A note on the realignment criterion

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Abstract. For a quantum state in a bipartite system represented as a density matrix, researchers used the realignment matrix and functions on its singular values to study the separability of the quantum state. We obtain bounds for elementary symmetric functions of singular values of realignment matrices. This answers some open problems proposed by Lupo, Aniello, and Scardicchio. As a consequence, we show that the proposed scheme by these authors for testing separability would not work if the two subsystems of the bipartite system have the same dimension.

1. Introduction

Quantum entanglement was first proposed by Einstein, Podolsky, and Rosen [3] and Schrödinger [17] as a strange phenomenon of quantum mechanics, criticizing the completeness of the quantum theory. Nowadays, entanglement is not only regarded as a key for the interpretation of quantum mechanics or as a mere scientific curiosity, but also as a resource for various applications, like quantum cryptography [4], quantum teleportation [1], and quantum computation [14].

Suppose quantum states of two quantum systems are represented by density matrices (positive semidefinite matrices with trace 1) of sizes m and n, respectively. States of their *bipartite composition system* are represented by $mn \times mn$ density matrices. Such a state is *separable* if there are positive numbers p_j summing up to 1, $m \times m$ density matrices ρ_j^1, and $n \times n$ density matrices ρ_j^2 such that

$$\rho = \sum_{j=1}^{k} p_j \, \rho_j^1 \otimes \rho_j^2.$$

A state is *entangled* if it is not separable. In quantum information science, it is important to determine the separability of a state. However, the problem of characterizing separable states is *NP-hard* [5]. Therefore, researchers focus on finding effective criterion to determine whether a density matrix is separable or not.

A simple and strong criterion for separability of density matrix is the *computable cross norm or realignment (CCNR) criterion*. The name CCNR comes from the fact that this criterion has been discovered in two different forms, namely, by cross norms [15, 16] and by realignment of density matrices [2].

To describe the realignment criterion, let M_N be the set of $N \times N$ complex matrices. $\mathcal{D}(m, n)$ will denote the set of all $mn \times mn$ density matrices and $\mathcal{D}_s(m, n)$ the set of separable density matrices in $\mathcal{D}(m, n)$. For any $X = [x_{ij}] \in M_n$, let

$$\text{vec}\,(X) = (x_{11}, x_{12}, \ldots, x_{1n},\ x_{21}, x_{22}, \ldots, x_{2n},\ \ldots,\ x_{n1}, x_{n2}, \ldots, x_{nn}).$$

If $\rho = [X_{rs}]_{1 \leq r,s \leq m} \in \mathcal{D}(m, n)$ with $X_{rs} \in M_n$, then the realignment of ρ is the $m^2 \times n^2$ matrix ρ^R with rows

$$\text{vec}\,(X_{11}), \text{vec}\,(X_{12}), \ldots, \text{vec}\,(X_{1m}), \text{vec}\,(X_{21}), \ldots, \text{vec}\,(X_{2m}), \ldots, \text{vec}\,(X_{m1}), \ldots \text{vec}\,(X_{mm}).$$

For example, if $(m, n) = (2, 3)$ and $\rho = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \in D(2, 3)$ with $X_{rs} \in M_3$, then

$$\rho^R = \begin{bmatrix} \text{vec} (X_{11}) \\ \text{vec} (X_{12}) \\ \text{vec} (X_{21}) \\ \text{vec} (X_{22}) \end{bmatrix}.$$

The realignment criterion asserts that if $\rho \in \mathcal{D}_s(m, n)$ then the sum of the singular values of ρ^R is at most 1. Recall that the *singular values* of an $M \times N$ matrix A are the nonnegative square roots of the $k = \min\{M, N\}$ largest eigenvalues of the matrix $AA^\dagger$.

For convenience of notation, we assume that $m \leq n$ in the following discussion. For $\rho \in \mathcal{D}(m, n)$, let $s_1 \geq \cdots \geq s_{m^2}$ be the singular values of ρ^R. The realignment criterion can be stated as

$$s_1 + \cdots + s_{m^2} \leq 1 \qquad \text{for } \rho \in \mathcal{D}_s(m, n).$$

In [10], Lupo, Aniello, and Scardicchio suggest further study of the symmetric functions on the singular values of ρ^R, in order to find conditions beyond the realignment criterion to identify entanglement.

Let

$$\mathcal{S}(m, n) = \{(s_1, \ldots, s_{m^2}) : \quad s_1 \geq \cdots \geq s_{m^2} \text{ are the singular values of } \rho^R,$$
$$\text{for some } \rho \in \mathcal{D}(m, n)\}$$

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$$\text{for some } \rho \in \mathcal{D}_s(m, n)\}.$$

For each $1 < \ell \leq m^2$, define the ℓ-th elementary symmetric function

$$f_\ell (s_1, \ldots, s_{m^2}) = \sum_{1 \leq i_1 < \cdots < i_\ell \leq m^2} \Pi_{j=1}^\ell s_{i_j}.$$

Following [10], we define for each $1 < \ell \leq m^2$,

$$\tilde{B}_\ell(m, n) = \max\{f_\ell (\mathbf{s}) : \mathbf{s} \in \mathcal{S}(m, n), \ \mathbf{s} = (s_1, \ldots, s_{m^2}) \text{ with } \textstyle\sum_{i=1}^{m^2} s_i \leq 1\},$$
$$B_\ell(m, n) = \max\{f_\ell (\mathbf{s}) : \mathbf{s} \in \mathcal{S}_s(m, n)\}.$$

The bounds $\tilde{B}_\ell(m, n)$ and $B_\ell(m, n)$ were introduced in [10] using different notations, namely, $\tilde{x}_\ell(d, D)$ and $x_\ell(d, D)$ with $(d, D) = (m^2, n^2)$.

It follows from the definitions that if $\tilde{B}_\ell(m, n) > B_\ell(m, n)$, then there exists an entangled density matrix ρ such that the sum of singular values of ρ^R is at most 1

but $f_\ell(s_1, \ldots, s_{m^2}) > B_\ell(m, n)$. Therefore, the bound $B_\ell(m, n)$ can be used to detect entanglement for which the realignment criterion fails. Numerical estimations for these bounds were given for $(m, n) = (2, 2)$ and $(2, 3)$ in [10]. The numerical results also suggest that $\tilde{B}_\ell(2, 2) = B_\ell(2, 2)$ and $\tilde{B}_\ell(2, 3) > B_\ell(2, 3)$. The authors of [10] raised the following two open problems in the search for criterion for entanglement beyond the realignment criterion:

(P1) To determine the actual values of the upper bounds $B_\ell(m, n)$ and $\tilde{B}_\ell(m, n)$.

(P2) To determine if $\tilde{B}_\ell(m, n) > B_\ell(m, n)$.

In this paper, we study the singular values of ρ^R for a density matrix ρ. We refine some inequalities given in [10]. This leads to an explicit formula for $\tilde{B}_\ell(m, n)$, for all $n \geq m$, except for $m^3 - m/2 < n < m^3$, that gives a partial solution to (P1). Furthermore, we show that $\tilde{B}_\ell(n, n) = B_\ell(n, n)$ for all n and this implies that the answer to (P2) is negative if $m = n$.

We conclude this section with a reformulation of another simple and strong criterion for separability in terms of the singular values. Let $X = [X_{rs}]_{1 \leq r, s \leq m} \in \mathcal{D}(m, n)$ with $X_{rs} \in M_n$. The *partial transpose* of X with respect to the second subsystem is given by $X^{T_2} = [X_{rs}^t]_{1 \leq r, s \leq m}$, where X_{rs}^t is the transpose of X_{rs}. The PPT criterion [12] states that if $X \in \mathcal{D}_s(m, n)$, then X^{T_2} is positive semi-definite. For $m + n \leq 5$, PPT criterion is a necessary and sufficient condition for separability [7], i.e. $X \in \mathcal{D}_s(m, n)$ if and only if $X^{T_2} \in \mathcal{D}(m, n)$. For $m, n > 1$ and $m + n > 5$, the PPT criterion and the CCNR criterion are independent. Note that for $X \in \mathcal{D}(m, n)$, X^{T_2} is Hermitian. So the singular values of X^{T_2} are the absolute values of the eigenvalues of X^{T_2}. Since the sum of all eigenvalues of X^{T_2} is equal to trace $(X^{T_2}) = \text{trace}(X) = 1$, X^{T_2} is positive semi-definite if and only if the sum of the singular values of X^{T_2} is at most 1, cf. [8, Corollary 1]. Thus the PPT criterion shares a similar form with the CCNR criterion.

2. Main results and their implications

In this section, we continue to use the notations introduced in Section 1 and assume that $m \leq n$. We will describe the results and their implications. The proofs will be given in the next section.

For any density matrix ρ, we obtain the following lower bound for the largest singular value for ρ^R, the realigned matrix of ρ.

Lemma 2.1 *Let* $\mathbf{s} = (s_1, \ldots, s_{m^2}) \in \mathcal{S}(m, n)$. *Then* $s_1 \geq \dfrac{1}{\sqrt{mn}}$.

Recall that for two vectors $\mathbf{x}$, $\mathbf{y} \in \mathbf{R}^N$, $\mathbf{x}$ is *majorized* by $\mathbf{y}$, denotes by $\mathbf{x} \prec \mathbf{y}$, if for all $1 \leq k \leq N$, the sum of the k largest entries of $\mathbf{x}$ is not larger than that of $\mathbf{y}$, and the sum of all entries of $\mathbf{x}$ is equal to that of $\mathbf{y}$. A function $f : \mathbf{R}^N \to \mathbf{R}$ is *Schur concave* if $f(\mathbf{y}) \leq f(\mathbf{x})$ whenever $\mathbf{x} \prec \mathbf{y}$.

Using Lemma 2.1, we will show that if $n \leq m^3$, then the vector $\mathbf{s}$ in $\mathcal{S}(m, n)$ always *marojize* a vector of the form $(\alpha, \beta, \ldots, \beta)$. One can then apply the theory of majorization and Schur concave functions (see [11]) to obtain the inequality $f_\ell(\mathbf{s}) \leq f_\ell(\alpha, \beta, \ldots, \beta)$, as shown in Lemma 2.2.

For $1 \leq r \leq N$, $\binom{N}{r}$ will denote the binomial coefficient $\frac{N!}{r!(N-r)!}$.

Lemma 2.2 *Suppose $n \leq m^3$ and $\mathbf{s} = (s_1, \ldots, s_{m^2}) \in \mathcal{S}(m, n)$ with $\sum_{i=1}^{m^2} s_i \leq 1$. Let*

$$\alpha = \frac{1}{\sqrt{mn}} \quad and \quad \beta = \frac{1 - \alpha}{m^2 - 1} = \frac{\sqrt{mn} - 1}{\sqrt{mn}(m^2 - 1)}.$$

Then

$$(\alpha, \overbrace{\beta, \ldots, \beta}^{m^2 - 1}) \prec \frac{1}{\sum_{i=1}^{m^2} s_i}(s_1, \ldots, s_{m^2}),$$

and for $1 < \ell \leq m^2$,

$$f_\ell(\mathbf{s}) \leq f_\ell(\alpha, \beta, \ldots, \beta) \leq \binom{m^2}{\ell}\left(\frac{1}{m^2}\right)^\ell.$$

Furthermore,

(a) *$f_\ell(\mathbf{s}) = f_\ell(\alpha, \beta, \ldots, \beta)$ if and only if $\mathbf{s} = (\alpha, \beta, \ldots, \beta)$;*

(b) *$f_\ell(\alpha, \beta, \ldots, \beta) = \binom{m^2}{\ell}\left(\frac{1}{m^2}\right)^\ell$ if and only if $n = m^3$.*

It follows from Lemma 2.2 that $\tilde{B}_\ell(m, n) \leq \binom{m^2}{\ell}\left(\frac{1}{m^2}\right)^\ell$ for all $m \leq n \leq m^3$ and the equality holds if and only if $n = m^3$, which has been shown in [10, Proposition 4]. The following result gives an explicit formula for $\tilde{B}_\ell(m, n)$ for all $n \geq m$, except for $m^3 - m/2 < n < m^3$. This provides a partial solution to problem (P1).

Theorem 2.3 *Suppose $m \leq n \leq m^3 - m/2$. Then for $1 < \ell \leq m^2$,*

$$\tilde{B}_\ell(m, n) = f_\ell(\alpha, \beta, \ldots, \beta), \quad with \quad \alpha = \frac{1}{\sqrt{mn}} \quad and \quad \beta = \frac{1 - \alpha}{m^2 - 1}.$$

If $n \geq m^3$, then $\tilde{B}_\ell(m, n) = f_\ell(1/m^2, \ldots, 1/m^2) = \binom{m^2}{\ell}\left(\frac{1}{m^2}\right)^\ell$.

Theorem 2.3 gives the values of $\tilde{B}_\ell(m, n)$ for all $n \geq m$, except for $m^3 - m/2 < n < m^3$. In particular, it holds for all n which is divisible by m. In application, both n and m are powers of 2. Therefore, n is always divisible by m and $\tilde{B}_\ell(m, n)$ is given by the above theorem.

When $m = n$, following our proof of Theorem 2.3 in the next section, one actually gives explicit formulas for $B_\ell(n, n)$ and $\tilde{B}_\ell(n, n)$.

Theorem 2.4 *For any n and $1 \leq \ell \leq n^2$,*

$$B_\ell(n, n) = \tilde{B}_\ell(n, n) = f_\ell(\alpha, \beta, \ldots, \beta) \quad with \quad \alpha = \frac{1}{n} \quad and \quad \beta = \frac{n-1}{n(n^2 - 1)}.$$

Theorem 2.4 provides partial solutions to both problems (P1) and (P2). In particular, it gives a negative answer to problem (P2) for the case when $m = n$. As a result, if $m = n$, the upper bounds of the elementary symmetric functions of realignment matrices cannot be used to derive new conditions for detecting separability beyond the realignment criterion.

3. Proofs

Proof of Lemma 2.1. Define $\mathbf{x} = (x_1, \ldots, x_{m^2})^t$, $\mathbf{y} = (y_1, \ldots, y_{n^2})^t$ by

$$x_i = \begin{cases} 1 & \text{if } i = k(m+1) + 1 \text{ for some } 0 \leq k \leq m - 1, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$y_j = \begin{cases} 1 & \text{if } j = k(n+1) + 1 \text{ for some } 0 \leq k \leq n - 1, \\ 0 & \text{otherwise}. \end{cases}$$

Then $\frac{1}{\sqrt{m}} \mathbf{x}$ and $\frac{1}{\sqrt{n}} \mathbf{y}$ are unit vectors and

$$\frac{1}{\sqrt{m}} \mathbf{x}^t \rho^R \frac{1}{\sqrt{n}} \mathbf{y} = \frac{\text{trace}\, \rho}{\sqrt{mn}} = \frac{1}{\sqrt{mn}}.$$

Because

$$s_1 = \max\left\{ \mathbf{u}^t \rho^R \mathbf{v} : \mathbf{u} \in \mathbf{C}^{m^2} \text{ and } \mathbf{v} \in \mathbf{C}^{n^2} \text{ are unit vectors} \right\},$$

we conclude that $s_1 \geq \frac{1}{\sqrt{mn}}$. $\qquad\qquad\square$

Proof of Lemma 2.2. Note that

$$n \leq m^3 \quad \Longleftrightarrow \quad mn \leq m^4 \quad \Longleftrightarrow \quad \sqrt{mn} - 1 \leq m^2 - 1 \quad \Longleftrightarrow \quad \beta \leq \alpha.$$

Suppose $\mathbf{s} = (s_1, \ldots, s_{m^2}) \in \mathcal{S}(m,n)$ with $s = \sum_{i=1}^{m^2} s_i \leq 1$. Let $\tilde{\mathbf{s}} = (1/s)\,\mathbf{s}$. Then $\tilde{s}_1 \geq s_1 \geq \alpha$. Therefore, $(1/m^2, \ldots, 1/m^2) \prec (\alpha, \beta, \ldots, \beta) \prec \tilde{\mathbf{s}}$. Since f_ℓ is strictly concave [11], we have

$$f_\ell(\mathbf{s}) \leq f_\ell(\tilde{\mathbf{s}}) \leq f_\ell(\alpha, \beta, \ldots, \beta) \leq f_\ell(1/m^2, \ldots, 1/m^2) = \binom{m^2}{\ell}\left(\frac{1}{m^2}\right)^\ell,$$

and the equality $f_\ell(\mathbf{s}) = f_\ell(\alpha, \beta, \ldots, \beta)$ holds if and only if $\mathbf{s} = (\alpha, \beta, \ldots, \beta)$. This proves (a). Assertion (b) follows readily from (a). $\qquad\square$

Proof of Theorem 2.3. We first consider the simpler case when $n \geq m^3$. It suffices to construct $\rho \in \mathcal{D}(m,n)$ for which ρ^R has singular values $1/m^2, \ldots, 1/m^2$. Suppose $\{E_{1,1}, \ldots, E_{m,m}\}$ is the standard basis of $m \times m$ matrices. For $1 \leq k, \ell \leq m$, let $F_{k,\ell} = (E_{k,\ell} \otimes I_{m^2}) \oplus O_{n-m^3}$. Then $\rho = \dfrac{1}{m^3} \sum_{k,\ell=1}^{m} E_{k,\ell} \otimes F_{k,\ell}$ is an $mn \times mn$ density matrix while ρ^R has singular values $1/m^2, \ldots, 1/m^2$.

Next, suppose $m \leq n \leq m^3 - m/2$. By Lemma 2.2, we have $\tilde{B}_\ell(m,n) \leq f_\ell(\alpha, \beta, \ldots, \beta)$ for all $1 \leq \ell \leq m^2$. We will construct $\rho \in \mathcal{D}(m,n)$ for which ρ^R has singular values $\alpha, \beta, \ldots, \beta$. Suppose $n = mq + r$ with $0 \leq r < m$. For $1 \leq k, \ell \leq m$, let $F_{k,\ell} = (E_{k,\ell} \otimes I_q) \oplus O_r$. Define

$$\rho_1 = \sum_{k,\ell=1}^{m} E_{k,\ell} \otimes F_{k,\ell}, \quad \rho_2 = I_m \otimes (I_{mq} \oplus O_r), \quad \text{and} \quad \rho_3 = I_m \otimes (O_{mq} \oplus I_r)$$

and

$$\rho = s_1\rho_1 + s_2\rho_2 + s_3\rho_3 \quad \text{with} \quad s_1 = \frac{\beta}{\sqrt{q}}, \quad s_2 = \alpha^2 - \frac{\beta}{m\sqrt{q}}, \quad \text{and} \quad s_3 = \alpha^2.$$

Denote $J_{m,n}$ by the $m \times n$ matrix with all entries equal to one. Then the realigned matrix ρ^R is (under permutation of rows and columns) given by

$$A = \left[\begin{array}{cccc} \overbrace{s_1I_m + s_2J_{m,m} \mid \cdots \mid s_1I_m + s_2J_{m,m}}^{q-\text{terms}} & s_3J_{m,r} & O_{m,(m^2-m)q} & O \\[2ex] O & O & \underbrace{s_1I_{m^2-m} \mid \cdots \mid s_1I_{m^2-m}}_{q-\text{terms}} & O \end{array}\right].$$

Note that

$$AA^\dagger = \left(qs_1^2 I_m + (2qs_1s_2 + qms_2^2 + rs_3^2)J_{m,m}\right) \oplus qs_1^2 I_{m^2-m}.$$

Since $J_{m,m}$ has only one non-zero eigenvalue m, a matrix of the form $\mu I_m + \nu J_{m,m}$ has eigenvalues $\mu + m\nu$ and μ with multiplicity 1 and $m - 1$, respectively. As a result $AA^\dagger$ has one eigenvalue equal to

$$qs_1^2 + m(2qs_1s_2 + qms_2^2 + rs_3^2) = \alpha^4(m^2q + mr) = \alpha^2$$

and $m^2 - 1$ eigenvalues equal to

$$qs_1^2 = \beta^2.$$

Hence, taking square roots, we see that the matrix ρ^R has the desired singular values $\alpha, \beta, \ldots, \beta$.

It remains to show that ρ is a density matrix. Notice that

$$\mathrm{trace}\,(\rho) = s_1(mq) + s_2(m^2q) + s_3(mr) = \alpha^2 m(mq + r) = 1.$$

Since ρ_1, ρ_2, and ρ_3 are all positive semi-definite and both s_1 and s_3 are nonnegative, ρ is a density matrix if s_2 is nonnegative. Notice that

$$s_2 \geq 0 \iff \frac{1}{mn} \geq \frac{\sqrt{mn} - 1}{\sqrt{mn}\sqrt{qm(m^2 - 1)}} \iff m^2 - 1 \geq \sqrt{\frac{n^2}{q}} - \sqrt{\frac{n}{mq}}.$$

For a fixed m, let

$$f(q, r) = \sqrt{\frac{(mq + r)^2}{q}} - \sqrt{\frac{(mq + r)}{mq}} \quad \text{for} \quad q \geq 1 \text{ and } 0 \leq r \leq m - 1.$$

Then

$$\frac{\partial f}{\partial q} = \frac{mq - r}{2q^{3/2}} + \frac{r}{2q\sqrt{mq(mq + r)}} > 0 \quad \text{and} \quad \frac{\partial f}{\partial r} = \frac{1}{\sqrt{q}} - \frac{1}{2mq}\sqrt{\frac{mq}{mq + r}} > 0$$

for all $q \geq 1$ and $0 \leq r \leq m - 1$. Therefore,

(a) $f(q, r) \leq f(m^2 - 2, m - 1)$ for all $1 \leq q \leq m^2 - 2$ and $r \leq m - 1$; and

(b) $f(m^2 - 1, r) \leq f(m^2 - 1, m/2)$ for all $r \leq m/2$.

So, it suffices to prove that

$$(1) \quad f(m^2 - 2, m - 1) \leq m^2 - 1 \quad \text{and} \quad (2) \quad f(m^2 - 1, m/2) \leq m^2 - 1.$$

To prove (1), since $m \geq 2$, we have

$$m^4(m^2 - 2) - (m(m^2 - 2) + m - 1)^2 = 2m^4 + 2m^3 - m^2 - 2m - 1 > 0.$$

It follows that $\sqrt{\dfrac{(m(m^2-2)+m-1)^2}{m^2-2}} < m^2$ and hence

$$f(m^2-2, m-1) = \sqrt{\frac{(m(m^2-2)+m-1)^2}{m^2-2}} - \sqrt{\frac{(m(m^2-2)+m-1)}{m(m^2-2)}} \le m^2-1.$$

To prove (2), since $m^2 - 1 \le \left(m - \dfrac{1}{2m}\right)^2$, i.e., $\sqrt{m^2-1} \le \left(m - \dfrac{1}{2m}\right)$, we have

$$\sqrt{\frac{(m(m^2-1)+m/2)^2}{m^2-1}} = m\sqrt{m^2-1} + \frac{m}{2\sqrt{m^2-1}} \le m\left(m - \frac{1}{2m}\right) + \frac{m}{2\sqrt{m^2-1}},$$

and

$$\sqrt{\frac{(m(m^2-1)+m/2)}{m(m^2-1)}} = \frac{1}{2}\sqrt{1 + \frac{m^2}{m^2-1} + \frac{2m\left(m - \frac{1}{2m}\right)}{m^2-1}}$$

$$\ge \frac{1}{2}\sqrt{1 + \frac{m^2}{m^2-1} + \frac{2m}{\sqrt{m^2-1}}} = \frac{1}{2}\left(1 + \frac{m}{\sqrt{m^2-1}}\right).$$

Consequently,

$$f(m^2-1, m/2) = \sqrt{\frac{(m(m^2-1)+m/2)^2}{m^2-1}} - \sqrt{\frac{(m(m^2-1)+m/2)}{m(m^2-1)}}$$

$$\le m\left(m - \frac{1}{2m}\right) + \frac{m}{2\sqrt{m^2-1}} - \frac{1}{2}\left(1 + \frac{m}{\sqrt{m^2-1}}\right) = m^2-1. \qquad \square$$

Remark The smallest values of m, n which do not satisfy the conditions in Theorem 2.3 are $m = 3$ and $n = 26$. For these values, the proof in Theorem 2.3 does not work because $s_2 < 0$. In this case, the question about the exact value of $\tilde{B}_\ell(m, n)$ is still open.

Proof of Theorem 2.4. Suppose $m = n$. Then the matrix ρ constructed in the proof of Theorem 2.3 has the form

$$\rho = \frac{1}{n(n+1)}\left(I_{n^2} + xx^t\right)$$

where

$$x_i = \begin{cases} 1 & \text{if } i = k(n+1)+1 \text{ for some } 0 \le k \le n-1, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from [13] that ρ is separable. $\qquad \square$

4. Conclusion

The main goal of this paper is to investigate the open problems (P1) and (P2) proposed in [10] in the search for a new criterion for separability. We study the singular values of the realignment of density matrices and obtain new bounds on the elementary symmetric functions. The results are applied to find explicit formulas for $\tilde{B}_\ell(m,n)$, for all $n \geq m$, except $m^3 - m/2 < n < m^3$ and $B_\ell(n,n)$. This provides a partial answer to the open problem (P1). Furthermore, we show that $\tilde{B}_\ell(n,n) = B_\ell(n,n)$ for all n so that one cannot use $\tilde{B}_\ell(m,n)$ to differentiate separable matrices from density matrices whose realignment matrix has trace norm at most 1 when $m = n$. This gives a negative answer to problem (P2) when $m = n$. For $m \neq n$, numerical results in [10] suggested that $\tilde{B}_\ell(m,n) > B_\ell(m,n)$. If this strict inequality holds, then we would have a new criterion for separability. Our explicit formula for $\tilde{B}_\ell(m,n)$ will be useful in this study.

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