=1}^{\ell} c_j \lambda_j(H) - \sum_{j=1}^{k-\ell} c_{k-j+1} \lambda_j(-H) : 0 \le \ell \le k \right\}.$$

For a general operator $A \in \mathcal{B}(\mathcal{H})$, we have

$$\mathbf{cl}\,(W_c(A)) = \bigcap_{t \in [0,2\pi)} \left\{ \mu \in \mathbb{C} : \mathrm{Re}\,\left(e^{it}\mu\right) \le M_c\left(\mathrm{Re}\,(e^{it}A)\right) \right\}. \tag{1}$$

For a quadratic operator $A \in \mathcal{B}(\mathcal{H})$, it is easy to determine $\lambda_m\left(\mathrm{Re}\,(e^{it}A)\right)$. Thus, it is not hard to determine $W_c(A)$ using (1). It turns out that $\mathbf{cl}\,(W_c(A))$ can always be expressed as the sum of a finite number of elliptical disks, namely,

$$\mathbf{cl}\,(W_c(A)) = W(A_1) + \cdots + W(A_t) + d$$

for some constant $d \in \mathbb{C}$ and $A_1, \dots, A_t \in M_2$ with $t \le k$.

To simplify the statement of our results, we will impose the following assumption on the vector $c = (c_1, \dots, c_k)$:

$$\begin{aligned} &c_1 \ge \cdots \ge c_k \quad \text{with} \quad c_{p+1} = 0, \quad \text{where} \\ &\dim \mathcal{H} = \infty > k = 2p \quad \text{or} \quad \dim \mathcal{H} = k \in \{2p, 2p+1\}. \end{aligned} \tag{2}$$

Note that it is easy to reduce the general case to the study of the special vector c with assumption (2). In the infinite dimensional case, this can be achieved by adding zeros to the vector $c = (c_1, \dots, c_k)$. In the finite dimensional case, we can first assume that $k = \dim \mathcal{H}$ by adding zeros to the vector c, and then replace c with $\hat{c} = c - c_{p+1}(1, \dots, 1)$. One can then use the fact that $W_c(A) = W_{\hat{c}}(A) + c_{p+1}\mathrm{tr}\, A$ to determine the shape of $W_c(A)$. Note also that the advantage of this assumption on c is that the supremum in the definition of $M_c(H)$ is always attained at $\ell = p$.

Theorem 4.1. *Let $A \in \mathcal{B}(\mathcal{H})$ be a quadratic operator with operator matrix in the form described in Theorem 1.1. Suppose $c = (c_1, \dots, c_k)$ satisfies* (2) *and $t = \min\{p, r\}$. For $j = 1, \dots, t$, let*

$$B_j = (c_j - c_{k-j+1}) \begin{bmatrix} a & \lambda_j(P) \\ 0 & b \end{bmatrix} + c_{k-j+1}(a+b)I_2.$$

Then $W_c(A) = \mathcal{E}$ *or* $\mathbf{int}(\mathcal{E})$, *where*

$$\mathcal{E} = W(B_1) + \cdots + W(B_t) + \gamma \sum_{j=t+1}^{k-t} c_j.$$

The equality $W_c(A) = \mathcal{E}$ *holds if and only if* $P : \mathcal{H}_1 \to \mathcal{H}_1$ *has a compression* $\operatorname{diag}(\lambda_1(P), \ldots, \lambda_t(P))$.

Proof. Suppose $\dim \mathcal{H} = n$ is finite. So we have $k = n$ and $r \le p$. Notice that A is unitarily similar to

$$A_1 \oplus \cdots \oplus A_r \oplus \gamma I_s,$$

where $A_j = \begin{bmatrix} a & \lambda_j(P) \\ 0 & b \end{bmatrix}$ for $j = 1, \ldots, r$. By the argument in the proof of Theorem 2.1, we have

$$\lambda_j\left(\operatorname{Re}(e^{i\xi}A)\right) = \begin{cases} \lambda_1\left(\operatorname{Re}(e^{i\xi}A_j)\right) & \text{if } j \le r, \\ \operatorname{Re}(e^{i\xi}\gamma) & \text{if } r < j \le r+s, \\ \lambda_2\left(\operatorname{Re}(e^{i\xi}A_{n-j+1})\right) & \text{if } r+s < j \le n. \end{cases}$$

Under assumption (2) and $k = n$, we have

$$M_c\left(\operatorname{Re}(e^{i\xi}A)\right) = \sum_{j=1}^{n} c_j\lambda_j\left(\operatorname{Re}(e^{i\xi}A)\right).$$

On the other hand,

$$\operatorname{Re}(e^{i\xi}\gamma)\sum_{j=r+1}^{n-r} c_j = \sum_{j=r+1}^{n-r} c_j\lambda_j\left(\operatorname{Re}(e^{i\xi}A)\right)$$

and

$$\begin{aligned} & \sum_{j=1}^{r} M_{(c_j, c_{n-j+1})}\left(\operatorname{Re}(e^{i\xi}A_j)\right) \\ = & \sum_{j=1}^{r}\left[c_j\lambda_1\left(\operatorname{Re}(e^{i\xi}A_j)\right) + c_{n-j+1}\lambda_2\left(\operatorname{Re}(e^{i\xi}A_j)\right)\right] \\ = & \sum_{j=1}^{r} c_j\lambda_j\left(\operatorname{Re}(e^{i\xi}A)\right) + \sum_{j=n-r+1}^{n} c_j\lambda_j\left(\operatorname{Re}(e^{i\xi}A)\right). \end{aligned}$$

Thus, $M_c\left(\mathrm{Re}\,(e^{i\xi}A)\right)$ equals

$$\sum_{j=1}^{r} M_{(c_j,c_{n-j+1})}\left(\mathrm{Re}\,(e^{i\xi}A_j)\right) + \mathrm{Re}\,(e^{i\xi}\gamma)\sum_{j=r+1}^{n-r} c_j.$$

By (1) and the above equation, the two compact convex sets

$$W_c(A) \quad \text{and} \quad W_{(c_1,c_n)}(A_1) + \cdots + W_{(c_r,c_{n-r+1})}(A_r) + \gamma \sum_{j=r+1}^{n-r} c_j$$

always share the same support line in each direction. Thus, the two sets are the same. Since $W_{(c_j,c_{n-r+j})}(A_j) = W(B_j)$ for $j = 1,\ldots,r$, it follows that

$$W_c(A) = W(B_1) + \cdots + W(B_r) + \gamma \sum_{j=r+1}^{n-r} c_j.$$

Next, suppose $\dim \mathcal{H}$ is infinite. Suppose r is finite or $\lambda_p(P) = 0$. Let $t = \min\{p, r\}$. Then $P : \mathcal{H}_1 \to \mathcal{H}_1$ has a compression $\mathrm{diag}\,(\lambda_1(P),\ldots,\lambda_t(P))$. Take

$$\tilde{A} = A_1 \oplus \cdots \oplus A_t \oplus \gamma I_{k-2t} \in M_k$$

with $A_j = \begin{bmatrix} a & \lambda_j(P) \\ 0 & b \end{bmatrix}$ for $j = 1,\ldots,t$. Then we have $\lambda_m(\mathrm{Re}\,(e^{i\xi}A)) = \lambda_m(\mathrm{Re}\,(e^{i\xi}\tilde{A}))$ for each $\xi \in [0, 2\pi)$ and $m = 1,\ldots,p$. Thus, $M_c(\mathrm{Re}\,(e^{i\xi}A)) = M_c(\mathrm{Re}\,(e^{i\xi}\tilde{A}))$ for all $\xi \in [0, 2\pi)$ and so $W_c(A) = W_c(\tilde{A})$. The result follows from the finite dimensional case.

Suppose r is infinite and $\lambda_p(P) > 0$. For $\mu_1 \geq \cdots \geq \mu_p > 0$, let

$$\mathcal{E}(\mu_1,\ldots,\mu_p) = W_{(c_1,c_k)}\begin{bmatrix} a & \mu_1 \\ 0 & b \end{bmatrix} + \cdots + W_{(c_p,c_{k-p+1})}\begin{bmatrix} a & \mu_p \\ 0 & b \end{bmatrix}.$$

Notice that $\mathcal{E}(\lambda_1(P),\cdots\lambda_p(P)) = W(B_1) + \cdots + W(B_p)$.

By Lemma 2.4, there exist a k-dimensional subspace $\mathcal{V}$ of $\mathcal{H}$ and $X : \mathcal{V} \to \mathcal{H}_1$ satisfying $X^*X = I_k$ and $X^*PX = \mathrm{diag}\,(\lambda_1,\ldots,\lambda_p)$ with $\lambda_j(P) - \varepsilon < \lambda_j \leq \lambda_j(P)$ for $j = 1,\ldots,p$. Let $Z = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \oplus I_s$. Then Z^*AZ is unitary similar to

$$\begin{bmatrix} a & \lambda_1 \\ 0 & b \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} a & \lambda_p \\ 0 & b \end{bmatrix} \oplus \gamma I_s.$$

Note that $W_c(B) \subseteq W_c(A)$ if B is a compression of A. Applying the result for finite $r = p$, we have

$$\mathcal{E}(\lambda_1, \ldots, \lambda_p) = W_c(Z^*AZ) \subseteq W_c(A).$$

As $\lambda_j \to \lambda_j(P)$ and hence $\begin{bmatrix} a & \lambda_j \\ 0 & b \end{bmatrix} \to \begin{bmatrix} a & \lambda_j(P) \\ 0 & b \end{bmatrix}$ when $\varepsilon \to 0$, we see that all the interior points of $\mathcal{E}(\lambda_1(P), \ldots, \lambda_p(P))$ lie in $W_c(A)$.

Conversely, suppose $\mu \in W_c(A)$. Then there exist a k-dimensional subspace $\mathcal{V}_1$ of $\mathcal{H}$ and $X : \mathcal{V}_1 \to \mathcal{H}$ such that $X^*X = I_k$ and $\mu \in W_c(X^*AX)$. By Lemma 2.5, there are a $(4k + \ell)$-dimensional subspace $\mathcal{V}_2$, containing $\mathcal{V}_1$ and $Y : \mathcal{V}_2 \to \mathcal{H}$ such that $Y^*Y = I_{\mathcal{V}_2}$ and Y^*AY has operator matrix

$$\begin{bmatrix} aI_{2k} & P' \\ 0 & bI_{2k} \end{bmatrix} \oplus \gamma I_\ell \in M_{4k+\ell},$$

where of $P' = \operatorname{diag}(p_1, \ldots, p_{2k})$ is a $2k$-dimensional compression of P, with $p_1 \geq \cdots \geq p_{2k}$ and $p_i \leq \lambda_i(P)$ for $1 \leq i \leq 2k$. Since X^*AX is a compression of Y^*AY, we have $\mu \in W_c(X^*AX) \subseteq W_c(Y^*AY)$. By the finite dimensional result, we have

$$\mu \in W_c(Y^*AY) = \mathcal{E}(\lambda_1(P'), \ldots, \lambda_p(P')) \subseteq \mathcal{E}(\lambda_1(P), \ldots, \lambda_p(P)).$$

So, we have shown that

$$\mathbf{int}(\mathcal{E}(\lambda_1(P), \ldots, \lambda_p(P))) \subseteq W_c(A) \subseteq \mathcal{E}(\lambda_1(P), \ldots, \lambda_p(P)).$$

Also, it follows from the above argument that if $W_c(A)$ contains a boundary point of $\mathcal{E}(\lambda_1(P), \ldots, \lambda_p(P))$, then $\lambda_i(P) = \lambda_i(P') = p_i$ for all $i = 1, \ldots, p$. Then P has a k-dimensional compression $\operatorname{diag}(\lambda_1(P), \ldots, \lambda_p(P))$. Conversely, it is clear that if P has a k-dimensional compression of the above diagonal from, $\Lambda_k(A)$ contains all the boundary point of $\mathcal{E}(\lambda_1(P), \ldots, \lambda_p(P))$. The proof is complete. □

In Theorem 4.1, if $\lambda_m(P) = 0$ for some $m \leq t$, then $W(B_m) + \cdots + W(B_t)$ becomes a line segment joining

$$a \sum_{j=m}^{t} c_j + b \sum_{j=k-t+1}^{k-m+1} c_j \quad \text{and} \quad b \sum_{j=m}^{t} c_j + a \sum_{j=k-t+1}^{k-m+1} c_j.$$

Thus, $W_c(A)$ is a sum of $m - 1$ nondegenerate elliptical disks with one line segment. Therefore, we have the following corollary.

Corollary 4.2. *Let $c = (c_1, \ldots, c_k)$ and $A \in \mathcal{B}(\mathcal{H})$ satisfy the hypotheses of Theorem 4.1.*

(a) *If $\sigma(A)$ is a singleton, i.e., $a = b$, then $W_c(A)$ is a circular disk with radius $\sum_{j=1}^{\min\{p,r\}} (c_j - c_{k-r+j})\lambda_j(P)$.*

(b) *If $\sigma(A) = \{a, b\}$ has two distinct elements, then $W_c(A)$ is the sum of elliptical disks such that all boundary points are differentiable. If $\lambda_m(P) = 0$ for some $m \le \min\{p, r\}$, then there are exactly two flat portions on the boundary. Otherwise, there is no flat portion on the boundary.*

Our results on the c-numerical ranges cover and refine those in [19, Section 2.3]. Specializing the results to the k-numerical range, we have the following corollary covering the results in [3], where the authors proved that $W_k(A)$ is a union of infinitely many circular disks. Here, we show that $W_k(A)$ is the sum of at most k elliptical disks with at most one point.

Corollary 4.3. *Suppose A is a quadratic operator with operator matrix in the form described in Theorem 1.1. Let $t = \min\{k, r\}$ and*

$$A_j = \begin{bmatrix} a & \lambda_j(P) \\ 0 & b \end{bmatrix} \qquad j = 1, \ldots, t.$$

(a) *If $k \le r + s$, then $W_k(A) = \mathcal{E}$ or $W_k(A) = \mathbf{int}(\mathcal{E})$, where*

$$\mathcal{E} = W(A_1) + \cdots + W(A_t) + (k - t)\gamma.$$

The equality $W_k(A) = \mathcal{E}$ holds if and only if $P : \mathcal{H}_1 \to \mathcal{H}_1$ has a compression $\mathrm{diag}\,(\lambda_1(P), \ldots, \lambda_t(P))$.

(b) *If $k > r + s$, then $W_k(A)$ equals*

$$W(A_1) + \cdots + W(A_{2r+s-k}) + (k - r - s)(a + b) + s\gamma.$$

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