Linear Preservers of Finite Reflection Groups

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Abstract

Let G be the finite reflection groups $\mathbf{H}_3$, $\mathbf{H}_4$, $\mathbf{F}_4$, $\mathbf{E}_8$, $\mathbf{E}_7$ or $\mathbf{E}_6$, acting irreducibly on the Euclidean space V. We show that there exists an overgroup $\tilde{G}$ of G such that a linear operator $\phi : \mathrm{End}(V) \to \mathrm{End}(V)$ satisfies $\phi(G) = G$ if and only if ϕ has the form $X \mapsto PXQ$ or $X \mapsto PX^tQ$ for some $P, Q \in \tilde{G}$. From our result, we can show that $\tilde{G}$ is actually the normalizer of G in the group of orthogonal operators on V. Moreover, $\tilde{G} = G$ except when $G = \mathbf{F}_4$.

1 Introduction

Let V be a Euclidean space and let $\mathrm{End}(V)$ be the algebra of linear endomorphisms on V. An operator $T \in \mathrm{End}(V)$ is a *reflection* if there exists a unit vector $u \in V$ such that $T(v) = v - 2(v, u)u$ for all $v \in V$. A group G of invertible operators in $\mathrm{End}(V)$ is a *reflection group* if it is generated by a set of reflections. The study of reflection groups has motivations and applications in many areas, and the theory is quite well developed; see [1, 2].

Recently, there has been considerable interest in characterizing those linear operators $\phi : \mathrm{End}(V) \to \mathrm{End}(V)$ such that

$$\phi(G) = G. \tag{1}$$

For $G = O(V)$, the group of orthogonal operators on V, Wei [7] showed that a linear operator $\phi : \mathrm{End}(V) \to \mathrm{End}(V)$ satisfying (1) if and only if there exist $P, Q \in G$ such that ϕ has the form

$$X \mapsto PXQ \quad \text{or} \quad X \mapsto PX^*Q, \tag{2}$$

where X^* is the adjoint operator of X acting on V so that $(Xu, v) = (u, X^*v)$ for all $u, v \in V$. In [5], it was shown that the same result holds for $G = \mathbf{A}_n$. In [4], the authors reproved this result using a different approach, and consider the problems for the cases when $G = \mathbf{B}_n, \mathbf{D}_n$ and $\mathbf{I}_2(n)$. For $\mathbf{D}_n$ and $\mathbf{I}_2(n)$, $\phi : \mathrm{End}(V) \to \mathrm{End}(V)$ is a linear operator satisfying (1) if and only if there exist P and Q in the normalizer $N(G)$ of G in $O(V)$ such that ϕ has the form (2). The same statement is true for $G = O(V)$ and $\mathbf{A}_n$ because $N(G) = G$ in these cases. However, the situation for $G = \mathbf{B}_n$ is different. Suppose $G = \mathbf{B}_n$ is viewed as the group of $n \times n$ signed permutation matrices, i.e., product of diagonal orthogonal matrices and permutation matrices, acting on $V = \mathbb{R}^n$, and $\mathrm{End}(V)$ is identified with the set $M_n(\mathbb{R})$ of $n \times n$ real matrices. Then a linear operator $\phi : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ satisfies (1) if and only if there exist $P, Q \in G$ and $R = (r_{ij}) \in M_n(\mathbb{R})$ with $r_{ij} \in \{1, -1\}$ such that ϕ has the form

$$X \mapsto R \circ (PXQ) \qquad \text{or} \qquad X \mapsto R \circ (PX^tQ),$$

where $Y \circ Z$ denotes the Schur (entry-wise) product of two matrices $Y, Z \in M_n(\mathbb{R})$.

In this paper, we consider the problem for the remaining cases, namely, $G = \mathbf{H}_3$, $\mathbf{H}_4$, $\mathbf{F}_4$, $\mathbf{E}_8$, $\mathbf{E}_7$ and $\mathbf{E}_6$, and confirm that a linear operator $\phi : \mathrm{End}(V) \to \mathrm{End}(V)$ satisfies (1) if and only if there exist $P, Q \in N(G)$ such that (2) holds.

One may also study the more difficult problem of characterizing linear operators $\phi : \mathrm{End}(V) \to \mathrm{End}(V)$ such that $\phi(G) \subseteq G$. When $G = O(n)$, such a linear map has the usual form (2) except when $n = 2, 4, 8$, and there are singular maps ϕ satisfying (1) in these cases; see [7] for details. Furthermore, one may consider other subsets $\mathcal{S}$ of $\mathrm{End}(V)$ related to G and linear maps $\phi : \mathrm{End}(V) \to \mathrm{End}(V)$ such that $\phi(\mathcal{S}) = \mathcal{S}$ and $\phi(\mathcal{S}) \subseteq \mathcal{S}$; see [3, 4, 5]). All of these can be viewed as studies of linear preserver problems related to groups and algebraic sets; see [6, Chapter 4].

Our paper is organized as follows. We present some preliminary results and describe some basic strategies of our proofs in the next section. In Sections 3 – 8, we prove our preserver results for $G = \mathbf{H}_3$, $\mathbf{H}_4$, $\mathbf{F}_4$, $\mathbf{E}_8$, $\mathbf{E}_7$, and $\mathbf{E}_6$, respectively. In each of these sections, we describe a natural matrix realization of G, and possible inner products (X, Y) for elements $X, Y \in G$. These results are then used to solve the corresponding preserver problem. For $G = \mathbf{E}_7$ and $\mathbf{E}_6$, we work on their 8×8 matrix realizations (as subgroups of $\mathbf{E}_8$). Some matlab programs used in our proofs are included in Section 9.

In our discussion, denote by $\{e_1, \dots, e_n\}$ the standard basis for $\mathbb{R}^n$, $e = \sum_{j=1}^n e_j$, and $E_{ij} = e_i e_j^t \in M_n(\mathbb{R})$. If V is equal to (or identified with) $\mathbb{R}^n$, then $\mathrm{End}(V)$ is equal to (or identified with) $M_n(\mathbb{R})$, which is also a Euclidean space with inner product defined by $(X, Y) = \mathrm{tr}\,(XY^t)$.

It is worth noting that even though the general strategies of our proofs can be easily described, see Section 2, it requires a lot of effort and technical details to prove our results. It would be nice if there are shorter conceptual proofs for our results.

2 Preliminary Results

Denote by $O(\mathrm{End}(V))$ the group of orthogonal operators on $\mathrm{End}(V)$ preserving the inner product. We have the following result; see [4, Corollaries 2.2].

Proposition 2.1 *Let G be a finite reflection group acting irreducibly on V. The collection of linear maps $\phi : \mathrm{End}(V) \to \mathrm{End}(V)$ satisfying $\phi(G) = G$ form a subgroup of $O(\mathrm{End}(V))$.*

General Procedures and Strategies

We briefly describe some general procedures and strategies in our proofs in the next few Sections.

GP1. To find a matrix realization of the given reflection group G, we use the standard root systems in $\mathbb{R}^n$ described in [1, p.76] to construct some basic reflections $I_n - 2xx^t$, and their products until we get all the elements in G. Very often, we partition the group G into different subsets to facilitate future discussion.

GP2. Using the matrix realization in GP1, we determine some possible inner products $r = (X, Y)$ for elements $X, Y \in G$. For each r, we define

$$\mathcal{S}_r = \{X \in G : (I, X) = r\}$$

which is used in the proof of the linear preserver result.

GP3. To characterize $\phi : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ such that $\phi(G) = G$, we can always assume that $\phi(I_n) = I_n$. Otherwise, we can replace ϕ by a mapping of the form

$$X \mapsto \phi(I_n)^{-1}\phi(X).$$

By Proposition 2.1, we see that $\phi(\mathcal{S}_r) = \mathcal{S}_r$, where $\mathcal{S}_r$ is defined as in GP2. Then we show that there is an overgroup $\tilde{G}$ of G so that one can strategically modify ϕ by a finite sequence of mappings of the form

$$X \mapsto P^t\phi(X)P \qquad \text{or} \qquad X \mapsto P^t\phi(X)^tP \tag{3}$$

by $P \in \tilde{G}$ so that the resulting map is the identity map on $M_n(\mathbb{R})$. It will then follow that the original ϕ has the desired form.

GP4. Using our results, one can show that the group $\tilde{G}$ in GP3 is $N(G)$, the normalizer of G in $O(V)$, as follows. By our linear preserver result, if ϕ satisfies $\phi(I_n) = I_n$ and $\phi(G) = G$ then ϕ has the form

$$X \mapsto P^tXP \qquad \text{or} \qquad X \mapsto P^tX^tP,$$

for some P in a certain group $\tilde{G}$. Since the mapping $X \mapsto P^tXP$ sends G onto itself for any $P \in \tilde{G}$, we see that $\tilde{G} \le N(G)$. Now, if $Q \in N(G)$ then the mapping ϕ defined by $X \mapsto Q^tXQ$ satisfies $\phi(G) = G$. By our linear preserver result, there exists $P \in \tilde{G}$ such that

$$P^tXP = Q^tXQ \text{ for all } X \in G \qquad \text{or} \qquad P^tX^tP = Q^tXQ \text{ for all } X \in G.$$

If the latter case holds, then $XPQ^tX = PQ^t$ for all $X \in G$, which is impossible; if the former case holds, then one readily shows that $P = Q$. Thus, we get the reverse inclusion $N(G) \le \tilde{G}$.

GP5. To study $\mathbf{E}_8, \mathbf{E}_7$ and $\mathbf{E}_6$, we first use strategies GP1 - GP4 to handle $\mathbf{E}_8 \subseteq M_8(\mathbb{R})$. Then we identify $\mathbf{E}_7 \subseteq M_7(\mathbb{R})$ as a subgroup $\mathcal{E}_7$ of $\mathbf{E}_8 \subseteq M_8(\mathbb{R})$ by the mapping

$$A \mapsto U^t \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} U \in \mathbf{E}_8$$

for some suitable orthogonal matrix $U \in M_8(\mathbb{R})$. To characterize a linear map $\psi : M_7(\mathbb{R}) \to M_7(\mathbb{R})$ such that $\psi(\mathbf{E}_7) = \mathbf{E}_7$, we consider an affine map ϕ induced by ψ on the affine space generated by $\mathcal{E}_7$. We use a similar idea to investigate $\mathbf{E}_6$.

3 $\mathbf{H}_3$

3.1 Matrix Realization

The group $\mathbf{H}_3$ has $2^3 \cdot 3 \cdot 5 = 120$ elements; see [1, p.80]. Using the standard root systems (see [1, p.76]) of $\mathbf{H}_3$ in $\mathbb{R}^3$, we see that $\mathbf{H}_3$ admits a matrix realization in $M_3(\mathbb{R})$ consisting of the following matrices:

(I) 24 matrices of the form PD, where $P \in M_3(\mathbb{R})$ is an even permutation (so P is either the identity or a length 3 cycle) and $D \in M_3(\mathbb{R})$ is a diagonal orthogonal matrix.

(II) 12 matrices of the form PHP^t, where P is a matrix of type (I) and

$$H = I_3 - 2(-b, c, a)^t(-b, c, a) = \begin{pmatrix} a & b & c \\ b & c & -a \\ c & -a & -b \end{pmatrix} \tag{4}$$

with $a = (1+\sqrt{5})/4$, $b = (-1+\sqrt{5})/4$, $c = 1/2$. Note that the diagonals of these matrices have the form $(a, c, -b)$, $(-b, a, c)$ or $(c, -b, a)$, and the sum of the diagonal entries is always one.

(III) 84 matrices of the form QD where Q is a type (II) matrix and D is a diagonal orthogonal matrix not equal to I_3. In fact, each of these seven diagonal matrices D generates a class of twelve matrices, and we get seven different classes. Note that the absolute values of the diagonals are (a, c, b), (b, a, c) or (c, b, a).

3.2 Inner product

Since $(X, Y) = (I_3, X^tY)$ for any $X, Y \in \mathbf{H}_3$, we focus on the possible values of (I_3, X) with $X \in \mathbf{H}_3$. If $X \in \mathbf{H}_3$ is type (I), then $(I_3, X) \in \{0, \pm 1, \pm 3\}$; $X \in \mathbf{H}_3$ is of type (II), then $(I_3, X) = 1$; if X is type (III), then $(I_3, X) \in \{0, -1, \pm\sqrt{5}/2, \pm(1+\sqrt{5})/2\}$. Thus, if $X \in \mathbf{H}_3$, then

$$(I, X) \in \{0, \pm 1, \pm\sqrt{5}/2, \pm(1+\sqrt{5})/2, \pm 3\}.$$

By GP2 in Section 2, for each r in the above set, define

$$\mathcal{S}_r = \{X \in \mathbf{H}_3 : (I_3, X) = r\}. \tag{5}$$

For example, let $r = 1$. If $X \in \mathcal{S}_1$, then X must of one of the following two forms.

(a) The 3 matrices of type (I), namely,

$$D_1 = \mathrm{diag}(-1, 1, 1),\ D_2 = \mathrm{diag}(1, -1, 1),\ D_3 = \mathrm{diag}(1, 1, -1), \tag{6}$$

(b) The 12 type (II) matrices.

3.3 Linear Preservers

Theorem 3.1 *A linear operator* $\phi : M_3(\mathbb{R}) \to M_3(\mathbb{R})$ *satisfies* $\phi(\mathbf{H}_3) = \mathbf{H}_3$ *if and only if there exist* $P, Q \in \mathbf{H}_3$ *such that* ϕ *has the form*

$$X \mapsto PXQ \quad or \quad X \mapsto PX^tQ.$$

Consequently, $N(\mathbf{H}_3) = \mathbf{H}_3$.

Proof. The assertion on $N(\mathbf{H}_3)$ follows from GP4 in Section 2. The ($\Leftarrow$) part of the first assertion is clear. We consider the ($\Rightarrow$) part. Define $\mathcal{S}_r$ as in (5). By Proposition 2.1, if ϕ preserves $\mathbf{H}_3$, then ϕ preserves the inner product $(X, Y) = \operatorname{tr}(XY^t)$. By GP3 in Section 2, we may assume that $\phi(I_3) = I_3$ and $\phi(\mathcal{S}_r) = \mathcal{S}_r$ for each r. In the following, we will show that ϕ has the form $X \mapsto P^tXP$ or $X \mapsto P^tX^tP$ for some $P \in \mathbf{H}_3$. We shall use the matrices D_1, D_2, D_3 and H defined in §3.1 – 3.2.

First, consider $\phi(D_j) = Y_j$ for some $j = 1, 2, 3$. Then $Y_j \in \mathcal{S}_1$. Since $\phi(I_3) = I_3$ and $(D_1 + D_2 + D_3)/2 = I$, we have $(Y_1 + Y_2 + Y_3)/2 = I$. We consider 2 cases depending on whether $\phi(D_1)$ is a type (a) or type (b) matrix defined in §3.2.

Case 1. Suppose Y_1 is a type (a) matrix. Then $(Y_1 + Y_2 + Y_3)/2 = I$ implies that all Y_j are type (a) matrices. We can assume that $Y_1 = D_1$; otherwise, replace ϕ by a mapping of the form $X \mapsto Q\phi(X)Q^t$ for a suitable even permutation matrix Q. Then

$$\{\phi(D_2), \phi(D_3)\} = \{D_2, D_3\}.$$

We will show that $\phi(D_i) = D_i$ for $i = 2, 3$.

Suppose that $\phi(X) = Y$ for some $X, Y \in \mathbf{H}_3$. Since ϕ fixes I_3 and D_1, $(X, I_3) = (Y, I_3)$ and $(X, D_1) = (Y, D_1)$. It follows that $\operatorname{tr}(X) = \operatorname{tr}(Y)$ and

$$2(X, E_{11}) = (X, I_3 - D_1) = (Y, I_3 - D_1) = 2(Y, E_{11}).$$

Now, consider

$$\mathcal{T} = \{X \in \mathcal{S}_1 : (I_3, X) = 1,\ (X, E_{11}) = a\} = \{H\} \cup \{D_iHD_i : i = 1, 2, 3\}$$

where H is the matrix in (4). Then $\phi(\mathcal{T}) = \mathcal{T}$, and thus $\phi(H) \in \mathcal{T}$. We may assume that $\phi(H) = H$, otherwise replace ϕ with $X \mapsto D_i\phi(X)D_i$. Since

$$(H, \phi(D_2)) = (\phi(H), \phi(D_2)) = (H, D_2) \neq (H, D_3),$$

it follows that $\phi(D_2) = D_2$, and thus $\phi(D_3) = D_3$. So, we have shown that the modified mapping ϕ fixes X for $X = I_3, D_1, D_2, D_3, H$.

Since $\phi(D_i) = D_i$ and ϕ preserves inner product, we see that $(D_i, X) = (D_i, \phi(X))$ for all $i = 1, 2, 3$. Thus $\phi(X)$ and X have the same diagonal. Consider the four matrices with diagonal $(-a, c, -b)$, namely,

$$X_1 = D_1H, \quad X_2 = HD_1 = X_1^t, \quad X_3 = -D_2HD_3, \quad X_4 = -D_3HD_2 = X_3^t.$$

Then $\phi(X_1) = X_j$ for some $j \in \{1,2,3,4\}$. Since

$$(\phi(X_1), H) = (\phi(X_1), \phi(H)) = (X_1, H) = (X_2, H) \neq (X_3, H) = (X_4, H),$$

we see that $\phi(X_1) \in \{X_1, X_2\}$. We may assume that $\phi(X_1) = X_1$; otherwise, replace ϕ with the mapping $X \mapsto \phi(X)^t$. Then $\phi(X_2) = X_2$. Furthermore, we have $\phi(X_3) \in \{X_3, X_4\}$. Since

$$(X_1, \phi(X_3)) = (\phi(X_1), \phi(X_3)) = (X_1, X_3) \neq (X_1, X_4),$$

we see that $\phi(X_3) = X_3$. As a result, we have $\phi(X_i) = X_i$ for $i = 1,2,3,4$.

Next, consider the four matrices with diagonal $(a, -c, -b)$, namely,

$$X_5 = D_2 H, \quad X_6 = H D_2, \quad X_7 = -D_1 H D_3, \quad X_8 = -D_3 H D_1.$$

Since $(H - X_1, X_i) \neq (H - X_1, X_j)$ for $5 \leq i < j \leq 8$, we have $\phi(X_i) = X_i$ for $i = 5,6,7,8$.

Now, we have $\phi(X) = X$ for $X \in \{D_1, D_2, D_3, H, X_1, \ldots, X_8\}$, which is a spanning set of $M_3(\mathbb{R})$; for example, it can be checked using MATLAB as shown in the last section. Thus $\phi(X) = X$ for all $X \in M_3(\mathbb{R})$.

Case 2. If Y_1 is a type (b) matrix, then we may replace ϕ by a mapping of the form $X \mapsto P\phi(X)P^t$ for a type (I) matrix P and assume that $Y_1 = H$. Then replace ϕ by the mapping $X \mapsto HQ^t D_1 \phi(X) D_1 Q H$ with $Q = E_{12} + E_{23} + E_{31}$; we see that $\phi(D_1) = D_1$, and we are back to case 1. □

4 $\mathbf{H}_4$

4.1 Matrix Realization

Note that $\mathbf{H}_4$ has $2^6 \cdot 3^2 \cdot 5^2$ elements; see [1, p.80]. Let

$$a = (1+\sqrt{5})/4, \quad b = (-1+\sqrt{5})/4, \quad c = 1/2.$$

Using GP1 in Section 2 and the standard root systems of $\mathbf{H}_4$ in $\mathbb{R}^4$ (see [1, p.76]), we see that $\mathbf{H}_4$ contains the following two matrices:

$$A = I - ee^t/2 = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \tag{7}$$

and

$$B = I_4 - 2(0,-b,c,a)^t(0,-b,c,a) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & c \\ 0 & b & c & -a \\ 0 & c & -a & -b \end{pmatrix}. \tag{8}$$

Using these two matrices, we can describe the matrices in $M_4(\mathbb{R})$ as follows.

(I) $4!2^3 = 2^6 3$ matrices of the form PD, where P is an even permutation, and D is a diagonal orthogonal matrix. Note that $(I_4, P) \in \{0, \pm 1, \pm 4\}$.

(II) $2^7 3$ matrices of the form PAQ, where A is the matrix in (7), and P, Q are matrices of type (I). Note that

$$(I_4, PAQ) \in \{0, \pm 1, \pm 2\}.$$

(III) $2^{10}3$ matrices of the form PBQ, where B is the matrix in (8), and P, Q are matrices of type (I). The counting is done as follows. For a matrix of the form PBQ, each P and Q has $2^4 12$ choices. However, $PBQ = RBS$ if and only if $R^t PBQS^t = B$. So, we have to count pairs of (X, Y) such that $XBY = B$. One can check that $XBY = B$ if and only if $X = [r] \oplus U$ and $Y = [r] \oplus V$, where $r = \pm 1$ and (U, V) is one of the following pairs:

$$\pm(I_3, I_3),\ \pm(-E_{13} + E_{21} + E_{32}, E_{13} + E_{21} - E_{32}),$$

$$\pm(E_{12} + E_{23} - E_{31}, E_{12} + E_{23} - E_{31}).$$

So, there are 12 pairs of matrices (X, Y), and the total number of type (III) matrices is $(2^4 12)^2/12 = 2^{10}3$. Note that

$$(I_4, PBQ) \in \{0, \pm 1, \pm 2, (\pm 1 \pm \sqrt{5})/2, (\pm 3 \pm \sqrt{5})/2\}.$$

(IV) $2^{11}3$ matrices of the form PCQ, where P, Q are matrices of type (I), and

$$C = B \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} B = \begin{pmatrix} 0 & a & b & c \\ a & b & 0 & -c \\ b & 0 & -a & c \\ c & -c & c & c \end{pmatrix}.$$

The counting is done as follows. For a matrix of the form PCQ, each P and Q has $2^4 12$ choices. However, $PCQ = RCS$ if and only if $R^t PCQS^t = C$. So, we have to count pairs of (X, Y) such that $XCY = C$. One can check that $XCY = C$ if and only if $X = Y = \pm(U \oplus [1])$ with $U = I_3$, $E_{13} - E_{21} - E_{32}$, or $-E_{12} - E_{23} + E_{31}$. So, there are 6 pairs of such matrices (X, Y), and the total number of class (IV) matrices is $(2^4 12)^2/6 = 2^{11}3$. Note that

$$(I_4, PCQ) \in \{0, \pm 1, (\pm 1 \pm \sqrt{5})/2, (\pm 3 \pm \sqrt{5})/2\}.$$

(V) $3 \cdot 2^6 \cdot 24$ matrices of the form PEQ, where P and Q are type (III) matrices and

$$\begin{aligned} E &= (E_{12} + E_{21} + E_{34} - E_{43})B(E_{14} + E_{23} - E_{32} + E_{41})B \\ &= \begin{pmatrix} c & 0 & b & -a \\ 0 & c & -a & -b \\ -b & a & c & 0 \\ a & b & 0 & c \end{pmatrix}. \end{aligned} \tag{9}$$

The counting is done as follows. For a matrix of the form PEQ, each P and Q has $2^4 12$ choices. However, $PEQ = RES$ if and only if $R^t PEQS^t = E$. So, we have to count pairs of (X, Y) such that $XEY = E$. One can check that $XEY = E$ if and only if $X = Y^t$ is the plus or minus of one of the following:

$$I_4, E_{12} - E_{21} + E_{34} - E_{43}, E_{13} + E_{24} - E_{31} - E_{42}, E_{14} - E_{23} + E_{32} - E_{41}.$$

So, there are 8 pairs of such matrices (X, Y), and the total number of class (V) matrices is $(2^4 12)^2/8 = 3 \cdot 2^6 \cdot 24$. One checks that

$$(I_4, PEQ) \in \{0, \pm 1, \pm 2, (\pm 1 \pm \sqrt{5})/2, \pm 1 \pm \sqrt{5}\}.$$

4.2 Inner product

By the discussion in the last subsection, if $X \in \mathbf{H}_4$, then

$$(I_4, X) \in \{0, \pm 1, \pm 2, (\pm 1 \pm \sqrt{5})/2, (\pm 3 \pm \sqrt{5})/2, \pm 1 \pm \sqrt{5}, \pm 4\}.$$

By GP2 in Section 2, for each r in the above set, define

$$\mathcal{S}_r = \{Y \in \mathbf{H}_4 : (I_4, Y) = r\}. \tag{10}$$

Then $\mathcal{S}_2$ consists of matrices of the following forms.

(a) The 4 diagonal matrices, namely

$$D_i = I_4 - 2E_{ii}, \qquad i = 1, 2, 3, 4. \tag{11}$$

(b) The 24 matrices of the form DA_iD for $i = 1, 2, 3$ where $D = \text{diag}(1, \pm 1, \pm 1, \pm 1)$, $A_1 = A$ defined in (7),

$$A_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad A_3 = A_2^t.$$

(c) The 48 type (III) matrices with diagonal entries $1, a, -b, c$ in a certain order. Note that these must be of the form PBP^t where P is of type (I). To see this, note that if P and Q are type (I) matrices such that PBQ has diagonal entries $1, a, -b, c$, then removing the row and column containing the entry 1, we get a type (II) matrix of Section 3.1. Hence, we see that $Q = P^t$.

(d) The 24 type (V) matrices of the form PEP^t where P is a type (I) matrix. This conjugation will leave the diagonal entries (namely c, c, c, c) on the diagonal.

4.3 Linear Preservers

Theorem 4.1 *A linear operator $\phi : M_4(\mathbb{R}) \to M_4(\mathbb{R})$ satisfies $\phi(\mathbf{H}_4) = \mathbf{H}_4$ if and only if there exist $P, Q \in \mathbf{H}_4$ such that ϕ has the form*

$$X \mapsto PXQ \quad \text{or} \quad X \mapsto PX^tQ.$$

Consequently, $N(\mathbf{H}_4) = \mathbf{H}_4$.

Proof. The assertion on $N(\mathbf{H}_4)$ follows from GP4 in Section 2. The ($\Leftarrow$) part of the first assertion is clear. We consider the ($\Rightarrow$) part. Define $\mathcal{S}_r$ as in (10). By Proposition 2.1, if ϕ preserves $\mathbf{H}_4$, then ϕ preserves the inner product $(X, Y) = \operatorname{tr}(XY^t)$. By GP3 in Section 2, we may assume that $\phi(I_4) = I_4$ and $\phi(\mathcal{S}_r) = \mathcal{S}_r$ for each r. In the following, we will show that ϕ has the form $X \mapsto P^tXP$ or $X \mapsto P^tX^tP$ for some $P \in \mathbf{H}_4$. We shall frequently use the matrices $D_1, D_2, D_3, D_4, A_1, A_2, A_3, B$ and E as defined in §4.1 – 4.2 as well as the classification of elements of $\mathcal{S}_2$ as types (a), (b), (c) and (d) as defined in §4.2. Furthermore, denote by $D_{ij} = D_iD_j$.

For E defined as in (9), since $D_{12}ED_{12} = E^t$ and $E + E^t = I_4$, the elements of (d) can be paired up such that $X + X^t = I_4$, where both X and X^t are in (d). Also, the same applies for those matrices in (b) of the form DA_iD, where $i = 2, 3$. (For example, $A_2 + A_3 = I_4$). Now consider $\phi(D_i) = Y_i$. Note that since there exists no $X \in \mathcal{S}_2$ such that $D_i + X = I$; thus, Y_i must be of type (a), (c), or type (b) of the form DA_1D. We consider three cases according to these.

Case 1. Suppose that $Y_1 = D_j$. Then replace ϕ with a mapping of the form $X \mapsto P\phi(X)P^t$ where P is an even permutation such that $\phi(D_1) = D_1$. Note that

$$2(\phi(X), E_{11}) = (\phi(X), I_4 - D_1) = (X, I_4 - D_1) = 2(X, E_{11}).$$

Let $D_1 + X_2 + X_3 + X_4 = 2I_4$ where $X_i \in \mathcal{S}_2$. Since $(X_2 + X_3 + X_4, E_{11}) = 3$, and since 1 is the largest possible value for (X_i, E_{11}), it follows that each X_i is either of type (a) or one of the 12 type (c) with the $(1,1)$ entry equal to one. Thus, either $\phi(D_2) = D_j$ or $\phi(D_2) = Z_2$ where Z_2 is one of the 12 type (c) matrices. If the first case happens, we may assume that $\phi(D_2) = D_2$; otherwise, replace ϕ by a mapping of the form $X \mapsto P\phi(X)P^t$ for a suitable even permutation matrix P such that $(P, E_{11}) = 1$. If the second case happens, then there exists a signed even permutation matrix Q with $(Q, E_{11}) = 1$ such that $Q^t\phi(D_1)Q = B$. Now, replace ϕ by a mapping of the form $X \mapsto BPD_2Q^t\phi(X)QD_2P^tB$ with $P = E_{11} + E_{24} + E_{32} + E_{43}$. Then the resulting map fixes I_4, D_1, D_2.

Recall that $\phi(D_i) = Y_i$ for $i = 1, \ldots, 4$, and $Y_1 + Y_2 + Y_3 + Y_4 = 3I_4$. Thus,

$$\{\phi(D_3), \phi(D_4)\} = \{D_3, D_4\}.$$

Moreover, for $i \in \{1, 2\}$, $(X, E_{ii}) = (\phi(X), \phi(E_{ii})) = (\phi(X), E_{ii})$. Consider the set

$$\mathcal{T} = \{X \in \mathcal{S}_2 : (X, E_{11}) = 1,\ (X, E_{22}) = a\} = \{D_iBD_i : i = 1, 2, 3, 4\}.$$

(Note that $B = D_1BD_1$). Then $\phi(\mathcal{T}) = \mathcal{T}$. If $\phi(B) = D_iBD_i$, then replace ϕ with $X \mapsto D_i\phi(X)D_i$. So, we may assume ϕ fixes B. Now, since $(D_3, B) \neq (D_4, B)$, we have $\phi(D_3) = D_3$ and $\phi(D_4) = D_4$. It then follow that

$$(X, E_{ii}) = (\phi(X), E_{ii}), \qquad i = 1, ..., 4.$$

Since $(B, D_iBD_i) \neq (B, D_jBD_j)$ for $i \neq j$, we have

$$\phi(D_iBD_i) = D_iBD_i, \qquad i = 2, 3, 4.$$

Let $P = E_{12} + E_{23} + E_{31} + E_{44}$, and consider the matrices

$$B_1 = B, \quad B_2 = PBP^t, \quad B_3 = P^tBP.$$

Consider those matrices in $\mathcal{S}_2$ with diagonal $(a, c, 1, -b)$, namely, $D_iB_2D_i$ for $i = 1, 2, 3, 4$ (note that $B_2 = D_3B_2D_3$). Then

$$(B, B_2) = (B, D_1B_2D_1) \neq (B, D_2B_2D_2) = (B, D_4B_2D_4).$$

We may assume that $\phi(B_2) = B_2$. Otherwise, $\phi(B_2) = D_1B_2D_1$ and replace ϕ with $X \mapsto D_1\phi(X)D_1$. Now $\phi(D_2B_2D_2) = D_iB_2D_i$ for either $i = 2$ or $i = 4$. Since $(B_2, D_2B_2D_2) \neq (B_2, D_4B_2D_4)$, we have

$$\phi(D_iB_2D_i) = D_iB_2D_i, \qquad i = 1, 2, 3, 4.$$

The matrices with diagonal $(c, 1, a, -b)$ are $D_iB_3D_i$ for $i = 1, 2, 3, 4$ (note that $B_3 = D_2B_3D_2$). We have

$$(B, B_3) = (B, D_1B_3D_1) \neq (B, D_3B_3D_3) = (B, D_4B_3D_4).$$

But $(B_2, B_3) = (B_2, D_3B_3D_3) \neq (B_2, D_1B_3D_1) = (B_2, D_4B_3D_4)$. Therefore,

$$\phi(D_iB_3D_i) = D_iB_3D_i, \qquad i = 1, 2, 3, 4.$$

Next, consider

$$B_4 = D_3B, \quad B_5 = D_2B_2, \quad \text{and} \quad B_6 = D_1B_3.$$

Their diagonals are $(1, a, -c, -b)$, $(a, -c, 1, -b)$ and $(-c, 1, a, -b)$ respectively. Note that for each $i = 4, 5, 6$, that $D_jB_iD_j$ share diagonal entries, where $j = 1, 2, 3, 4$. Since the triples $((B, D_jB_iD_j)\ (B_2, D_jB_iD_j)\ (B_3, D_jB_iD_j))$ are different for different i, j, one can see that each of these 12 matrices must be mapped to themselves. Thus $\phi(X) = X$ where $X = D_jB_iD_j$ for $j = 1, 2, 3, 4$ and $i = 1, ..., 6$. One readily checks that these 24 matrices span $M_4(\mathrm{IR})$; see the last section. So ϕ fixes every matrix in $M_4(\mathrm{IR})$.

Case 2. Suppose that Y_1 has the form PBP^t. Then replace ϕ by the mapping of the form $X \mapsto P\phi(X)P^t$. Thus $Y_1 = B$. Then replace ϕ by the mapping $X \mapsto BP^t\phi(X)PB$ where $P = E_{11} + E_{23} + E_{34} + E_{42}$. Thus $\phi(D_1) = D_1$, and we are back to case 1.

Case 3. Suppose that Y_1 has the form DA_1D. Then replace ϕ by the mapping of the form $X \mapsto D\phi(X)D$ where D is such that $\phi(D_1) = A_1$. Then replace ϕ with the mapping of the form $X \mapsto A_1D_1\phi(X)D_1A_1$. Thus $\phi(D_1) = D_1$, and we are back to case 1. □

5 $\mathbf{F}_4$

5.1 Matrix Realization

The group $\mathbf{F}_4$ has $4!2^4 3$ elements (see [1, p.80]) and is generated by $\mathbf{B}_4$ and the matrix

$$A = I - ee^t/2 = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}. \tag{12}$$

Let $\tilde{G}$ be the group in $O(4)$ generated by $\mathbf{B}_4$ and the matrix

$$B = \frac{1}{\sqrt{2}}\left\{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right\}. \tag{13}$$

Then $\tilde{G}$ has $4!2^5 3$ elements. Our result will show that $\tilde{G} = N(\mathbf{F}_4)$ as discussed in GP4 in Section 2.

5.2 Inner product

By the discussion in the last subsection, if $X \in \mathbf{F}_4$, then

$$(I_4, X) \in \{0, \pm 1, \pm 2, \pm 4\}.$$

By GP2 in Section 2, for each r in the above set, define

$$\mathcal{S}_r = \{Y \in \mathbf{F}_4 : (I_4, Y) = r\}. \tag{14}$$

The set $\mathcal{S}_2$ consists of matrices of the following forms.
(I) There are 4 diagonal matrices, namely, $D_i = I_4 - 2E_{ii}$, $i = 1, \ldots, 4$.
(II) There are 24 matrices of the form $DP_{ij} \in \mathbf{B}_4$, where P_{ij} is the matrix obtained from I_4 by interchanging the ith and jth rows for $1 \le i < j \le 4$, and D is a diagonal orthogonal matrix such that $\operatorname{tr}(DP_{ij}) = 2$. For example,

$$P_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(III) There are 48 matrices of the form $DA_1D, \ldots, DA_6D$, where

$$A_1 = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \; A_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

$$A_3 = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \; A_4 = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

$$A_5 = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}, \; A_6 = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix},$$

and

$$D \in \{\mathrm{diag}(1, \delta_1, \delta_2, \delta_3) : \delta_1, \delta_2, \delta_3 \in \{1, -1\}\}.$$

5.3 Linear Preservers

Theorem 5.1 *A linear operator $\phi : M_4(\mathbb{R}) \to M_4(\mathbb{R})$ on $M_4(\mathbb{R})$ satisfies $\phi(\mathbf{F}_4) = \mathbf{F}_4$ if and only if there exist P, Q in the group $\tilde{G}$ generated by $\mathbf{F}_4$ and B defined in* (13) *with $PQ \in \mathbf{F}_4$ such that ϕ has the form*

$$X \mapsto PXQ \quad or \quad X \mapsto PX^tQ.$$

Consequently, $N(\mathbf{F}_4) = \tilde{G}$.

Proof. The assertion on $N(\mathbf{F}_4)$ follows from the GP4 in Section 2. The ($\Leftarrow$) part of the first assertion is clear. We consider the ($\Rightarrow$) part. Define $\mathcal{S}_r$ as in (14). By Proposition 2.1, if ϕ preserves $\mathbf{H}_4$, then ϕ preserves the inner product $(X, Y) = \mathrm{tr}\,(XY^t)$. By GP3 in Section 2, we may assume that $\phi(I_4) = I_4$ and $\phi(\mathcal{S}_r) = \mathcal{S}_r$ for each r. In the following, we will show that ϕ has the form $X \mapsto P^tXP$ or $X \mapsto P^tX^tP$ for some $P \in \tilde{G}$. Throughout this proof we will use the matrices D_1, D_2, D_3, D_4, $A_1, ..., A_6$ and P_{ij} as defined in §5.1 – 5.2. We also refer to matrices in $\mathcal{S}_2$ as types (I), (II) and (III) as defined in §5.2.

Note that the four type (I) matrices $D_1, \ldots, D_4$ are mutually orthogonal matrices satisfying $(D_1 + \cdots + D_4)/2 = I_4$. Let $\phi(D_j) = Y_j$ for $j = 1, \ldots, 4$.

Case 1. If Y_1 is one of the four type (I) matrices, then $(Y_1 + Y_2 + Y_3 + Y_4)/2 = I$ implies that all Y_j are type (I) matrices. We can assume that $Y_j = D_j$ for all $j = 1, \ldots, 4$; otherwise, replace ϕ by a mapping of the form $X \mapsto Q\phi(X)Q^t$ for a suitable permutation matrix Q.

Now note that $(D_j, X) = 1$ for all X in type (III) of $\mathcal{S}_2$ and $j = 1, \ldots, 4$. For $1 \le i < j \le 4$, let $\mathcal{P}_{ij}$ be the set of 4 type (II) matrices of the form DP_{ij}. Then for all $X \in \mathcal{P}_{ij}$, $(D_k, X) = 2$ if $k = i, j$ and $(D_k, X) = 0$ otherwise. The same must be true of $\phi(X)$. Thus, $\phi(\mathcal{P}_{ij}) = \mathcal{P}_{ij}$.

Let

$$C_1 = I_2 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathcal{P}_{34} \quad \text{and} \quad \phi(C_1) = I_2 \oplus \begin{pmatrix} 0 & y_1 \\ y_2 & 0 \end{pmatrix} \in \mathcal{P}_{34}.$$

We claim that $y_1 = y_2$. Note that $Z \in \mathcal{S}_2$ satisfies $(C_1, Z) = 1$ if and only if one of the following holds:

(a) Z is one of the type (II) matrices with only one diagonal entry overlapping with those of C_1. There are 16 such matrices having the form $DP_{13}, DP_{14}, DP_{23}$ and DP_{24}, with four choices of diagonal orthogonal matrices D for each P_{ij}.

(b) Z is one of the type (III) matrices such that the $(3,4)$ and $(4,3)$ entries have different signs. There are 32 such matrices having the form DA_jD for $j = 3,4,5,6$, with eight choices of diagonal orthogonal matrices D for each A_j.

As a result, there should be 48 matrices Z in $\mathcal{S}_2$ such that $(\phi(C_1), Z) = 1$. However, if the $(3,4)$ and $(4,3)$ entries of $\phi(C_1)$ have different signs, then $Z \in \mathcal{S}_2$ satisfies $(\phi(C_1), Z) = 1$ can only happen if Z satisfies (a) or

(c) Z is one of the type (III) matrices such that the $(3,4)$ and $(4,3)$ entries have the same sign. there are 16 such matrices having the form DA_jD for $j = 1,2$, with eight choices of diagonal orthogonal matrices D for each A_j.

Thus, there are only 32 such matrices, which is a contradiction. Therefore, ϕ maps symmetric matrices in $\mathcal{P}_{34}$ to symmetric matrices in $\mathcal{P}_{34}$. Note that one can generalize this argument for all $\mathcal{P}_{ij}$ for all i,j.

We may assume that $\phi(C_1) = C_1$; otherwise, replace ϕ by a mapping of the form $X \mapsto D_4\phi(X)D_4$. Now,

$$C_2 = I_2 \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathcal{P}_{34}$$

is not a symmetric matrix in $\mathcal{P}_{34}$, and thus $\phi(C_2) \in \mathcal{P}_{34}$ is not symmetric. We may assume that $\phi(C_2) = C_2$; otherwise, replace ϕ by the mapping $X \mapsto \phi(X)^t$.

Divide type (III) matrices into two subclasses:

$\mathcal{T}_1$ is the set of type (III) matrices of the form PA_iP^t, where $i = 1,2$ (i.e., $(X, E_{34}) = (X, E_{43})$), and

$\mathcal{T}_2$ is the set of type (III) matrices of the form PA_iP^t, where $i = 3,4,5,6$ (i.e., $(X, E_{34}) = -(X, E_{43})$).

Then $(C_1, X) = 1$ for a type (III) matrix X if and only if $X \in \mathcal{T}_2$. Let

$$C_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_2, \quad C_4 = [1] \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus [1].$$

Since the symmetric matrices in $\mathcal{P}_{12}$ are mapped to themselves, we may assume that $\phi(C_3) = C_3$; otherwise, replace ϕ by a mapping of the form $X \mapsto D_1\phi(X)D_1$. Since symmetric elements of $\mathcal{P}_{23}$ are mapped to themselves, we may assume that $\phi(C_4) = C_4$; otherwise, replace ϕ by a mapping of the form $X \mapsto D_{12}\phi(X)D_{12}$.

Next, we show that ϕ fixes $E = E_{12} + E_{23} + E_{34} + E_{41}$. Since $(E, D_j) = 0$ for all $j \in \{1, \dots, 4\}$, and $(E, X) = 0$ for all $X \in \mathcal{P}_{ij}$ with $(i,j) \in \{(1,3),(2,4)\}$, it follows that $\phi(E)$ has a zero in all eight of the nonzero entry found among these eight matrices. Since $(E, C_i) = (-1)^{i-1}$ for $i = 1,2$, we have $(\phi(E), E_{34}) = 1$ and $(\phi(E), E_{43}) = 0$. So $\phi(E) = \pm E_{12} \pm E_{23} + E_{34} \pm E_{41}$. Since $(E, C_i) = 1$ for $i = 3,4$, $\phi(E) = E_{12} + E_{23} + E_{34} + E_{41} = E$ or

$\phi(E) = E_{12} + E_{23} + E_{34} - E_{41} = \hat{E}$. But $(E, X) = 2$ for exactly one matrix, namely $D_2 A_4 D_2$ in $\mathcal{T}_2$, whereas $(\hat{E}, X) = 2$ for 3 different matrices $D_{24} A_3 D_{24}$, $D_1 A_5 D_1$, and $D_{24} A_6 D_{24}$ in $\mathcal{T}_2$. Thus, $\phi(E) = E$. A similar argument shows that $\phi(E^t) = E^t$.

Let

$$C_5 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus I_2 \quad \text{and} \quad C_6 = [1] \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus [1].$$

Since $(E, C_i) = -1$ and $(E, C_i^t) = 1$ for $i \in \{5, 6\}$, it follows that $\phi(C_i) = C_i$ and $\phi(C_i^t) = C_i^t$ for $i \in \{5, 6\}$.

For each $X \in \mathcal{T}_1$ define

$$f(X) = [(I, X), (E, X), (E^t, X), (D_1, X), \ldots, (D_4, X), (C_1, X), \ldots, (C_6, X)].$$

Then one can show (say, using MATLAB) that $f(X) \neq f(Y)$ whenever $X \neq Y$ in $\mathcal{T}_1$. Since ϕ fixes the matrices $I_4, E, E^t, D_1, \ldots, D_4, C_1, \ldots, C_6$, it follows that $\phi(X) = X$ for all $X \in \mathcal{T}_1$. One can check that

$$\mathcal{T}_1 \cup \{D_1, \ldots, D_4, C_1, \ldots, C_6\}$$

span $M_n(\mathbb{R})$; see the last section. Thus, $\phi(X) = X$ for all $X \in M_n(\mathbb{R})$.

Case 2. Suppose Y_1 is a type (II) matrix. Then we may assume that Y_1 has the form $I_2 \oplus \begin{pmatrix} 0 & y_1 \\ y_2 & 0 \end{pmatrix}$. We claim that $y_1 y_2 = 1$. Otherwise, there are only 32 matrices Z in $\mathcal{S}_2$ satisfying $(Y_1, Z) = 1$, whereas all the 48 type (III) matrices Z satisfy $(D_1, Z) = 1$. Now, replace ϕ by a mapping of the form $X \mapsto B\phi(X)B$. The modified mapping will satisfy $\phi(D_1) = D_j$ with $j = 3$ or 4. Thus, we are back to case 1.

Case 3. Suppose Y_1 is one of the type (III) matrix. Note that Y_1 cannot have the form PA_jP for $j = 5, 6$, because

$$\min\{(D_1, Z) : Z \in \mathcal{S}_2\} = 0,$$

but for $\{j, k\} = \{5, 6\}$ and we have

$$(PA_jP, PD_1A_kD_1P) = -2$$

and hence

$$\min\{(PA_jP, Z) : Z \in \mathcal{S}_2\} < 0.$$

Now, suppose $Y_1 = PA_jP$ for $j = 1, 2, 3$, or 4. We may assume that $Y_1 = A_j$; otherwise, replace ϕ by $X \mapsto P\phi(X)P$. If $Y_1 = A_1$, replace ϕ by $A \mapsto B\phi(A)B$. The resulting mapping satisfies $\phi(D_1) = E_{11} + E_{23} + E_{32} + E_{44}$. We are back to Case 2.

If $Y_1 = A_2$, A_3 or A_4, replace ϕ by $A \mapsto B\phi(A)B$. The resulting mapping satisfies $\phi(D_1) = E_{11} - E_{23} + E_{32} + E_{44}$, which is impossible by the argument in Case 2. $\square$

6 $\mathbf{E}_8$

6.1 Matrix Realization

The group $\mathbf{E}_8$ has $2^{14}3^5 5^2 7 = 8!2^7 3^3 5$ elements which can be divided into the following 3 classes.

(I) The $8!2^7$ elements of $\mathbf{D}_8$.

(II) The $8!2^{13}$ matrices of the form XAY, where $X, Y \in \mathbf{D}_8$ and

$$A = I_8 - ww^t/4 \qquad \text{with} \qquad w = e - 2e_8 = (1, \cdots, 1, -1)^t \in \mathbb{R}^8. \tag{15}$$

The counting is done by: $2^7 8!$ choices for each of X and Y, and there are $2 \cdot 8!$ pair of (P, Q) in $\mathbf{D}_8 \times \mathbf{D}_8$ satisfying $PAQ = A$.

(III) The $8!2^8 \cdot 35$ matrices of the form XBY, where $X, Y \in \mathbf{D}_8$ and

$$B = B_1 \oplus B_2 = A(-I_4 \oplus I_4)A, \tag{16}$$

where

$$B_1 = (1,1,1,1)^t(1,1,1,1)/2 - I_4, \quad B_2 = I_4 - (1,1,1,-1)^t(1,1,1,-1)/2.$$

The counting can be done as follows. First choose 4 rows and 4 columns in $\binom{8}{4}^2$ ways. Then put matrix pairs $(X_1 B_1 Y_1, X_2 B_2 Y_2)$ in the two complementary blocks, where

(i) $X_1, Y_1, X_2, Y_2 \in \mathbf{D}_4$,

(ii) $X_1, Y_1, X_2, Y_2 \in (\mathbf{B}_4 \setminus \mathbf{D}_4)$,

(iii) $X_1, X_2 \in \mathbf{D}_4$ and $Y_1, Y_2 \in (\mathbf{B}_4 \setminus \mathbf{D}_4)$, or

(iv) $X_1, X_2 \in (\mathbf{B}_4 \setminus \mathbf{D}_4)$ and $Y_1, Y_2 \in \mathbf{D}_4$.

The number of choices for $X_i B_i Y_i$ in each case is $|\mathbf{F}_4 \setminus \mathbf{B}_4|/4 = 4!2^3$. Since $DB_1D = B_2$ with $D = \text{diag}(1,1,1,-1)$, we see that cases (i) and (ii) yield the same matrices, and also cases (iii) and (iv) yield the same matrices. So, there are $2(4!2^3)^2$ so many choices for the pairs. Consequently, the total number of this class is $2(4!2^3)^2\binom{8}{4}^2 = 8!2^7 70$.

6.2 Maximum inner product

Let $X \in \mathbf{E}_8$ with $X \neq I$. Then $(I, X) \leq n - 2$. The equality holds if and only if $X = I - (e_i \pm e_j)(e_i \pm e_j)^t$ for some $1 \leq i < j \leq 8$ or $X = P^t AP$ for some $P \in \mathbf{D}_8$. By GP2 in Section 2, for each possible value of $r = (I, X)$, define

$$\mathcal{S}_r = \{X \in \mathbf{E}_8 : (I_8, X) = r\}. \tag{17}$$

Note that the largest value for r is 6, and $\mathcal{S}_6$ consists of matrices of the following forms.

(a) The 56 matrices of the form

$$X_{ij} = I_8 - (e_i - e_j)(e_i - e_j)^t \qquad \text{or} \qquad Y_{ij} = I_8 - (e_i + e_j)(e_i + e_j)^t,$$

where $1 \leq i < j \leq 8$.

(b) The 64 matrices of the form DAD^t, where A is defined in (15) and D is a diagonal orthogonal matrix in $\mathbf{D}_8$.

6.3 Linear Preservers

Theorem 6.1 *A linear operator $\phi : M_8(\mathbb{R}) \to M_8(\mathbb{R})$ satisfies $\phi(\mathbf{E}_8) = \mathbf{E}_8$ if and only if there exist $P, Q \in \mathbf{E}_8$ such that ϕ has the form*

$$X \mapsto PXQ \quad \text{or} \quad X \mapsto PX^tQ.$$

Consequently, $N(\mathbf{E}_8) = \mathbf{E}_8$.

Proof. The assertion on $N(\mathbf{E}_8)$ follows from GP4 in Section 2. The ($\Leftarrow$) part of the first assertion is clear. We consider the ($\Rightarrow$) part. Define $\mathcal{S}_r$ as in (17). By Proposition 2.1, if ϕ preserves $\mathbf{E}_8$, then ϕ preserves the inner product $(X, Y) = \mathrm{tr}\,(XY^t)$. By GP3 in Section 2, we may assume that $\phi(I_8) = I_8$ and $\phi(\mathcal{S}_r) = \mathcal{S}_r$ for each r. In the following, we will show that ϕ has the form $X \mapsto P^tXP$ or $X \mapsto P^tX^tP$ for some $P \in \mathbf{E}_8$. We shall use the matrices A, X_{ij} and Y_{ij} as defined in §6.1 − 6.2 as well as the classification of elements of $\mathcal{S}_6$ as type (a) and (b) as defined in §6.2.

Define $D_i = I_8 - 2E_{ii}$, and $D_{ij} = D_iD_j$. Note that those D described in (b) have one of three forms:

$$D_{ij}, \; D_{ijkl} = D_{ij}D_{kl}, \quad \text{or} \quad -D_{ij}.$$

Note the following four types of conjugations will be used extensively throughout this first part. For i, j, k distinct,

$$X_{ik}X_{jk}X_{ik}^t = X_{ij} \qquad \text{and} \qquad Y_{ik}Y_{jk}Y_{ik}^t = X_{ij}.$$

Since $X_{78} \in \mathcal{S}_6$, it follows that $\phi(X_{78}) = Z \in \mathcal{S}_6$. If $Z = X_{ij}$ or Y_{ij}, then replace ϕ by the mapping $X \mapsto P\phi(X)P^t$ with

$$P = \begin{cases} X_{i7}X_{j8} & \text{if } Z = X_{ij}, \\ X_{i7}Y_{j8} & \text{if } Z = Y_{ij}. \end{cases}$$

Then $\phi(X_{78}) = X_{78}$. If $\phi(X_{78}) = DAD^t$, then replace ϕ by the mapping $X \mapsto D\phi(X)D^t$ so that $\phi(X_{78}) = A$. Furthermore, replace ϕ by the mapping $X \mapsto Q_7\phi(X)Q_7^t$, where $Q_7 = D_{78}AD_{78}^t$, so that $\phi(X_{78}) = X_{78}$.

Now consider those $X \in \mathcal{S}_6$ such that $(X, X_{78}) = 5$. They are of the following two forms.

(c) 24 matrices of the form X_{ij} or Y_{ij} where $i < 7 \leq j \leq 8$.

(d) 32 matrices of the form DAD^t where $(D, E_{77}) = (D, E_{88}) = 1$.

If $X \in \mathcal{S}_6$ is not of these forms, then $(X, X_{78}) = 4$. It is important to note the sign pattern of the diagonal D in type (d). For any i, j, if $(DAD^t, X_{ij}) = 5$, then (DAD^t, E_{ij}) must be positive. Thus,

$$(D, E_{ii}) = (D, E_{88}) \qquad \text{if } i < j = 8$$

but

$$(D, E_{ii}) = -(D, E_{jj}) \qquad \text{if } i < j < 8.$$

We change the signs in this argument if we are interested in $(DAD^t, X_{ij}) = 4$.

If $\phi(X_{67}) = X_{i7}$ then replace ϕ be the mapping $X \mapsto X_{i6}\phi(X)X_{i6}^t$. If $\phi(X_{67}) = X_{i8}$ then replace ϕ be the mapping $X \mapsto X_{78}\phi(X)X_{78}^t$, thus reducing the problem to the previous case. If $\phi(X_{67}) = Y_{ij}$ then replace ϕ be the mapping $X \mapsto D_{ik}\phi(X)D_{ik}^t$ for $k \neq i, 7, 8$. If $\phi(X_{67}) = DAD^t$ where $(D, E_{77}) = (D, E_{88})$, then replace ϕ be the mapping $X \mapsto \hat{D}\phi(X)\hat{D}^t$ where $\hat{D}$ is a diagonal orthogonal matrix such that $\phi(X_{67}) = Q_7$ where Q_7 is defined as before. Now replace ϕ by the mapping $X \mapsto Q_6\phi(X)Q_6^t$ where $Q_6 = D_{68}AD_{68}^t$ Therefore $\phi(X_{67}) = X_{67}$.

Now consider those $X \in \mathcal{S}_6$ such that $(X, X_{78}) = 4$ and $(X, X_{67}) = 5$. They are of the following two forms.

(e) 10 matrices of the form X_{i6} or Y_{i6}, $i < 6$.
(f) 16 matrices of the form DAD^t where $D = D_{68}$, D_{ij68} or $-D_{ij}$ for $i < j < 6$.

Since X_{56} is in this set, so must $\phi(X_{56})$. If $\phi(X_{56}) = X_{i6}$, then replace ϕ by the mapping $X \mapsto X_{i5}\phi(X)X_{i5}^t$. If $\phi(X_{56}) = Y_{i6}$, then replace ϕ by the mapping $X \mapsto D_{ik}\phi(X)D_{ik}$ where $k < 6$. If $\phi(X_{56}) = DAD^t$ where $(D, E_{66}) = -(D, E_{77}) = (D, E_{88})$, then replace ϕ by the mapping $X \mapsto \hat{D}\phi(X)\hat{D}^t$ where $\hat{D} = DD_{68}$. Next replace ϕ by the mapping $X \mapsto Q_5\phi(X)Q_5^t$ where $Q_5 = D_{58}AD_{58}^t$.

We can fix first, X_{45}, second, X_{34}, and third, X_{23}, in the same way as we fixed X_{56} by the following arguments. For $k = 5$, 4 and then 3, consider those $X \in \mathcal{S}_6$ that have inner product of 5 with the matrix $X_{k,k+1}$ (which has just been shown to be fixed by ϕ) but inner product of 4 with $X_{i,i+1}$ for $i \geq k+1$. Then $\phi(X_{k-1,k})$ have one of three forms: $X_{i,k-1}$, $Y_{i,k-1}$ and DAD where $D = \hat{D}D_{k-1,8}$. If it is one of the first two forms, then replace ϕ by a mapping of the form $X \mapsto P\phi(X)P^t$ where P is a appropriate matrix of one of the first two forms. If it is of the third form, replace ϕ by the mapping

$$X \mapsto Q_{k-1}\hat{D}\phi(X)\hat{D}Q_{k-1} \qquad \text{for} \qquad Q_{k-1} = D_{k-1,8}AD_{k-1,8}.$$

Now define $Q_1 = D_{18}AD_{18}^t$. Then Q_1 is the only type (b) matrix such that $Q_1 \in \mathcal{S}_6$ and $(Q_1, X_{i,i+1}) = 4$ for $i = 2, ..., 7$. There are no type (a) matrices where this property holds, thus $\phi(Q_1) = Q_1$. Consider those $X \in \mathcal{S}_6$ such that $(X, X_{23}) = 5$ and $(X, X_{i,i+1}) = 4$ for $i = 3, ..., 7$. Then $X = X_{12}$, Y_{12} or Q_2. Inspecting the sign pattern, we have $(X_{ij}, Q_1) \neq (Y_{ij}, Q_1)$ for all i, j. Thus, we may assume that $\phi(X_{12}) = X_{12}$. Otherwise $\phi(X_{12}) = Q_2$ and replace ϕ

by the mapping $X \mapsto Q_1\phi(X)Q_1$. Thus, $\phi(Z) = Z$ for $Z = I_8, Q_1, X_{i,i+1}$ for $i = 1, ..., 7$. One can check that this is sufficient to show that $\phi(X) = X$ for all $X \in \mathcal{S}_6$; see the last section.

Suppose that $\phi(X) = Y$ for some $X \in \mathbf{E}_8$. Then

$$2(X, E_{ij} + E_{ji}) = (X, X_{ij} - Y_{ij}) = (Y, X_{ij} - Y_{ij}) = 2(Y, E_{ij} + E_{ji}).$$

Also, for all $i \neq j$, $I - (X_{ij} + Y_{ij})/2 = E_{ii} + E_{jj}$. Therefore, $(X, E_{ii}) = (Y, E_{ii})$ for all i and thus

$$\phi(X) = X \text{ or } X^t \qquad \text{for all } X \in \mathbf{E}_8.$$

Let $X_{ijk} = X_{ij}X_{ik} \in \mathcal{S}_5$ for each $i < j < k$. Then X_{ijk} is the type (I) matrix with the following principal submatrices

$$X_{ijk}(i,j,k) = I_5, \qquad X_{ijk}[i,j,k] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Here $Z[i,j,k]$ denotes the submatrix of Z lying in rows and columns i, j, and k; and $Z(i,j,k)$ denotes the matrix obtained from Z by deleting its rows and columns indexed by i, j, and k. Then we may assume that $\phi(X_{678}) = X_{678}$. Otherwise, $\phi(X_{678}) = X_{678}^t$, and replace ϕ by the mapping $X \mapsto \phi(X)^t$. Note that $(X_{ijk}, X_{678}) = 5$ if and only if $i < 6 \leq j < k \leq 8$. But then $(X_{ijk}^t, X_{678}) = 4$. Thus $\phi(X_{ijk}) = X_{ijk}$ for all X_{ijk} where $i < 6 \leq j < k \leq 8$. Continuing in this manner, we can fix all matrices of the form X_{ijk}. Therefore $\phi(Z) = Z$ whenever X_{ijk} for all $1 \leq i < j < k \leq 8$.

Let

$$P = E_{18} + \sum_{j=2}^{8} E_{j,j-1} \in M_8(\mathrm{I\!R}).$$

Then, for all $Q \in \mathcal{S}_6$ of type (b), $\phi(PQ) = PQ$ or $(PQ)^t$. But clearly, $(PQ, X_{123}) \neq ((PQ)^t, X_{123})$, therefore, $\phi(PQ) = PQ$ for all $Q \in \mathcal{S}_6$ of type (b). Thus, $\phi(Z) = Z$ for $Z = I_8, X_{ijk}$ where $i < j < k$, X_{ij} where $i < j$ and for $Z = Q$ and PQ for all $Q \in \mathcal{S}_6$ of type (b). One can check that these matrices span $M_8(\mathrm{I\!R})$; see the last section. Thus $\phi(X) = X$ for all $X \in \mathbf{E}_8$. □

7 $\mathbf{E}_7$

Let

$$w = e - 2e_8 \in \mathrm{I\!R}^8.$$

Then $\mathbf{E}_7$ has a natural realization as a subgroup of $\mathbf{E}_8 \subseteq M_8$ defined and denoted by

$$\mathcal{E}_7 = \{X \in \mathbf{E}_8 : Xw = w\}$$

acting on the 7-dimensional subspace $w^\perp$ in $\mathbb{R}^8$. Suppose U is an orthogonal matrix with $w/\sqrt{8}$ as the first column. Then for every $A \in \mathcal{E}_7$, we have

$$U^t A U = \begin{pmatrix} 1 & 0 \\ 0 & \hat{A} \end{pmatrix}. \tag{18}$$

The collection of such $\hat{A} \in M_7(\mathbb{R})$ will form a matrix realization of $\mathbf{E}_7$ in $M_7(\mathbb{R})$. Moreover, for any $A, B \in \mathcal{E}_7$ and the corresponding $\hat{A}, \hat{B} \in \mathbf{E}_7$, we have

$$(A, B) = \mathrm{tr}\,(AB^t) = 1 + \mathrm{tr}\,(\hat{A}\hat{B}^t) = 1 + (\hat{A}, \hat{B}). \tag{19}$$

Of course, one may have different realizations of $\mathbf{E}_7$ in $M_7(\mathbb{R})$ by a different choice of U. Nonetheless, it is well known that all the realizations of $\mathbf{E}_7$ in $O(7)$ are orthogonally similar. In this section, we will study $\mathbf{E}_7$ via $\mathcal{E}_7$ as mentioned in GP5 of section 2.

7.1 Matrix Realization

The group $\mathcal{E}_7 = \{X \in \mathbf{E}_8 : Xw = w\}$ has $8!72$ elements which can be divided into the following three classes.

(I) The $8!$ elements in $\mathbf{D}_8 \cap \mathcal{E}_7$, those elements $X \in \mathbf{D}_8$ such that $Xw = w$.

(II) The $8!36$ matrices of the form X^tAY satisfying $X^tAYw = w$, where $X, Y \in \mathbf{D}_8$ and $A = I_8 - ww^t/4$ with $w = e - 2e_8 = (1, \cdots, 1, -1)^t \in \mathbb{R}^8$. The counting is done as follows. Consider the equation $AYw = Xw$, i.e., $Yw - Xw = w(w^tYw)/4$. There are 3 cases.

(i) $w^tYw = 8$, $Yw - Xw = 2w$. Then $Yw = w = -Xw$. There are $8!$ choices for each of X and Y and there are $8!$ pairs (P, Q) in $\mathbf{D}_8 \times \mathbf{D}_8$ such that $PAQ = A$ with $Qw = w$ and $w^tP = w^t$. So, there are $8!$ elements in this case. Clearly, these must coincide with the $8!$ elements of the form $-PA$ where P is a matrix of type (I).

(ii) $w^tYw = -8$, $Yw - Xw = -2w$. Then $Yw = -w = -Xw$. Every pair (X, Y) in (b.i) can be converted to $(-X, -Y)$ to this case, and we actually get the same $XAY = (-X)A(-Y)$ matrix. So, no new addition in this case.

(iii) $w^tYw = 0$, $Yw = Xw$. For each of the 70 choices of $w_i \in w^\perp$, where all entries of w_i are ± 1, we have a fixed $P_i \in \mathbf{D}_8$ such that $P_iw = w_i$, $Y = P_i\hat{Y}$ and $X = P_i\hat{X}$ with $\hat{Y}w = w = \hat{X}w$. Now, there are $8!$ choices for each of $\hat{X}$ and $\hat{Y}$, and we have to factor out the $8!$ so many (R, S) pairs such that $R^t(P_i^tAP_i)S = P_i^tAP_i$ with $Sw = w = Rw$. Thus, there are $8!$ so many X^tAY corresponding to each choice of w_i. However, for each w_i, the $8!$ matrices X^tAY corresponding to w_i are the same as the $8!$ matrices corresponding to $-w_i$. Thus, we have $8!70/2 = 8!35$ matrices in this case. These are the matrices of the form $PDAD^t$ where P is a type (I) matrix and D is a diagonal matrix whose diagonal entries are permutations of $(1, 1, 1, 1, -1, -1, -1, -1)$. In other words, $Dw \in w^\perp$.

(III) The $8!35$ matrices of the form X^tBY satisfying $X^tBYw = w$, where $X, Y \in \mathbf{D}_8$ and $B = B_1 \oplus B_2 = A(-I_4 \oplus I_4)A$, where

$$B_1 = (1, 1, 1, 1)^t(1, 1, 1, 1)/2 - I_4, \quad B_2 = I_4 - (1, 1, 1, -1)^t(1, 1, 1, -1)/2.$$

The counting is done as follows. In order to have $X^tBYw = w$, the last row of X^tBY must contain either a row of $X_2B_2Y_2$ with a nonzero $(8,8)$ entry or a row of $X_1B_1Y_1$ with the $(8,8)$ equal to zero. In the first case, we have $\binom{7}{4}\binom{7}{4}4!$ ways to put $X_1B_1Y_1$ so as to make the first 4 entries of X^tBYw equal to 1, and then 4! ways to put the $X_2B_2Y_2$ matrices so that the last 4 entries of X^tBYw are $1,1,1,-1$. In the second case, we have $\binom{7}{4}\binom{7}{4}4!$ ways to put $X_2B_2Y_2$ so as to make the first 4 entries of X^tBYw equal to 1, and then 4! ways to put the $X_1B_1Y_1$ matrices so that the last 4 entries of X^tBYw are $1,1,1,-1$. Thus, total number is $2(\binom{7}{4}\binom{7}{4}(4!)^2) = 8!35$.

7.2 Maximum inner product

Let $X \in \mathcal{E}_7$ with $X \neq I$. Then $(I_8, X) \leq 6$ and hence the inner product on the irreducible subspace $\mathbf{E}_7$ is bounded by 5. Using the matrix realization in $M_8(\mathbb{R})$ and by GP2 in Section 2, for each possible value of $r = (I_8, X)$, define

$$\mathcal{S}_r = \{X \in \mathcal{E}_7 : (I_8, X) = r\}. \tag{20}$$

Note that $\mathcal{S}_6$ consists of matrices in one of the following two forms.

(a) The 28 matrices X_{ij} of the form

$$X_{ij} = I_8 - (e_i - e_j)(e_i - e_j)^t \text{ for } 1 \leq i < j \leq 7$$

$$X_{i8} = I_8 - (e_i + e_8)(e_i + e_8)^t \text{ for } 1 \leq i \leq 7.$$

(b) The 35 matrices of the form $X = DAD^t$ for some diagonal orthogonal D such that $Dw \in w^{\perp}$.

7.3 Linear Preservers

Theorem 7.1 *A linear operator $\psi : M_7(\mathbb{R}) \to M_7(\mathbb{R})$ satisfies $\psi(\mathbf{E}_7) = \mathbf{E}_7$ if and only if there exist $P, Q \in \mathbf{E}_7$ such that ψ has the form*

$$X \mapsto PXQ \quad \text{or} \quad X \mapsto PX^tQ.$$

Consequently, $N(\mathbf{E}_7) = \mathbf{E}_7$.

Proof. The assertion on $N(\mathbf{E}_7)$ follows from GP4 in Section 2. The $(\Leftarrow)$ part of the first assertion is clear. We consider the $(\Rightarrow)$ part. Let $\psi : M_7(\mathbb{R}) \to M_7(\mathbb{R})$ be a linear map satisfying $\psi(\mathbf{E}_7) = \mathbf{E}_7$. By Proposition 2.1, if ψ preserves $\mathbf{E}_7$, then ψ preserves the inner product $(\hat{X}, \hat{Y}) = \mathrm{tr}\,(\hat{X}\hat{Y}^t)$. Also, by GP3 in Section 2, we may assume that $\psi(I_7) = I_7$.

Let V_7 be the affine space generated by $\mathcal{E}_7$, and let U be an orthogonal matrix establishing the correspondence between $\mathcal{E}_7$ and $\mathbf{E}_7$ as described in (18). Consider an affine map $\phi : V_7 \to V_7$ defined by

$$\phi\left(U\begin{pmatrix}1 & 0\\ 0 & \hat{X}\end{pmatrix}U^t\right) = U\begin{pmatrix}0 & 0\\ 0 & \psi(\hat{X})\end{pmatrix}U^t + U\begin{pmatrix}1 & 0\\ 0 & 0_7\end{pmatrix}U^t.$$

Then $\phi(\mathcal{E}_7) = \mathcal{E}_7$. Since ψ preserves inner product in $M_7(\mathrm{IR})$, we have $(\phi(X), \phi(Y)) = (X, Y)$ for all $X, Y \in \mathcal{E}_7$ by (19). Define $\mathcal{S}_r$ as in (20). Since $\psi(I_7) = I_7$, therefore, $\phi(I_8) = I_8$ and by GP3 in Section 2, $\phi(\mathcal{S}_r) = \mathcal{S}_r$ for each r. In the following, we will show that for some $P \in \mathcal{E}_7$, ϕ has the form

$$X \mapsto P^tXP \text{ for all } X \in \mathcal{E}_7 \qquad \text{or} \qquad X \mapsto P^tX^tP \text{ for all } X \in \mathcal{E}_7.$$

We shall use the matrices A and X_{ij} for $1 \le i < j \le 8$ as defined in §7.1−7.2. We also refer to matrices in $\mathcal{S}_6$ as type (a) and (b) matrices as defined in §7.2. Furthermore, let $D_i = I_8 - 2E_{ii}$, and $D_{ij} = D_iD_j$. Note that those D described in (b) will be of the form $D_{ijkl} = D_{ij}D_{kl}$ where i, j, k, l are all distinct. If i', j', k', l' are such that $\{i, i', j, j', k, k', l, l'\} = \{1, ..., 8\}$, then

$$D_{ijkl}AD_{ijkl} = D_{i'j'k'l'}AD_{i'j'k'l'}.$$

Also, for i, j, k distinct and X_{jk}, X_{ik} and X_{ij} all of type (a),

$$X_{ik}X_{jk}X_{ik}^t = X_{ij}.$$

We may assume that $\phi(X_{78}) = X_{78}$. Otherwise $\phi(X_{78}) = X_{ij}$ or $\phi(X_{78}) = DAD$ for an appropriate D. If $\phi(X_{78}) = X_{ij}$, then replace ϕ by the mapping $X \mapsto P\phi(X)P^t$ where P is an appropriate type (a) matrix. If $\phi(X_{78}) = DAD^t$, then consider D. If $D = D_{ijk7}$ for $i, j, k < 7$, then replace ϕ by the mapping $X \mapsto QAQ^t$ where $Q = D_{ijk8}AD_{ijk8}$. If $D = D_{ij78}$, Then replace ϕ by the mapping $X \mapsto X_{k8}\phi(X)X_{k8}^t$ for some $k \neq i, j$. Thus $\phi(X_{78}) = D_{ijk7}AD_{ijk7}$, which has already been discussed. Therefore, $\phi(X_{78}) = X_{78}$.

Now, consider those $X \in \mathcal{S}_6$ such that $(X, X_{78}) = 5$. They are of two forms.

(c) 12 matrices of the form X_{ij} for $i < 7 \le j$.

(d) 20 matrices of the form DAD where $D = D_{ijk7}$ and $i < j < k < 7$.

We may assume that $\phi(X_{67}) = X_{67}$. Otherwise $\phi(X_{67}) = X_{ij}$ where $i < 6$ and $7 \le j$ or $\phi(X_{67}) = DAD$ where $D = D_{ijk7}$ and $i < j < k < 7$. If $\phi(X_{67}) = X_{i7}$ where $i < 6$ then replace ϕ by the mapping $X \mapsto X_{i6}\phi(X)X_{i6}^t$. If $\phi(X_{67}) = X_{i8}$ where $i < 6$, then replace ϕ by the mapping $X \mapsto X_{78}\phi(X)X_{78}^t$ and we are back to the previous case. If $\phi(X_{67}) = DAD$ where $D = D_{ijk7}$ and $i < j < k < 7$, then either $k = 6$ or $k \neq 6$. If $k \neq 6$, then replace ϕ by the mapping of the form $X \mapsto QAQ^t$ where $Q = D_{ijk6}AD_{ijk6}$. If $k = 6$, replace ϕ by the mapping $X \mapsto X_{k'6}\phi(X)X_{k'6}$ where $k' \neq i, j$ and also $k' < 6$. Therefore, $\phi(X_{67}) = X_{67}$.

Now, consider those $X \in \mathcal{S}_6$ such that $(X, X_{78}) = 4$ and $(X, X_{67}) = 5$. They are of two forms.

(e) 5 matrices of the form X_{i6} for $i < 6$.

(f) 10 matrices of the form DAD where $D = D_{ijk6}$ and $i < j < k < 6$.

We may assume that $\phi(X_{56}) = X_{56}$. Otherwise $\phi(X_{56}) = X_{i6}$ where $i < 5$ or $\phi(X_{56}) = DAD$ where $D = D_{ijk6}$ and $i < j < k < 6$. If $\phi(X_{56}) = X_{i6}$ where $i < 5$ then replace ϕ by the mapping $X \mapsto X_{i5}\phi(X)X_{i5}^t$. If $\phi(X_{56}) = DAD$ where $D = D_{ijk6}$ and $i < j < k < 6$, then either $k = 5$ or $k \neq 5$. If $k \neq 5$, then replace ϕ by the mapping of the form $X \mapsto QAQ^t$ where $Q = D_{ijk5}AD_{ijk5}$. If $k = 5$, replace ϕ by the mapping $X \mapsto X_{k'5}\phi(X)X_{k'5}$ where $k' \neq i, j$ and also $k' < 5$. Therefore, $\phi(X_{56}) = X_{56}$. We may also fix X_{45} in a similar manner, using those $X \in \mathcal{S}_6$ such that $(X, X_{56}) = 5$, but has inner product 4 with the other fixed matrices.

Now, consider those $X \in \mathcal{S}_6$ such that

$$[(X, X_{78}), (X, X_{67}), (X, X_{56}), (X, X_{45})] = [4, 4, 4, 4].$$

Then $X = X_{12}, X_{13}$ or X_{23}. We may assume that $\phi(X_{12}) = X_{12}$. Otherwise $\phi(X_{12}) = X_{i3}$ for $i = 1$ or 2, in which case, replace ϕ by the mapping $X \mapsto X_{j3}\phi(X)X_{j3}$ where $\{i, j\} = \{1, 2\}$. We may also assume that $\phi(X_{23}) = X_{23}$; otherwise $\phi(X_{23}) = X_{13}$. If this is the case, replace ϕ by the mapping $X \mapsto X_{12}\phi(X)X_{12}$. Thus $\phi(Z) = Z$ for

$$Z = X_{12}, X_{23}, X_{45}, X_{56}, X_{67} \text{ and } X_{78}.$$

One can check that this requires that $\phi(X) = X$ for all $X \in \mathcal{S}_6$; see the last section.

Consider those $X \in \mathcal{E}_7$ such that $X_{ij}X \in \mathcal{S}_6$ and $(X_{ij}X, X_{ij}) = 5$. In other words, $X \in \mathcal{S}_5$ and $(X, X_{ij}) = 6$. So X is of the form $X_{ij}X_{lk}$, where $k \notin \{i, j\}$ but $l \in \{i, j\}$, or the form $X_{ij}DAD^t$ where $Dw \in w^\perp$ and $(D, E_{ii}) = -(D, E_{jj})$. If we add the condition that $(X, X_{ik}) = 6$ and $(X, X_{jk}) = 6$, then X must be of the form $X_{ij}X_{ik}$ or $X_{ij}X_{jk} = (X_{ij}X_{ik})^t$. Let $X_{ijk} = X_{ij}X_{ik} \in \mathcal{S}_5$ for each $i < j < k$. Then X_{ijk} is the type (I) matrix with the following principal submatrices.

$$X_{ijk}(i, j, k) = I_5, \qquad X_{ijk}[i, j, k] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \epsilon_1 \\ \epsilon_2 & 0 & 0 \end{pmatrix},$$

where $\epsilon_1 = \epsilon_2 = 1$ if $k < 8$ and -1 if $k = 8$. Thus, $\phi(X_{ijk}) = X_{ijk}$ or $\phi(X_{ijk}) = X_{ijk}^t$ for all $i < j < k$. Then we may assume that $\phi(X_{678}) = X_{678}$. Otherwise, $\phi(X_{678}) = X_{678}^t$ and replace ϕ by the mapping $X \mapsto \phi(X)^t$. Thus, also $\phi(X_{678}^t) = X_{678}^t$. Consider those X_{ijk} such that $(X_{ijk}, X_{678}) = 5$. Then either $\{j, k\} = \{6, 7\}$ or $\{j, k\} = \{7, 8\}$. But for those X_{ijk}, $(X_{ijk}, X_{678}^t) = 4$. So $\phi(X_{ijk}) = X_{ijk}$ for all such X_{ijk}. Using these newly fixed matrices, continue in the same manner until $\phi(X_{ijk}) = X_{ijk}$ for all X_{ijk} such that $1 \leq i < j < k \leq 8$.

We have shown that $\phi(X) = X$ for all $X \in \mathcal{S}_6$, $X = I_8$ and all X of the form X_{ijk}. It can be shown (see the last section) that there are 50 linearly independent matrices in this collection. Given this, and the fact that

$$\phi(0) = U \begin{pmatrix} 1 & 0 \\ 0 & 0_7 \end{pmatrix} U^t,$$

we see that the linear map

$$\phi(X) - \phi(0)$$

is completely determined. In particular, $\phi(X) = X$ for all $X \in \mathcal{E}_7$. It follows that the original affine map ϕ on V_7 has the form

$$X \mapsto P^t X P \qquad \text{or} \qquad X \mapsto P^t X^t P$$

for some $P \in \mathcal{E}_7$. Note that if $P, X \in \mathcal{E}_7$, there exists $\hat{P}, \hat{X} \in \mathbf{E}_7$ such that

$$P^t X P = U \begin{pmatrix} 0 & 0 \\ 0 & \hat{P}^t \hat{X} \hat{P} \end{pmatrix} U^t + U \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U^t.$$

Thus, there exists a $\hat{P} \in \mathbf{E}_7$ such that

$$\psi(\hat{X}) = \hat{P}^t \hat{X} \hat{P} \text{ for all } \hat{X} \in \mathbf{E}_7 \qquad \text{or} \qquad \psi(\hat{X}) = \hat{P}^t \hat{X}^t \hat{P} \text{ for all } \hat{X} \in \mathbf{E}_7.$$

Since $\mathbf{E}_7$ spans $M_7(\mathbb{R})$, ψ on $M_7(\mathbb{R})$ has the asserted form. □

Note that in the above proof, we showed that an affine map ϕ on V_7 satisfies $\phi(\mathcal{E}_7) = \mathcal{E}_7$ and preserves the inner product on V_7 if and only if there exists $P, Q \in \mathcal{E}_7$ such that ϕ has the form

$$X \mapsto PXQ \qquad \text{or} \qquad X \mapsto PX^tQ \tag{21}$$

on V_7. The same proof can actually be used to show that a linear map $\hat{\phi}$: span $\mathcal{E}_7 \mapsto$ span $\mathcal{E}_7$ satisfies $\hat{\phi}(\mathcal{E}_7) = \mathcal{E}_7$ and preserves the inner product on span $\mathcal{E}_7$ if and only if there exists $P, Q \in \mathcal{E}_7$ such that $\hat{\phi}$ has the form (21).

8 $\mathbf{E}_6$

In this section, we let

$$w = e - 2e_8 \in \mathbb{R}^8 \qquad \text{and} \qquad v = e_7 - e_8 \in \mathbb{R}^8.$$

Then $\mathbf{E}_6$ has a natural realization as a subgroup of $\mathbf{E}_8 \subseteq M_8$ defined and denoted by

$$\mathcal{E}_6 = \{X \in \mathcal{E}_7 : Xv = v\} = \{X \in \mathbf{E}_8 : Xv = v \text{ and } Xw = w\}$$

acting on the 6-dimensional subspace span $(v, w)^\perp$ in $\mathbb{R}^8$. Suppose U is an orthogonal matrix with $w/\sqrt{8}$ as the first column and the normalization of the component of v orthogonal to w as the second column. Then for every $A \in \mathcal{E}_6$, we have

$$U^t A U = \begin{pmatrix} I_2 & 0 \\ 0 & \hat{A} \end{pmatrix}. \tag{22}$$

The collection of such $\hat{A} \in M_6(\mathbb{R})$ will form a matrix realization of $\mathbf{E}_6$ in $M_6(\mathbb{R})$. Moreover, for any $A, B \in \mathcal{E}_6$ and the corresponding $\hat{A}, \hat{B} \in \mathbf{E}_6$, we have

$$(A, B) = \operatorname{tr}(AB^t) = 2 + \operatorname{tr}(\hat{A}\hat{B}^t) = 2 + (\hat{A}, \hat{B}). \tag{23}$$

Of course, one may have different realizations of $\mathbf{E}_6$ in $M_6(\mathbb{R})$ by a different choice of U. Nonetheless, it is well known that all the realizations of $\mathbf{E}_6$ in $O(6)$ are orthogonally similar. In this section, we will study $\mathbf{E}_6$ via $\mathcal{E}_6$ as mentioned in GP5 of section 2.

8.1 Matrix Realization

The group $\mathcal{E}_6 = \{X \in \mathbf{E}_8 : Xw = w,\ Xv = v\}$ has 6!72 elements which can be divided into the following 3 classes of matrices arising from $\mathcal{E}_7$.

(I) The 6!2! elements in $\mathbf{D}_8 \cap \mathbf{E}_6$, namely, those elements $X \in \mathbf{D}_8$ of the form $X = X_1 \oplus X_2$ for suitable choices of $X_1 \in \mathbf{B}_6$ and

$$X_2 = I_2 \qquad \text{or} \qquad X_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

(II) The 6!40 matrices of the form X^tAY satisfying $X^tAYw = w$ and $X^tAYv = v$, where $X, Y \in \mathbf{D}_8$ and $A = I_8 - ww^t/4$ with $w = e - 2e_8 = (1, \cdots, 1, -1)^t \in \mathbb{R}^8$. The counting is done as follows. Consider the equations $AYw = Xw$ and $AYv = Xv$, i.e.,

$$Yw - Xw = w(w^tYw)/4 \text{ and } Yv - Xv = w(w^tYv)/4.$$

Clearly, we must have $w^tYv = 0$. Thus, we are studying the (II.iii) matrices of $\mathcal{E}_7$ in §7.1. First, if $w_i \in \{w, v\}^\perp$ such that all entries of w_i are ± 1, then the last two entries of w_i have the same sign, and 3 of the first six entries equal to 1. So, there are $2 \cdot \binom{6}{3} = 40$ possibilities. For each of the 40 choices of $w_i \in \{w, v\}^\perp$, where all entries of w_i are ± 1, we have a fixed $P_i \in \mathbf{D}_8$ such that $P_iw = w_i$, $Y = P_i\hat{Y}$. There are 8! choices of such $\hat{Y}$, and for a fixed $\hat{Y}$ there are 6!2! choices of $\hat{X} \in \mathbf{D}_8$ so that $\hat{X}w = w$ and $P\hat{X}v = P\hat{Y}v$. We have to factor out the 8! so many (R, S) pairs such that $R(P_i^tAP_i)S = P_i^tAP_i$ with $Sw = w = R^tw$. Thus, there are 6!2! so many X^tAY corresponding to each choice of w_i. However, for each w_i, the 6!2! matrices X^tAY are the same as those corresponding to $-w_i$. Thus, we have $20(6!2!) = 6!40$ matrices of $\mathbf{E}_6$ in this class. And we also see that they are equivalent to matrices of the form $YDAD$ where Y is a matrix of type (I) and D is a diagonal orthogonal matrix such that $Dw \in w^\perp$ and $(D, E_{77}) = -(D, E_{88})$.

(III) The 6!30 matrices of the form X^tBY satisfying $X^tBYw = w$ and $X^tBYv = v$, where $X, Y \in \mathbf{D}_8$ and $B = B_1 \oplus B_2 = A(-I_4 \oplus I_4)A$, where

$$B_1 = (1,1,1,1)^t(1,1,1,1)/2 - I_4, \quad B_2 = I_4 - (1,1,1,-1)^t(1,1,1,-1)/2.$$

The counting is done as follows. In order to have $X^tBYw = w$ and $X^tBYv = v$, the 2×2 submatrix of X^tBY at the right bottom corner cannot contain zero entries. Thus, we have to choose from the first 6 rows and the first 6 columns a 4×4 submatrix to accommodate an $X_1B_1Y_1$ as described in $\mathbf{E}_7$, and there are $\binom{6}{4}^2 4!$ ways and there are 4 ways to fixed the matrix $X_2B_2Y_2$. Thus there are $\binom{6}{4}^2 4!4 = 6!30$ matrices in this case.

8.2 Maximum inner product

Let $X \in \mathcal{E}_6$ with $X \neq I_8$. Then $(I_8, X) \leq 6$. Therefore the inner product on the irreducible subspace $\mathbf{E}_6$ is bounded by 4. Using the matrix realization in $M_8(\mathbb{R})$ and by GP2 in Section 2, for each possible value of $r = (I_8, X)$, define

$$\mathcal{S}_r = \{X \in \mathcal{E}_6 : (I_8, X) = r\}. \tag{24}$$

Note that $\mathcal{S}_6$ consists of matrices in one of the following two forms.

(a) The 16 matrices of the form $X_{ij} = I_8 - (e_i - e_j)(e_i - e_j)^t$ for some $1 \leq i < j \leq 6$ and $X_{78} = I_8 - (e_7 + e_8)(e_7 + e_8)^t$.

(b) The 20 matrices of the form $X = DAD$ where D is an orthogonal diagonal matrix such that $Dw \in w^\perp$ and $(D, E_{77}) = -(D, E_{88})$.

8.3 Linear Preservers

Theorem 8.1 *A linear operator $\psi : M_6(\mathbb{R}) \to M_6(\mathbb{R})$ satisfies $\psi(\mathbf{E}_6) = \mathbf{E}_6$ if and only if there exist $P, Q \in \mathbf{E}_6$ such that ψ has the form*

$$X \mapsto PXQ \quad \text{or} \quad X \mapsto PX^tQ.$$

Consequently, $N(\mathbf{E}_6) = \mathbf{E}_6$.

Proof. The assertion on $N(\mathbf{E}_6)$ follows from GP4 in Section 2. The $(\Leftarrow)$ part of the first assertion is clear. We consider the $(\Rightarrow)$ part. Let $\psi : M_6(\mathbb{R}) \to M_6(\mathbb{R})$ be a linear map satisfying $\psi(\mathbf{E}_6) = \mathbf{E}_6$. By Proposition 2.1, if ψ preserves $\mathbf{E}_6$, then ψ preserves the inner product $(\hat{X}, \hat{Y}) = \operatorname{tr}(\hat{X}\hat{Y}^t)$ on $M_6(\mathbb{R})$. Also, by GP3 in Section 2, we may assume that $\psi(I_6) = I_6$.

Let V_6 be the affine space generated by $\mathcal{E}_6$, and let U be an orthogonal matrix establishing the correspondence between $\mathcal{E}_6$ and $\mathbf{E}_6$ as described in (22). Consider an affine map $\phi : V_6 \to V_6$ defined by

$$\phi\left(U\begin{pmatrix} I_2 & 0 \\ 0 & \hat{X}\end{pmatrix}U^t\right) = U\begin{pmatrix} 0_2 & 0 \\ 0 & \psi(\hat{X})\end{pmatrix}U^t + U\begin{pmatrix} I_2 & 0 \\ 0 & 0_6\end{pmatrix}U^t.$$

Then $\phi(\mathcal{E}_6) = \mathcal{E}_6$. Since ψ preserves inner product in $M_6(\mathbb{R})$, we have $(\phi(X), \phi(Y)) = (X, Y)$ for all $X, Y \in \mathcal{E}_6$ by (23). Define $\mathcal{S}_r$ as in (24). Since $\psi(I_6) = I_6$, therefore, $\phi(I_8) = I_8$ and by GP3 in Section 2, $\phi(\mathcal{S}_r) = \mathcal{S}_r$ for each r. In the following, we will show that for some $P \in \mathcal{E}_6$, ϕ has the form

$$X \mapsto P^tXP \text{ for all } X \in \mathcal{E}_6 \qquad \text{or} \qquad X \mapsto P^tX^tP \text{ for all } X \in \mathcal{E}_6.$$

We shall use the matrices A and X_{ij} as defined in §8.1 − 8.2. Also, we shall use the classification of matrices in $\mathcal{S}_6$ into types (a) and (b) as defined in §8.2.

Define $D_i = I_8 - 2E_{ii}$, and $D_{ij} = D_i D_j$. Note that those D described in (b) will be of the form $D_{ijk7} = D_{ij}D_{k7}$ where $i, j, k \neq 7$ are all distinct. If i', j', k' are such that $\{i, i', j, j', k, k'\} = \{1, ..., 6\}$, then

$$D_{ijk7}AD_{ijk7} = D_{i'j'k'8}AD_{i'j'k'8}.$$

Also, for i, j, k distinct, $i, j, k < 7$ and X_{jk}, X_{ik} and X_{ij} all of type (a),

$$X_{ik}X_{jk}X_{ik}^t = X_{ij}.$$

Let $\phi(X_{78}) = Z$. If $Z = X_{78}$ then we are done. If $Z = DAD$, where $D = D_{ijk7}$, then replace ϕ by the mapping $X \mapsto Q\phi(X)Q^t$ where $Q = D_{ijk8}AD_{ijk8}$. And so $\phi(X_{78}) = X_{78}$. If, on the other hand, $Z = X_{ij}$ for $i < j < 7$, then replace ϕ by the mapping $X \mapsto Q\phi(X)Q^t$ where $Q = D_{ikl7}AD_{ikl7}$ for $k, l \neq i, j, 7, 8$. Thus $\phi(X_{87}) = DAD$ where $D = D_{jkl7}$, and this case has already been covered. Therefore, $\phi(X_{78}) = X_{78}$.

Note that if $X \in \mathcal{S}_6$ is of type (a), then $(X, X_{78}) = 4$, while if X is of type (b), then $(X, X_{78}) = 5$. Thus, those $X \in \mathcal{S}_6$ that are of type (a) are mapped to themselves, and those of type (b) are mapped to themselves.

We may assume that $\phi(X_{56}) = X_{56}$. Otherwise, $\phi(X_{56}) = X_{ij}$ where $(i, j) \neq (5, 6)$ and $i < j \leq 6$. Then replace ϕ by the mapping $X \mapsto P\phi(X)P^t$ where

$$P = \begin{cases} X_{i5} & \text{if } j = 6, \\ X_{i6} & \text{if } j = 5, \\ X_{i5}X_{i6} & \text{if } i < j < 5. \end{cases}$$

Now consider those $X \in \mathcal{S}_6$ of type (a) such that $(X, X_{56}) = 5$. Then $X = X_{ij}$ where $i < j$ and $j \in \{5, 6\}$. We may assume that $\phi(X_{45}) = X_{45}$. Otherwise $\phi(X_{45}) = X_{ij}$ where $j \in \{5, 6\}$. If $j = 5$, then replace ϕ by the mapping $X \mapsto X_{i4}\phi(X)X_{i4}$. If $j = 6$, then replace ϕ by the mapping $X \mapsto X_{56}\phi(X)X_{56}$ and so we are back to the case where $j = 5$.

Now consider those $X \in \mathcal{S}_6$ of type (a) such that $(X, X_{56}) = 4$ and $(X, X_{45}) = 5$. They must be of the form X_{i4}. We may assume that $\phi(X_{34}) = X_{34}$. Otherwise it equals X_{i4} for $i \in \{1, 2\}$, in which case, replace ϕ by the mapping $X \mapsto X_{i3}\phi(X)X_{i3}$.

Now consider those $X \in \mathcal{S}_6$ of type (a) such that $(X, X_{56}) = 4$, $(X, X_{45}) = 4$ and $(X, X_{34}) = 5$. Then $X = X_{13}$ or X_{23}. If $\phi(X_{23}) = X_{13}$, then replace ϕ by the mapping $X \mapsto X_{12}\phi(X)X_{12}$. And thus, $\phi(X_{23}) = X_{23}$. By considering the inner products, we also see that $\phi(X_{12}) = X_{12}$. Thus $\phi(Z) = Z$ for $Z = I_8, X_{78}, X_{56}, X_{45}, X_{34}, X_{23}$ andX_{12}. This is sufficient to show that $\phi(X_{ij}) = X_{ij}$ whenever $i < j \leq 6$ and that $\phi(D_{ijk7}AD_{ijk7}) = D_{ijk7}AD_{ijk7}$ or $D_{ijk8}AD_{ijk8}$.

Now consider those $X \in \mathcal{S}_6$ that are of type (b). In particular, consider $\phi(D_{4567}AD_{4567}) = Z$. If $Z = D_{4567}AD_{4567}$, then we are done. If $Z = D_{4568}AD_{4568}$, then replace ϕ by the mapping $X \mapsto X_{78}\phi(X)X_{78}$. It can be shown (see the last section) that

$$(D_{ijk7}AD_{ijk7}, D_{4567}AD_{4567}) \neq (D_{ijk8}AD_{ijk8}, D_{4567}AD_{4567}).$$

Thus, for all $X \in \mathcal{S}_6$, $\phi(X) = X$.

Let $X_{ijk} = X_{ij}X_{ik}$. Then X_{ijk} is the type (I) matrix as defined in §8.1 with the following principal submatrices.

$$X_{ijk}(i,j,k) = I_5, \qquad X_{ijk}[i,j,k] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

In a manner similar to that of section 7.3, we consider those matrices $X \in \mathcal{S}_5$ such that $(X, X_{ij}) = 6$, $(X, X_{ik}) = 6$ and $(X, X_{jk}) = 6$ for $i < j < k$. Then $X = X_{ijk}$ or X_{ijk}^t. If $\phi(X_{123}) = X_{123}^t$, then replace ϕ by the mapping $X \mapsto \phi(X)^t$. Thus $\phi(X_{123}) = X_{123}$. Note that $(X_{ijk}, X_{123}) = 5$ if and only if $i + 1 = j < 3 < k \leq 6$. But if $(X_{ijk}, X_{123}) = 5$, then $(X_{ijk}, X_{123}^t) = 4$. So $\phi(X_{12k}) = X_{12k}$ for $k = 4, 5, 6$. Using these newly fixed matrices, continue in the same manner until $\phi(X_{ijk}) = X_{ijk}$ for all X_{ijk} such that $1 \leq i < j < k \leq 6$.

Note that for any $X \in \mathbf{E}_6$, if $Y \in \mathbf{E}_6$ and $(Y, X) = 6$, then $Y = XZ$ for some $Z \in \mathcal{S}_6$. Thus, for all $i < j < k \leq 6$, if Y is such that $(Y, X_{ijk}) = 6$ and $(Y, X_{78}) = 6$, then $Y = X_{ijk}X_{78}$. Thus, $\phi(X_{ijk}X_{78}) = X_{ijk}X_{78}$ for all $i < j < k \leq 6$. In particular, let $Y = X_{123}X_{78}$, and consider those X such that $(YX, I_8) = 6$ and $(YX, X_{78}) = 5$. They will be of the form $X = YDAD$ where $D = D_{ijk7}$ and $i < j < k \leq 6$. For each such X, define

$$f(X) = [(X_{12}, X), ..., (X_{56}, X), (D_{1237}AD_{1237}, X), ..., (D_{4567}AD_{4567}, X)].$$

One can show that $f(X) \neq f(Z)$ whenever $X \neq Z$ where X and Z are both of the form $YDAD$ with $D \neq D_{1237}$, D_{1238}; see the last section.

These 18 matrices together with those $X \in \mathcal{S}_6$ and those matrices of the form X_{ijk} for $i < j < k \leq 6$ all have the property that $\phi(X) = X$. It can be shown that there are 37 linearly independent matrices among this group; see the last section. Given this, and the fact that

$$\phi(0) = U^* \begin{pmatrix} I_2 & 0 \\ 0 & 0_6 \end{pmatrix} U,$$

we see that

$$\phi(X) - \phi(0)$$

is completely determined. In particular, $\phi(X) = X$ for all $X \in \mathcal{E}_6$. It follows that the original affine map ϕ on V_6 has the form

$$X \mapsto P^tXP \qquad \text{or} \qquad X \mapsto P^tX^tP$$

for some $P \in \mathcal{E}_6$. Note that if $P, X \in \mathcal{E}_6$, there exists $\hat{P}, \hat{X} \in \mathbf{E}_6$ such that

$$P^tXP = U^* \begin{pmatrix} 0_2 & 0 \\ 0 & \hat{P}^t\hat{X}\hat{P} \end{pmatrix} U + U^* \begin{pmatrix} I_2 & 0 \\ 0 & 0_6 \end{pmatrix} U.$$

Thus, there exists a $\hat{P} \in \mathbf{E}_6$ such that

$$\psi(\hat{X}) = \hat{P}^t\hat{X}\hat{P} \text{ for all } \hat{X} \in \mathbf{E}_6 \qquad \text{or} \qquad \psi(\hat{X}) = \hat{P}^t\hat{X}^t\hat{P} \text{ for all } \hat{X} \in \mathbf{E}_6.$$

Since $\mathbf{E}_6$ spans $M_6(\mathbb{R})$, ψ on $M_6(\mathbb{R})$ has the desired form. $\square$

As in the case of $\mathbf{E}_7$, the above proof would also show a similar result if we replace the linear map ψ on $M_6(\mathbb{R})$ satisfying $\psi(\mathbf{E}_6) = \mathbf{E}_6$ with either an affine map $\phi : V_6 \to V_6$ or a linear map $\hat{\phi}:$ span $\mathcal{E}_6 \to$ span $\mathcal{E}_6$ satisfying $\phi(\mathcal{E}_6) = \mathcal{E}_6$ and preserving inner product on V_6.

9 Matlab Programs

Matlab Program for $\mathbf{H}_3$

In the proof of the linear preserver of $\mathbf{H}_3$, we stated that 12 matrices

$$D_1, D_2, D_3, H, X_1, \ldots, X_8$$

span $M_3(\mathbb{R})$. We put these 12 matrices as row vectors of the matrix "R". The rank command will then show that there are 9 linearly independent vectors among these 12 matrices.

```
a=(1+sqrt(5))/4;b=(-1+sqrt(5))/4;c=1/2;
R=[-1 0 0 0 1 0  0 0 1; 1 0 0  0 -1 0 0 0 1;
  1 0 0 0 1 0  0 0 -1; a b c  b c -a  c -a -b;
 -a -b -c   b  c -a   c -a -b; -a b  c   -b  c -a   -c -a -b;
 -a -b  c   b  c  a   -c  a -b; -a b -c  -b  c  a    c  a -b;
  a  b  c  -b -c  a   c -a -b;  a -b c   b -c -a    c  a -b;
  a  b -c  -b -c -a  -c  a -b;  a -b -c  b -c  a   -c -a -b];
rank(R)
```

Matlab Program for $\mathbf{H}_4$

In the proof of the linear preserver of $\mathbf{H}_4$, we stated that 24 specific matrices could be shown to span $M_4(\mathbb{R})$. We put these twelve matrices in row vector form stared in "R". The rank command will then show that there are 16 linearly independent vectors among these 24 matrices.

```
a=(1+sqrt(5))/4;b=(-1+sqrt(5))/4;c=1/2;
D(:,:,1)=[-1 0 0 0; 0  1 0 0; 0 0  1 0; 0 0 0  1];
D(:,:,2)=[ 1 0 0 0; 0 -1 0 0; 0 0  1 0; 0 0 0  1];
D(:,:,3)=[ 1 0 0 0; 0  1 0 0; 0 0 -1 0; 0 0 0  1];
D(:,:,4)=[ 1 0 0 0; 0  1 0 0; 0 0  1 0; 0 0 0 -1];
B(:,:,1)=[1 0 0 0; 0 a b c; 0 b c -a;  0  c -a -b];
B(:,:,2)=[a b 0  c; b c 0 -a; 0 0 1  0; c -a  0 -b];
B(:,:,3)=[c 0 b -a; 0 1 0  0; b 0 a  c; -a 0  c -b];
B(:,:,4)=D(:,:,3)*B(:,:,1); B(:,:,5)=D(:,:,2)*B(:,:,2);
B(:,:,6)=D(:,:,1)*B(:,:,3); k=0; for i=1:4
  for j=1:6
    k=k+1; y=D(:,:,i)*B(:,:,j)*D(:,:,i);
    R(k,:)=[y(1,:) y(2,:) y(3,:) y(4,:)];
```

```
  end
end
rank(R)
```

Matlab Program for $\mathbf{F}_4$

In the proof of the linear preserver of $\mathbf{F}_4$, we stated that we could show that the 16 matrices of the form DA_iD for $i = 1, 2$ and $D = \text{diag}(1, \pm1, \pm1, \pm1)$ were mapped to themselves by comparing the inner products of these matrices with those already fixed by ϕ. Below follows the MATLAB code comparing the inner products of these 16 matrices with those of C_i for $i = 1, 3, 5$ and 6. A simple comparison of the inner products will verify that these matrices must indeed be mapped to themselves. We put the 16 matrices in row vector form, storing them in 'y'. The other matrices are also on row vector form, stored in 'x'. Finally, we use the 'rank' command to show that there are 16 linearly independent matrices among the 26 listed.

```
e1=[1 0 0 0];e2=[0 1 0 0];e3=[0 0 1 0];e4=[0 0 0 1];
D=[1 1 1  1; 1 -1 1  1; 1  1 -1  1; 1 -1 -1 1;
   1 1 1 -1; 1 1 -1 -1; 1 -1 1 -1; 1 -1 -1 -1];
A(:,:,1)=eye(4)-ones(4)/2;
A(:,:,2)=[1 1 -1 -1;1 1 1 1;1 -1 1 -1;1 -1 -1 1]/2;
k=0;
for j=1:8
  for i=1:2
    k=k+1;B=diag(D(j,:))*A(:,:,i)*diag(D(j,:));
    y(k,:)=[B(1,:) B(2,:) B(3,:) B(4,:)];
  end
end
x=[-e1 e2 e3 e4; e1 -e2  e3 e4; e1  e2 -e3 e4; e1 e2 e3 -e4;
    e1 e2 e4 e3; e1  e2 -e4 e3; e2  e1  e3 e4;
   -e2 e1 e3 e4; e1  e3  e2 e4; e1 -e3  e2 e4];
y*[x(5,:);x(7,:);x(9:10,:)]';
rank([x;y])
```

Matlab Program for $\mathbf{E}_8$

In the proof of the linear preserver of $\mathbf{E}_8$, we showed that $\phi(X) = X$ for

$$X = I_8, Q_1 = D_{18}AD_{18} \text{ and } X_{i(i+1)} \text{ for } i = 1, ..., 7.$$

We stated that by comparing the inner product of these matrices with the rest of the elements in $\mathcal{S}_6$, that we could show that $\phi(X) = X$ for all $X \in \mathcal{S}_6$. We store those X that are fixed in row vector form in "rset" and store matrices of the forms DAD, X_{ij} and Y_{ij} in row vector forms in "rA8", "rX8" and "rY8" respectively. Direct comparison of the inner product shows that each matrix must be fixed. We also stated that the matrices of the forms

$$I_8,\ DAD,\ X_{ij},\ Y_{ij},\ X_{ijk}, \text{ and } PDAD,$$

as defined in section 3.3, could be shown to span $M_8(\mathbb{R})$. We form these matrices and put them in row vector form stored in "rI", "rA8", "rX8", "rY8", "rP8" and 'rPA8" respectively. The rank command will then show that there are 64 linearly independent vectors among these matrices.

```
d2=[1 1 1 1 1 -1 -1 1;1 1 1 1 -1 1 -1 1;1 1 1 -1 1 1 -1 1;1 1 -1 1 1 1 -1 1;
    1 -1 1 1 1 1 -1 1;-1 1 1 1 1 1 -1 1;1 1 1 1 -1 -1 1 1;1 1 1 -1 1 -1 1 1;
    1 1 -1 1 1 -1 1 1;1 -1 1 1 1 -1 1 1;-1 1 1 1 1 -1 1 1;1 1 1 -1 -1 1 1 1;
    1 1 -1 1 -1 1 1 1;1 -1 1 1 -1 1 1 1;-1 1 1 1 -1 1 1 1;1 1 -1 -1 1 1 1 1;
    1 -1 1 -1 1 1 1 1;-1 1 1 -1 1 1 1 1;1 -1 -1 1 1 1 1 1;-1 1 -1 1 1 1 1 1;
    -1 -1 1 1 1 1 1 1];
d4=[1 1 1 -1 -1 -1 -1 1; -1 -1 -1 1 1 1 -1 1; 1 1 -1 1 -1 -1 -1 1;
    -1 -1 1 -1 1 1 -1 1; 1 -1 1 1 -1 -1 -1 1; -1 1 -1 -1 1 1 -1 1;
    -1 1 1 1 -1 -1 -1 1; 1 -1 -1 -1 1 1 -1 1; 1 1 -1 -1 1 -1 -1 1;
    -1 -1 1 1 -1 1 -1 1; 1 -1 1 -1 1 -1 -1 1; -1 1 -1 1 -1 1 -1 1;
    -1 1 1 -1 1 -1 -1 1; 1 -1 -1 1 -1 1 -1 1; 1 -1 -1 1 1 -1 -1 1;
    -1 1 1 -1 -1 1 -1 1; -1 1 -1 1 1 -1 -1 1; 1 -1 1 -1 -1 1 -1 1;
    -1 -1 1 1 1 -1 -1 1; 1 1 -1 -1 -1 1 -1 1; 1 1 -1 -1 -1 -1 1 1;
    1 -1 1 -1 -1 -1 1 1; -1 1 1 -1 -1 -1 1 1; 1 -1 -1 1 -1 -1 1 1;
    -1 1 -1 1 -1 -1 1 1; -1 -1 1 1 -1 -1 1 1; 1 -1 -1 -1 1 -1 1 1;
    -1 1 -1 -1 1 -1 1 1; -1 -1 1 -1 1 -1 1 1; -1 -1 -1 1 1 -1 1 1;
    1 -1 -1 -1 -1 1 1 1; -1 1 -1 -1 -1 1 1 1; -1 -1 1 -1 -1 1 1 1;
    -1 -1 -1 1 -1 1 1 1; -1 -1 -1 -1 1 1 1 1];
d6=[1 -1 -1 -1 -1 -1 -1 1; -1 1 -1 -1 -1 -1 -1 1; -1 -1 1 -1 -1 -1 -1 1;
    -1 -1 -1 1 -1 -1 -1 1; -1 -1 -1 -1 1 -1 -1 1; -1 -1 -1 -1 -1 1 -1 1;
    -1 -1 -1 -1 -1 -1 1 1];
d_8=[d6;d4;d2;ones(1,8)];w=[1 1 1 1 1 1 1 -1]'; A=eye(8) - w*w'/4;
for i=1:64;
  a=diag(d_8(i,:))*A*diag(d_8(i,:)); A_8(:,:,i)=a;
  rA8(i,:)=[a(1,:) a(2,:) a(3,:) a(4,:) a(5,:) a(6,:) a(7,:) a(8,:)];
end
k=0;
for j=2:8
  for i=1:(j-1);
    a=eye(8); a(i,i)=0; a(j,j)=0;
    b=zeros(8); b(i,j)=1; b(j,i)=1;
    X=a+b; Y=a-b; k=k+1;
    rX8(k,:)=[X(1,:) X(2,:) X(3,:) X(4,:) X(5,:) X(6,:) X(7,:) X(8,:)];
    rY8(k,:)=[Y(1,:) Y(2,:) Y(3,:) Y(4,:) Y(5,:) Y(6,:) Y(7,:) Y(8,:)];
  end
end
k=0;
for m=3:8;
  for j=2:(m-1)
    for i=1:(j-1)
```

```
      k=k+1;P=eye(8);P(i,i)=0;P(j,j)=0;P(m,m)=0;P(i,j)=1;P(j,m)=1;P(m,i)=1;
      rP8(k,:)=[P(1,:) P(2,:) P(3,:) P(4,:) P(5,:) P(6,:) P(7,:) P(8,:)];
      P_8(:,:,k)=P;
    end
  end
end
p=[zeros(1,7) 1; eye(7) zeros(7,1)];
for i=1:64
  P=p*A_8(:,:,i);
  rPA8(i,:)=[P(1,:) P(2,:) P(3,:) P(4,:) P(5,:) P(6,:) P(7,:) P(8,:)];
end
a=[1 zeros(1,8)]; rI=[a a a a a a a 1];
rset=[rI;rA8(1,:);rX8(1,:);rX8(3,:);rX8(6,:);rX8(10,:);
      rX8(15,:);rX8(21,:);rX8(28,:)];
ip=[rA8;rX8;rY8]*rset'
rank([rI;rA8;rX8;rY8;rP8;rPA8])
```

Matlab Program for $\mathbf{E}_7$

In the proof of the linear preserver of $\mathbf{E}_7$, we stated that we could show that the matrices of the form DAD and X_{ij}, both in $\mathcal{S}_6$ were mapped to themselves by comparing the inner products of these matrices with those already fixed by ϕ. Below follows the MATLAB code comparing the inner products of these 63 matrices with those of $X_{i(i+1}$ for $i = 1, 2, 4, 5, 6, 7$. A simple comparison of the inner products verifies that these matrices must indeed be mapped to themselves. Since the matrix realizations used for $\mathbf{E}_7$ form a subset of those used for $\mathbf{E}_8$, we use the matrices previously defined in for $\mathbf{E}_8$. We put the 63 matrices in row vector form, storing them in "rA7" and "rX8" respectively. The other matrices are also on row vector form, stored in "rset". We also stated that these matrices together with I_8 and matrices of the form X_{ijk} as defined in section 3.3, could be shown to span the 50 dimensional subspace of $M_8(\mathbb{R})$. We store these new matrices in row vector form in "rI" and "rP7"respectively. The rank command will then show that there are 50 linearly independent vectors among these matrices.

```
rA7=rA8(8:42,:); rX7=[rX8(1:21,:);rY8(22:28)]; rP7=rP8(1:35,:);
P_7=P_8(:,:,36:56);
for i=36:56
  P=diag([1,1,1,1,1,1,1,-1])*P_8(:,:,i)*diag([1,1,1,1,1,1,1,-1]);
  rP7(i,:)=[P(1,:) P(2,:) P(3,:) P(4,:) P(5,:) P(6,:) P(7,:) P(8,:)];
end
rset=[rI;rX7(1,:);rX7(3,:);rX7(10,:);rX7(15,:);rX7(21,:);rX7(28,:)];
ip=[rA7;rX7]*rset'
rank([rI;rA7;rX7;rP7])
```

Matlab Program for $\mathbf{E}_6$

In the proof of the linear preserver of $\mathbf{E}_6$, we stated that we could show that the matrices of the form X_{ij} were mapped to themselves and matrices of the form DAD were mapped to themselves or to $\hat{D}A\hat{A}$ (for D and $\hat{D}$ of particular forms) by comparing the inner products of these matrices with those already fixed by ϕ. Below follows the MATLAB code comparing the inner products of these matrices with those of $X_{i(i+1}$ for $i = 1, 2, 3, 4, 5, 7$. A simple comparison of the inner products will verify that these matrices must indeed be mapped to themselves. Since the matrix realizations used for $\mathbf{E}_7$ form a subset of those used for $\mathbf{E}_8$, we use the matrices previously defined in for $\mathbf{E}_8$. We put the matrices in row vector form, storing them in "rX6" and "rA6" respectively. The fixed matrices are also in row vector form, stored in "rset". Next, we fixed one of these matrices of the form DAD and compare inner products of the remaining with those fixed, whose row vectors are once again stored on "rset'. Comparison will once again verify that all matrices of the form DAD are mapped to themselves. We store the matrices of the form $PDAD$ (as defined in section 5.3) in row vector in "rZA6". Comparing inner products with those already fixed (whose row vector forms are once again stored in "rset"), shows that these matrices must be mapped to themselves. Finally, we stated that these matrices together with I_8 and matrices of the form X_{ijk} as defined in section 5.3, could be shown to span the 37 dimensional subspace of $M_8(\mathbb{R})$. We store these new matrices in row vector form in "rI" and "rP6"respectively. The rank command will then show that there are 37 linearly independent vectors among these matrices.

```
A_6=A_8(:,:,8:27); rA6=rA8(8:27,:); rX6=rX8(1:15,:);
Y=[eye(6) zeros(6,2);zeros(2,6) eye(2)-ones(2)]; rY=rY8(28,:);
rP6=rP8(1:20,:);
rset=[rI;rX6(1,:);rX6(3,:);rX6(6,:);rX6(10,:);rX6(15,:);rY];
ip=[rA6;rX6]*rset'
rset=[rI;rX6;rY;rA6(1,:)];
ip=[rA6]*rset'
z=P_6(:,:,1,2,3)*Y;
for i=1:20
  Z=z*A_6(:,:,i);
  rZA6(i,:)=[Z(1,:) Z(2,:) Z(3,:) Z(4,:) Z(5,:) Z(6,:) Z(7,:) Z(8,:)];
end
rset=[rI;rX6;rY;rA6];
ip=[rZA6]*rset'
rank([rI;rA6;rX6;rY;rP6;rZA6(3:20,:)])
```

Note added in proof.

Our results are on invertible linear preservers of reflection groups. A natural related problem is to characterize group automorphisms of reflection groups. By private communication, Professor Robert Guralnick informed us that such characterizations can be obtained readily from results in group theory.

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