Central Groupoids, Central Digraphs, and Zero-One Matrices A Satisfying $A^2 = J$ *

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1 Introduction

A directed graph with n vertices is called a central directed graph (central digraph) if every two vertices are connected by a unique length 2 walk. (There may be walks of other lengths.) In terms of the adjacency matrix A of the graph, this happens precisely when $A^2 = J_n$, where J_n is the $n \times n$ matrix of all ones. These two concepts are also related to the algebraic structure known as a central groupoid, which is a non-empty set with n elements and a binary operation $*$ such that

$$(x * y) * (y * z) = y$$

for any elements x, y, z in the set. One can establish the correspondence between a central digraph and a central groupoid as follows. Identify the vertices of the graph with the elements in the groupoid so that $i * j = k$ if and only if $i - k - j$ is the unique length two path from i to j.

Central digraphs, central groupoids and the matrix equation $A^2 = J$ have attracted the attention of many researchers in different areas because of their very rich and beautiful algebraic and combinatorial structures, and their connection to many pure and applied problems; e.g., see [5, 6, 7, 8] and their references.

In [5], Hoffman raised the question of characterizing all such matrices, which seems to be a very difficult problem. (See that references mentioned above.) Nevertheless, many interesting properties and techniques have been discovered. The purpose of this paper is to obtain more results on these subjects and to further develop techniques and insights in studying these concepts. In particular, we shall use approaches from algebra, combinatorial theory, matrix theory, and scientific computations. It is our hope that we can increase the collection of tools to study these concepts. We shall also mention many open problems.

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In our discussion, we shall denote by $\mathcal{A}_n$ the set of $n \times n$ zero-one matrices A satisfying $A^2 = J_n$.

2 Basic Properties

The properties in the following propositon are well-known. We include a short proof for the sake of completeness.

Proposition 2.1 *If $A \in \mathcal{A}_n$, then*

(a) $n = k^2$ *for some integer* k,

(b) *all row sums and column sums equal* k,

(c) A *has eigenvalues* $k, 0, \ldots, 0$.

(d) A *has exactly* k *1's on its main diagonal.*

Proof. First, note that if A has row sums $r_1, \ldots, r_n$ and column sums $c_1, \ldots, c_n$, then $A^3 = AA^2 = AJ = (r_1, \ldots, r_n)^t(1, \ldots, 1)$ and $A^3 = A^2A = JA = (1, \ldots, 1)^t(c_1, \cdots, c_n)$. Thus, all row sums and column sums are the same, say, equal to m. In particular, A has Perron root m with a positive (Perron) eigenvector $(1, \ldots, 1)^t$.

Now, since A^2 has eigenvalues $n, 0, \ldots, 0$, A has eigenvalues $\sqrt{n}, 0, \ldots, 0$, and hence $m = \sqrt{n}$ is the row sum of A. Since the trace of a matrix is equal to the sum of its eigenvalues, A has exactly k 1's on its main diagonal. □

We combine the results in [6] and [7] to get the following statement.

Proposition 2.2 *The matrices in $\mathcal{A}_n$ have the following Jordan forms (all attainable):*

$$[\sqrt{n}] \oplus \underbrace{B \oplus \cdots \oplus B}_{p} \oplus 0_{n-2p-1} \qquad \textit{with} \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Consequently, if $A \in \mathcal{A}_n$, then A

$$\sqrt{n} \leq \operatorname{rank}(A) \leq (n+1)/2.$$

The first equality holds if and only if A is permutationally similar to the standard matrix, which is an $\sqrt{n} \times \sqrt{n}$ block matrix (A_{ij}) in which each block A_{ij} is a $\sqrt{n} \times \sqrt{n}$ matrix all of whose entries are zero except for the entries in row j which are all ones.

Proof. Since A^2 is diagonalizable with eigenvalues $n, 0, \ldots, 0$, the Jordan blocks of A corresponding to the eigenvalue 0 have sizes at most 2. Thus, the rank of A is at most $(n+1)/2$. By the result in [7], all such ranks can be attained, and thus the prescribed Jordan structure can be attained.

Now, to prove the last assertion, note that AA^t has eigenvalues $n = \lambda_1 \geq \cdots \geq \lambda_n \geq 0$, and $\mathrm{tr}\,(AA^t) = k^3$. It follows that AA^t has at least k positive eigenvalues. Moreover, if there are exactly k positive eigenvalues, each of them is k^2 and thus A is permutationally similar to the standard matrix. □

Open problems in this area are to characterize maximal rank matrices in $\mathcal{A}_n$ and to invent new construction methods for matrices of all ranks.

The following observation, which can be readily verified, was made in [3] (see also [4]). We state it in terms of matrix language. Suppose $(i_1, \cdots, i_k)$ and $(j_1, \ldots, j_k)$ are subsequences of $(1, \ldots, n)$. Denote by $A[i_1, \ldots, i_k; j_1, \ldots, j_k]$ the submatrix of A lying in rows $i_1, \ldots, i_k$, and columns $j_1, \ldots, j_k$.

Proposition 2.3 *Let $A \in \mathcal{A}_n$, $1 \leq p < q \leq n$, and $1 \leq r < s \leq n$ satisfy*

$$A[p, q; r, s] \in \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Suppose $\tilde{A}$ is obtained from A by replacing $A[p, q; r, s]$ with $J_2 - A[p, q; r, s]$. Then $\tilde{A} \in \mathcal{A}_n$ if and only if the rth and sth rows of A are the same and the pth and qth columns of A are the same.

If $\tilde{A} \in \mathcal{A}_n$ is obtained from A by a change described in the above proposition, we say that $\tilde{A}$ is obtained from A by a "switch". The following conjecture was mentioned in [3] (see also [4]).

Conjecture: Every $A \in \mathcal{A}_n$ can be obtained from the standard matrix by a finite number of switches.

We also have the following observation.

Proposition 2.4 *Suppose $n = k^2 \geq 16$. Then up to permutation similarity, there are four matrices obtained from the standard matrix by applying one switch.*

Proof. Let $A \in \mathcal{A}_n$ be the standard matrix. Assume the switch take place at $A[p, q; r, s]$. We consider two cases. First, row r or row s contains a nonzero diagonal entry. In this case, we may apply a permutation similarity and assume that $r = 1$ and $s = k + 1$. Since column p and column q are identical, there exists $m \in \{1, \ldots, k\}$ such that $(m - 1)k < p, q \leq mk$. Now, it is easy to check that $m \neq 1, 2$. If $m \geq 3$, then we may assume that $m = 3$; otherwise, apply a permutation similarity involving row and column indices at least $2k$ to A. Now, in order to have

$$A[p, q; r, s] \in \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

we see that $p = 2k + 1$ and $2k + 1 < q \leq 3k$. Up to permutation similarity, we may assume that $q = 2k + 2$. So, up to permutation similarity, there is only one matrix with the desired property in this case.

Case 2. Neither row r nor row s contains a nonzero diagonal entry. By permutation similarity, we may assume that $r = 2$ and $s = 2k + 2$. Since column p and column q are identical, there exists $m \in \{1, \ldots, k\}$ such that $(m-1)k < p, q \leq mk$. It is now easy to check that up to permutation similarity, there is one desired matrix with $m = 1$, say, with $(p, q) = (1, 3)$; one matrix with $m = 2$, say, with $(p, q) = (k + 1, k + 3)$; one matrix with $m \geq 3$, say, with $(m, p, q) = (3, 2k + 1, 2k + 2)$. So, up to permutation similarity, there are three matrices of the desired form in this case.

Combining, up to permutation similarity, there are four matrices that differ by one switch from the standard matrix. □

Open problems in this area ask how many matrices can be obtained by applying two switches and why are there repeated rows/columns for any matrix $A \in \mathcal{A}_n$.

We now consider properties of the directed graphs of matrices in $\mathcal{A}_n$.

Proposition 2.5 *Let $A \in \mathcal{A}_n$ and let $G(A)$ be the directed graph of A with vertices $v_1, \ldots, v_n$. Then n of the vertices in $G(A)$ have self loops, while all other vertices in $G(A)$ are paired in two-cycles, with no vertex belonging to more than one two-cycle.*

Proof. Since A has k 1's on its main diagonal, $G(A)$ has k self loops. No other vertex can be in a two-cycle with an idempotent vertex (that is, one with a self loop) since then there would be two length two walks from the idempotent vertex to itself. Since there must be a length two walk from each non-idempotent vertex to itself, the walk must be a two-cycle with two non-idempotent vertices. No vertex v_i can be in two two-cycles, say with v_j and v_k, since then there would be at least two length two walks from v_j to v_k. □

3 Row and Column Multiplicities

Let $A \in \mathcal{A}_n$. An $n \times n$ matrix A is said to have row multiplicities $m_1, m_2, ..., m_s$ if A has m_1 rows that are equal, m_2 other rows that are equal, etc., where $m_1 + m_2 + \ldots m_s = n$. Similarly we can define the column multiplicities of A.

It has been conjectured by R.R. Fletcher III (see [3], [4]) that for each A such that $A^2 = J$, the multisets of integers representing the column and row multiplicities of A are equal. Although the conjecture is true for $n \leq 9$, it fails for $n = 16$, as is shown by the matrix A_1, see Appendix A. Note that A_1 has row multiplicities {4,4,3,3,1,1} and column multiplicities {4,4,2,2,2,2}.

The validity of Fletcher's conjecture for $n = 9$ can be checked by inspection of the solution set (see [6]), but we present the following general proof that does not require previous knowledge of the unique 9×9 central groupoids.

Proposition 3.1 *For $n = 9$, the multisets of row and column multiplicities for any 0-1 matrix satisfying $A^2 = J$ are equal.*

Proof. First, note that there is a one-to-one correspondence between groups of rows of multiplicity 3 and groups of columns with multiplicity 3. Any group of three rows that are equivalent (corresponding to vertices v_1, v_2, and v_3 that each have edges to vertices v_4, v_5, and v_6) have 1's in the same three columns, inherently creating a set of three columns with 1's in the same three rows. Since row and column sums are all equal to 3, each set of rows of multiplicity 3 corresponds directly and uniquely to a set of columns with multiplicity 3.

Second, we consider sets of rows with multiplicity 2. Consider a set of three columns, call them c_x, c_y, and c_z, that all have 1's in the same two rows, call them r_u and r_v, to create the set of rows with multiplicity 2. Note that each c_i for $i \in \{x,y,z\}$ has one more 1 in some row other than r_u and r_v, and not all in the same third row (or else the rows and columns would have multiplicity 3). We propose that two of the c_i's must have 1's in the same row, without loss of generality suppose c_x and c_y have 1's in row r_w, creating a corresponding set of two columns $\{c_x,c_y\}$ that are equivalent. As such, the given set of rows with multiplicity 2 combined with a row of multiplicity 1 has a one-to-one correspondence with a set of columns with multiplicity 2 and a column with multiplicity 1.

To prove that two such columns must always exist, consider the following proof by contradiction. If two columns c_x and c_y do not exist as stated above, then the remaining 1's required to complete the column sums of c_x, c_y, and c_z must have 1's in three distinct rows not equal to r_u or r_v. We have two cases to consider, first when r_u and r_v contain an idempotent element, and second when they do not.

Case 1: Up to permutation simularity, we may assume that such a matrix will have the following structure:

$$\begin{pmatrix}
1 & 1 & 1 & * & * & * & * & * & * \\
0 & 0 & 0 & * & * & * & * & * & * \\
0 & 0 & 0 & * & * & * & * & * & * \\
1 & 1 & 1 & * & * & * & * & * & * \\
0 & 0 & 0 & * & * & * & * & * & * \\
0 & 0 & 0 & * & * & * & * & * & * \\
1 & 0 & 0 & * & * & * & * & * & * \\
0 & 1 & 0 & * & * & * & * & * & * \\
0 & 0 & 1 & * & * & * & * & * & *
\end{pmatrix}$$

Where $r_u = r_1$, $r_v = r_4$, $c_x = c_1$, $c_y = c_2$, $c_z = c_3$, and r_1 contains an idempotent element, that is, an index that contains a 1 on the main diagonal of the matrix. Consider columns c_7, c_8, and c_9, which must all have column sum equal to 3. Multiplication of columns c_1, c_2, or c_3 by rows r_1, r_4, r_7 already provide that we have the following matrix:

$$
\begin{pmatrix}
1 & 1 & 1 & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & * & * \\
0 & 0 & 0 & * & * & * & * & * & * \\
1 & 1 & 1 & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & * & * \\
0 & 0 & 0 & * & * & * & * & * & * \\
1 & 0 & 0 & * & * & * & 0 & 0 & 0 \\
0 & 1 & 0 & * & * & * & * & * & * \\
0 & 0 & 1 & * & * & * & * & * & *
\end{pmatrix}
$$

Which implies that the three 1's in the rightmost three columns can occur only in rows 2, 3, 5, 6, 8, or 9. Furthermore, any position in column c_7 that contains a 1, call it $a_{k,7}$, must have that $a_{k,1} = a_{k,4} = 0$, or else matrix A^2 will have $a_{k,7} \geq 2$. However, this implies that positions $a_{k,8} = a_{k,9} = 1$, or else matrix A^2 will have $a_{k,8} = 0$ or $a_{k,9} = 0$. In other words, $a_{k,7} = 1$ implies $a_{k,8} = a_{k,8} = 1$, and since rows 8 and 9 both already have a 1 in column c_2 or c_3, we know that k cannot equal 8 or 9 (or else a row with row sum ≥ 4 would be created. One of the rows with this property must be either row 5 or 6, up to permutation similarity, it may be assumed to be row 5, leaving the following matrix:

$$
\begin{pmatrix}
1 & 1 & 1 & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & * & * \\
0 & 0 & 0 & * & * & * & * & * & * \\
1 & 1 & 1 & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & 1 & 1 & 1 \\
0 & 0 & 0 & * & * & * & * & * & * \\
1 & 0 & 0 & * & * & * & 0 & 0 & 0 \\
0 & 1 & 0 & * & * & * & 0 & 0 & 0 \\
0 & 0 & 1 & * & * & * & 0 & 0 & 0
\end{pmatrix}
$$

Simply stated, no completion to the above matrix will yield a matrix A such that $A^2 = J$, due to the fact that the product r_5 by c_7 yields a 0 entry in A^2 (there will in fact be many zero entries). Therefore, if at least one row contains an idempotent element, a set of two equivalent rows cannot correspond to three distinct columns.

Case 2: We now consider the final case when the two equivalent rows do not contain an idempotent element. Without loss of generality, we begin with the following structure:

$$\begin{pmatrix} * & 1 & 1 & 1 & * & * & * & * & * \\ * & 0 & 1 & 0 & * & * & * & * & * \\ * & 0 & 0 & 1 & * & * & * & * & * \\ * & 1 & 0 & 0 & * & * & * & * & * \\ * & 1 & 1 & 1 & * & * & * & * & * \\ * & 0 & 0 & 0 & * & * & * & * & * \\ * & 0 & 0 & 0 & * & * & * & * & * \\ * & 0 & 0 & 0 & * & * & * & * & * \\ * & 0 & 0 & 0 & * & * & * & * & * \end{pmatrix}$$

Note that rows r_1 and r_5 are equivalent, but columns c_2, c_3, and c_4 are each unique. We need not consider when a 1 occurs in any of $a_{2,2}$, $a_{3,3}$, or $a_{4,4}$ due to the following argument: consider when $a_{2,2} = 1$, then $a_{2,1} = a_{5,1} = 0$, but this yields a zero in entries $a_{2,3}$ and $a_{2,4}$ of A^2. Similar arguments apply when $a_{3,3} = 1$ or $a_{4,4} = 1$. Therefore, we may consider the above representation up to permutation simularity. However, in order to produce a matrix A such that $A^2 = J$, there must exist a 1 in either the first or fifth columns in each row r_i where $i \in \{2,3,4,6,7,8,9\}$, but this is impossible because the columns sums for c_1 and c_5 must be 3 (and there is no way to place a total of six 1's in seven required positions). Therefore, such an arrangement cannot exist.

Consequently, we have found that for every matrix A_9 such that $A^2 = J$, every group of rows with multiplicity 3 corresponds to uniquely to a group of columns with multiplicity 3, and similarly for multiplicities 2 and 1. Therefore, for 9x9 matrices, the row and column multiplicities are equivalent for matrices satisfying $A^2 = J$.

□

Proposition 3.2 *Let A_n be an $n \times n$ central groupoid in matrix form. Up to permutation similarity, there exists a unique PA_nQ form corresponding to A_n.*

Throughout our analysis of the significance of row and column multiplicities on the structure of central groupoids, we utilize restrictions on A determined by its PAQ form. Essentially, we define the PAQ form of a given central groupoid A to be the matrix obtained by row and column permutations resulting in the largest number predetermined rows. For example, consider a matrix with row multiplicity vector {4,4,3,3,1,1}. Given two sets of four and three equivalent rows, we apply row and column permutations defined by PAQ to obtain the following structure:

$$PAQ = \begin{pmatrix} J_{4,4} & 0 & 0 & 0 \\ 0 & J_{4,4} & 0 & 0 \\ 0 & 0 & J_{3,4} & 0 \\ 0 & 0 & 0 & J_{3,4} \\ 0 & 0 & *_{2,4} & *_{2,4} \end{pmatrix}.$$

In general, our desired PAQ form of a matrix A with row multiplicity vector $rm = \{m_1, m_2, m_3, m_4, \ldots\}$ is found by creating k distinct $J_{a_i,k}$ blocks where the a_i's, $i = 1, 2, \ldots, k$, correspond to the k largest members of rm. The blocks are then aligned in disjoint sets of rows in decreasing size across the top of PAQ, similar to the above.

Motivations behind this proposition relate directly to the basic properties of all central groupoids. First, note that the $\text{eig}(AA^t) = \text{eig}(PAQ(PAQ)^t)$, due to the following:

$$PAQ * (PAQ)^t = PAQQ^tA^tP^t = PAA^tP^t$$

Therefore, if a unique PAQ form does indeed exist for any central groupoid, the $\text{eig}(AA^t)$ of any two matrices with equivalent row and column multiplicity vectors can be shown to be equivalent, with possible extensions into characterizing words of the form $(AA^t)^{m/2}$ and $(A^tA)^{m/2}$. Furthermore, uniqueness in the PAQ of any given central groupoid A corresponds directly to equal permanent values for any matrices with the same row and column multiplicities.

4 Matrices close to the standard form

In this section, we consider matrices $A = (A_{ij}) \in A_{k^2}$ such that $A_{ij} \in M_k$ and A_{ij} are the same as that in the standard matrix unless $i = k-1$ and k. We begin with the cases when $k = 2, 3$. Since every matrix in $\mathcal{A}_4$ and $\mathcal{A}_9$ have this form up to permutation similarity, this will give a complete description of all matrices in $\mathcal{A}_4$ and $\mathcal{A}_9$. In particular, this verifies the result for $\mathcal{A}_9$ mentioned in [6] without proof.

4.1 $\mathcal{A}_4$

If $n = 4$, up to permutation similarity, A has the form:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Proof. Up to permutation similarity, we may assume that the first row of A is $(1,1,0,0)$. But then if we consider the $(1,1)$ and $(1,2)$ entries of A^2, we see that the second row of A must be $(0,0,1,1)$. Now, we may assume that the $(3,1)$ entry of A is 1 by permutation similarity. Since the second row of A^2 is the sum of the third and fourth rows of A, we see that the $A(3,4;1,2)$ is one of the following:

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad I_2.$$

If it were I_2, then the third row of A^2 will contain some 2 and 0, which is impossible. Thus, the first case holds, and the result follows.

4.2 $\mathcal{A}_9$

Suppose $A \in \mathcal{A}_9$. Let B be the 6-by-9 matrix formed by deleting the last 3 rows of A, and let C be the 6-by-9 matrix formed by deleting rows 4, 5 and 6 of A. We may assume by permutation similarity that A has the form $(A_{ij})_{1\leq i,j\leq 3}$, where

$$A_{11} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, A_{13} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Further, we may assume that $\text{rank}\,(B) \geq \text{rank}\,(C)$ since, if not, rows (and columns) 4, 5 and 6 of A can be interchanged with rows (and columns) 7, 8 and 9, and then rows (and columns) 2 and 3 of A can be interchanged, thus maintaining the desired form for the first 3 rows of A and yielding the desired inequality between the ranks of B and C.

We will find all solutions to $A^2 = J$ by considering the possible ranks of B. Denote row i and column i of A by r_i and c_i respectively, for $i = 1, \ldots, n$. Since $r_1+r_2+r_3 = r_4+r_5+r_6$, the rows of B are dependent and hence $3 \leq \text{rank}\,(B) \leq 5$. Clearly, if $\text{rank}\,(B) = 3$, then $\text{rank}\,(A) = 3$ and A can have no row that differs from one of its first three rows. It follows easily that if $\text{rank}\,(B) = 3$, then A satisfies (or can have its rows and columns permuted to satisfy) $A_{11} = A_{21} = A_{31}$, $A_{12} = A_{22} = A_{32}$, and $A_{13} = A_{23} = A_{33}$. We must now consider solutions for which $\text{rank}\,(B)$ is equal to 4 or 5.

Let A be a solution such that $\text{rank}\,(B) = 4$ or 5. Since the submatrix A_{21} contains three 1's, there are three cases to consider:

1) Each row of A_{21} contains exactly one 1,
2) One row of A_{21} contains exactly two 1's, and
3) All three 1's lie in one row of A_{21}.

In case 1) we may assume by permutation similarity that $A_{21} = I_3$. Since $r_4c_1 = 1$, $a_{44} = 0$. Similarly, since $r_4c_2 = 1$ and $r_4c_3 = 1$, $a_{45} = 0$ and $a_{46} = 0$. Because $r_5c_5 = 1$, $a_{55} = 0$. Since each column of A_{22} contains exactly one 1, $a_{65} = 1$. Then $r_6c_7 = r_6c_8 = r_6c_9 = 1$ implies that the second row of A_{23} is $(0, 0, 0)$. Since r_5 contains exactly three 1's, the second row of A_{22} is $(1, 0, 1)$ and thus $r_5c_5 \geq 2$, which is not possible. Hence there are no solutions in case 1).

In case 2) we may assume that row one of A_{21} is $(1, 1, 0)$, $(1, 0, 1)$, or $(0, 1, 1)$ and that row three of A_{21} is $(0, 0, 0)$. If row one of A_{21} is $(1, 1, 0)$, then row one of A_{22} must be $(0, 0, 0)$ or $(0, 0, 1)$, as will be shown in the following paragraph. In the former case there is one solution with $\text{rank}\,(A) = \text{rank}\,(B) = 4$ and one solution with $\text{rank}\,(B) = 4$ and $\text{rank}\,(A) = 5$. In the latter case there are two solutions with $\text{rank}\,(A) = \text{rank}\,(B) = 4$ and one solution with $\text{rank}\,(B) = 4$ and $\text{rank}\,(A) = 5$.

We now verify the above claim that if row one of A_{21} is $(1, 1, 0)$ and row two is $(0, 0, 1)$, then row one of A_{22} must be $(0, 0, 0)$ or $(0, 0, 1)$. We also show that if row one of A_{22} is $(0, 0, 0)$, then a single solution with $\text{rank}\,(A) = \text{rank}\,(B) = 4$ results. Other cases in this section can be examined in a similar manner. Since $r_4c_1 = 1$, $a_{44} = 0$. Similarly $a_{45} = 0$ since $r_4c_3 = 1$. Thus row one of A_{22} is either $(0, 0, 0)$ or $(0, 0, 1)$. Suppose the former. Since

r_4 contains three 1's, we may assume that the first row of A_{23} is $(1,0,0)$. Since rank $(A) = 4$, $r_4 + r_5 = r_1 + r_3$ and hence the second row of A_{23} is $(0,1,1)$. Then the third row of A_{22} must be $(1,1,1)$. Since $r_4 A = e$, the first row of A_{33} is $(1,1,1)$, implying $r_8 + r_9 = r_1 + r_2$. Because rank $(A) = 4$, r_8 and r_9 must belong to the set $r_1, \ldots, r_6$ and hence must be r_1 and r_2, thus verifying the existence of a unique solution in this case.

Continuing case 2), if row one of A_{21} is $(1,0,1)$ and rank $(B) = 4$, then it follows that the first two rows of A_{23} are $(1,0,0)$ and $(0,1,1)$, which results in two solutions. If row one of A_{21} is $(1,0,1)$ and rank $(B) = 5$, there is one solution. Concluding case 2), if row one of A_{21} is $(0,1,1)$ and rank $(B) = 4$, then the first two rows of A_{23} are $(0,0,1)$ and $(1,1,0)$, which yields one solution. If row one of A_{21} is $(0,1,1)$ and rank $(B) = 5$, there is one solution.

In case 3) we may assume that the three 1's lie in the first row of A_{21}. Since no row of A_{22} contains three 1's, we may also assume that row two of A_{22} contains exactly two 1's, being equal to $(1,1,0)$, $(1,0,1)$, or $(0,1,1)$. The first two possibilities yield one solution each while the third possibility yields two solutions. All four of these solutions have the property that rank $(A) =$ rank $(B) = 4$.

The cases considered above yield all possible solutions to $A^2 = J$ for $n = 9$. Fifteen solutions have been produced, but when it is required that no two solutions be permutation similar, only six solutions remain. We present these six permutationally different matrices $B_1, B_2, \ldots, B_6$, in Appendix A.

4.3 $\mathcal{A}_{16}$

Suppose $A = (A_{ij}) \in \mathcal{A}_{16}$ such that $A_{ij} \in M_4$ and A_{ij} is the same as that of the standard matrix whenever $i \leq 3$. Let $r_1, \ldots, r_{16}$ and $c_1, \ldots, c_{16}$ be the rows and columns of A.

Note that A_{41} cannot be a permutation matrix. Otherwise, we may assume that $A_{41} = I_4$. But then every row of A_{44} must be $[1\ 1\ 1\ 1]$ or $[0\ 0\ 0\ 0]$. In fact, if row p of A_{44} is different from the above two vectors, then for $r_q = [r_{q1} r_{q2} r_{q3} r_{q4}]$ with $q = 12 + p$, we have

$$[1\ 1\ 1\ 1] = r_q \begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \\ A_{41} \end{bmatrix} = r_{q1} A_{11} + r_{q2} A_{21} + r_{q3} A_{31} + r_{q4} A_{41}.$$

Now, $r_{q4} A_{41} = r_{q4}$ is just row p of A_{44}, which is neither $[1\ 1\ 1\ 1]$ nor $[0\ 0\ 0\ 0]$, and $r_{qj} A_{j1}$ must be $[1\ 1\ 1\ 1]$ or $[0\ 0\ 0\ 0]$. Thus,

$$[1\ 1\ 1\ 1] \neq r_{q1} A_{11} + r_{q2} A_{21} + r_{q3} A_{31} + r_{q4} A_{41},$$

which is impossible. On the other hand, since $A_{41} = I_4$, and all row sums of A equal 4, A_{44} cannot have a row $[1\ 1\ 1\ 1]$, which is a contradiction.

Now, we can permute the last four rows and the last four columns of A so that the row sums of A_{41} are non-increasing.

Suppose A_{41} has row sums $2,1,1,0$. Up to permutation similarity A_{41} can be any of the following:

$$C_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; C_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; C_3 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C_4 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; C_5 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; C_6 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that in each of the above cases, the rows of A_{44} can only be $[1\ 1\ 1\ *]$ or $[0\ 0\ 0\ *]$. Furthermore, the rows $[1\ 1\ 1\ 0]$ and $[0\ 0\ 0\ 1]$ cannot appear simultaneously. Thus,

$$A_{44} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Suppose $A_{41} = C_1$. Then $r_{13}A = [1 \cdots 1]$ implies that r_{13} has the form:

$$[1\,1\,0\,0 \mid 0\,0\ *\ * \mid 0\,0\ *\ * \mid 0\,0\,0\,0],$$

and hence r_{13} is one of the following cases:

(a) $[1\,1\,0\,0 \mid 0\,0\,1\,1 \mid 0\,0\,0\,0 \mid 0\,0\,0\,0]$, (b) $[1\,1\,0\,0 \mid 0\,0\,0\,0 \mid 0\,0\,1\,1 \mid 0\,0\,0\,0]$,

(c) $[1\,1\,0\,0 \mid 0\,0\,0\,1 \mid 0\,0\,1\,0 \mid 0\,0\,0\,0]$, (d) $[1\,1\,0\,0 \mid 0\,0\,1\,0 \mid 0\,0\,0\,1 \mid 0\,0\,0\,0]$.

Replace A by $RQAQ^tR^t$, where $Q = (Q_{ij})$ is a permutation matrix with $Q_{11} = Q_{23} = Q_{32} = Q_{44} = I_4$, and $R = R_1 \oplus R_1 \oplus R_1 \oplus R_1$ such that R_1 is obtained from I_4 by interchanging the second and the third rows. Then we can identify the first two cases and we can identify the last two cases.

If case (a) holds, the second rows only have four possible forms, and the matrix $[A_{41}\ A_{42}\ A_{43}\ A_{44}]$ is then completely determined according to the four different cases:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1&1&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0\\ 0&0&0&1&1&0&0&0&0&1&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&1&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&1&1&0&1&0&0&0&0\\ 0&0&0&1&1&1&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix}.$$

Similarly, if case (c) holds, then there are four choices for the second row of the matrix $[A_{41}\ A_{42}\ A_{43}\ A_{44}]$, and the matrix is then completely determined according to the four different cases:

$$\begin{pmatrix} 1&1&0&0&0&0&0&1&0&0&1&0&0&0&0&0\\ 0&0&1&0&1&1&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&1&0&0&1&0&1&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&1&0&0&0&0&0&1&0&0&1&0&0&0&0&0\\ 0&0&1&0&1&0&0&0&0&1&0&1&0&0&0&0\\ 0&0&0&1&0&1&1&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&1&0&0&0&0&0&1&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0\\ 0&0&0&1&1&0&0&0&0&1&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&1&0&0&0&0&0&1&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&1&1&0&1&0&0&0&0\\ 0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix}.$$

If $A_{41} = C_2$, replace A by $QRAR^tQ^t$ with $R = R_1 \oplus R_1 \oplus R_1 \oplus I_4$, where R_1 is obtained from I_4 by interchanging the second and third row, and $Q = (Q_{ij}) \in M_{16}$ is a permutation matrix with $Q_{11} = Q_{23} = Q_{32} = Q_{44} = I_4$. Then we are back to the case when $A_{41} = C_1$.

Suppose $A_{41} = C_3$. Then row 13 of A has the following choices.

(a) $[1\,0\,0\,1 \mid 0\,1\,1\,0 \mid 0\,0\,0\,0 \mid 0\,0\,0\,0]$, (b) $[1\,0\,0\,1 \mid 0\,0\,0\,0 \mid 0\,1\,1\,0 \mid 0\,0\,0\,0]$,

(c) $[1\,0\,0\,1 \mid 0\,0\,1\,0 \mid 0\,1\,0\,0 \mid 0\,0\,0\,0]$, (d) $[1\,0\,0\,1 \mid 0\,1\,0\,0 \mid 0\,0\,1\,0 \mid 0\,0\,0\,0]$.

Similar to the previous case, we can reduce case (b) to case (a), and reduce case (d) to (c). Moreover, if case (a) holds, then the matrix $[A_{41}\ A_{42}\ A_{43}\ A_{44}]$ has the four different forms:

$$\begin{pmatrix} 1&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&1&1&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&1&0&0&0&0&0&1&1&0&0&0&0\\ 0&0&1&0&0&0&0&1&1&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&1&1&0&1&0&0&0&0&0\\ 0&0&1&0&1&0&0&0&0&1&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&1&0&1&1&0&0&0&0\\ 0&0&1&0&1&0&0&1&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix}.$$

If case (c) holds, then the matrix $[A_{41}\ A_{42}\ A_{43}\ A_{44}]$ has the four different forms:

$$\begin{pmatrix} 1&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0\\ 0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0\\ 0&1&0&0&1&0&0&0&0&0&1&1&0&0&0&0\\ 0&0&1&0&0&1&0&1&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&1&1&0&1&0&0&0&0&0\\ 0&0&1&0&1&1&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&1&0&1&1&0&0&0&0\\ 0&0&1&0&1&1&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix}.$$

Next, if $A_{41} = C_4$, then Then row 13 of A has the following choices.

(a) $[0\ 1\ 1\ 0 \mid 1\ 0\ 0\ 1 \mid 0\ 0\ 0\ 0 \mid 0\ 0\ 0\ 0]$, (b) $[0\ 1\ 1\ 0 \mid 0\ 0\ 0\ 0 \mid 1\ 0\ 0\ 1 \mid 0\ 0\ 0\ 0]$,

(c) $[0\ 1\ 1\ 0 \mid 1\ 0\ 0\ 0 \mid 0\ 0\ 0\ 1 \mid 0\ 0\ 0\ 0]$, (d) $[0\ 1\ 1\ 0 \mid 0\ 0\ 0\ 1 \mid 1\ 0\ 0\ 0 \mid 0\ 0\ 0\ 0]$.

Similar to the previous case, we can reduce case (b) to case (a), and reduce case (d) to (c). Moreover, if case (a) holds, then the matrix $[A_{41}\ A_{42}\ A_{43}\ A_{44}]$ has the four different forms:

$$\begin{pmatrix} 0&1&1&0&1&0&0&1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&1&1&0&0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&1&1&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 0&1&1&0&1&0&0&1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&1&0&0&0&0&1&1&0&0&0&0\\ 0&0&0&1&0&0&1&0&1&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 0&1&1&0&1&0&0&1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&1&0&0&1&0&1&0&0&0&0\\ 0&0&0&1&0&1&0&0&1&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 0&1&1&0&1&0&0&1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0\\ 0&0&0&1&0&1&1&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix}.$$

If case (c) holds, then the matrix $[A_{41}\ A_{42}\ A_{43}\ A_{44}]$ has the four different forms:

$$\begin{pmatrix} 0&1&1&0&1&0&0&0&0&0&0&1&0&0&0&0\\ 1&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&1&1&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 0&1&1&0&1&0&0&0&0&0&0&1&0&0&0&0\\ 1&0&0&0&0&1&0&1&0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&1&0&1&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 0&1&1&0&1&0&0&0&0&0&0&1&0&0&0&0\\ 1&0&0&0&0&0&1&1&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&1&0&0&1&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 0&1&1&0&1&0&0&0&0&0&0&1&0&0&0&0\\ 1&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0\\ 0&0&0&1&0&1&1&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix}.$$

Suppose $A_{41} = C_5$. We can consider the first rows and second rows of the matrix $[A_{41}\ A_{42}\ A_{43}\ A_{44}]$ and conclude that there are basically eight different forms:

$$\begin{pmatrix} 0&1&0&1&1&0&1&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&1&0&1&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&1&1&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 0&1&0&1&1&0&1&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&1&0&0&0&0&1&1&0&0&0&0\\ 0&0&1&0&0&0&0&1&1&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 0&1&0&1&1&0&1&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0\\ 0&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 0&1&0&1&1&0&1&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0\\ 0&0&1&0&0&1&0&1&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 0&1&0&1&1&0&0&0&0&0&1&0&0&0&0&0\\ 1&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&1&1&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 0&1&0&1&1&0&0&0&0&0&1&0&0&0&0&0\\ 1&0&0&0&0&1&1&0&0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&1&1&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 0&1&0&1&1&0&0&0&0&0&1&0&0&0&0&0\\ 1&0&0&0&0&0&1&1&0&1&0&0&0&0&0&0\\ 0&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 0&1&0&1&1&0&0&0&0&0&1&0&0&0&0&0\\ 1&0&0&0&0&0&1&0&0&1&0&1&0&0&0&0\\ 0&0&1&0&0&1&0&1&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix}.$$

If $A_{41} = C_6$, we can reduce it to the case when $A_{41} = C_5$.

Now, suppose A_{41} has row sums $2, 2, 0, 0$. We may assume that

$$A_{41} = \begin{pmatrix} 1&1&0&0\\ 0&0&1&1\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}.$$

Note that if the first row of A_{41} is [1 0 1 0], one can replace A by $QRAR^tQ^t$, $R = R_1 \oplus R_1 \oplus R_1 \oplus I_4$, where R_1 is obtained from I_4 by interchanging row 2 and row 3, and $Q = (Q_{ij})$ is a permutation matrix, where $Q_{11} = Q_{23} = Q_{32} = Q_{44} = I_4$. Then row 13 of A has the form:

(a) [1 1 0 0 0 0 1 1 0 0 0 0 0 0 0 0],

(b) [1 1 0 0 0 0 1 0 0 0 0 1 0 0 0 0],

(c) [1 1 0 0 0 0 0 1 0 0 1 0 0 0 0 0],

(d) [1 1 0 0 0 0 0 0 0 0 1 1 0 0 0 0],

(e) [1 1 0 0 0 0 1 0 0 0 0 0 0 0 $*$ $*$] and row 3 or 4 of A_{44} equals [1 1 1 1],

(f) [1 1 0 0 0 0 0 0 0 1 0 0 0 * *] and row 3 or 4 of A_{44} equals [1 1 1 1],
(g) [1 1 0 0 0 0 0 1 0 0 0 0 0 0 * *] and row 3 or 4 of A_{43} equals [1 1 1 1],
(h) [1 1 0 0 0 0 0 0 0 0 1 0 0 * *] and row 3 or 4 of A_{43} equals [1 1 1 1],
(i) [1 1 0 0 0 0 0 0 0 0 0 0 0 0 1 1] and the sum of row 15 and 16 of A equals [0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1].

Now, suppose A_{41} has row sums $3, 1, 0, 0$. Then the first row of A_{41} is one of the following:

[1 1 1 0], [1 1 0 1], [1 0 1 1], [0 1 1 1].

The second case can be reduced to the first one. Row 13 of A has the form:
(a) [1 1 1 0 0 0 0 1 0 0 0 0 0 0 0 0],
(b) [1 1 1 0 0 0 0 0 0 0 1 0 0 0 0],
(b) [1 1 1 0 0 0 0 0 0 0 0 0 0 * * *] and row 2, 3, or 4 of A_{44} is [1 1 1 1].
(d) [1 1 0 1 0 0 1 0 0 0 0 0 0 0 0 0],
(e) [1 1 0 1 0 0 0 0 0 0 1 0 0 0 0 0],
(f) [1 1 0 1 0 0 0 0 0 0 0 0 0 * * *] and row 2, 3, or 4 of A_{43} is [1 1 1 1].
(g) [0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0],
(h) [0 1 1 1 0 0 0 0 1 0 0 0 0 0 0 0].

Finally, suppose A_{41} has row sums $4, 0, 0, 0$. We may then assume that row 13 of A is

[1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0].

5 Sub-Central Groupoids

In this and some of the following sections we present some conjectures and descriptions of some open questions that, once verified or disproven, may further deepen the understanding of central groupoids.

We discuss the topic of sub-central groupoids. The following conjecture has been verified with computer experimentation for all cases of A_n such that $A_n^2 = J$ for $n = 1, 4, 9$, and partially for $n = 16$. However, without a complete characterization of the 16×16 central groupoids and analysis of larger cases where $n \geq 5$, the following conjecture remains open.

Conjecture: Let $k = \sqrt{n}$. Given an $n \times n$ central groupoid with $n \geq 4$, there exists at least one (possibly more) sub-groupoid of size $(k-1)^2 \times (k-1)^2$ embedded within it.

We present a proof for the following proposition.

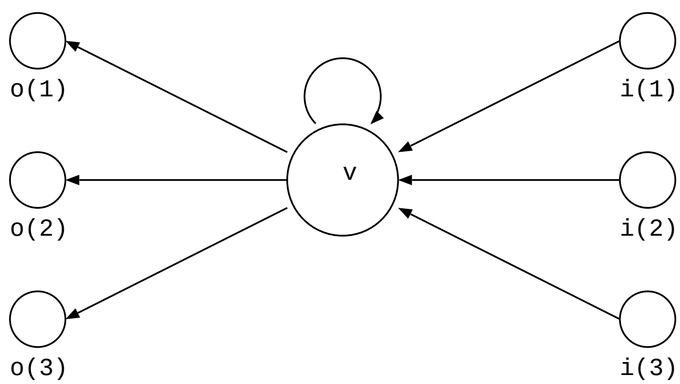
Proposition 5.1 *Let $k = \sqrt{n}$. Given A, an $n \times n$ central groupoid, we have*

(a) *a maximum of k sub-groupoids of size $(k-1)^2 \times (k-1)^2$ exist,*

(b) *if A is permutationally similar to the standard $n \times n$ construction, then k sub-central groupoids of size $(k-1)^2 \times (k-1)^2$ exist.*

Note that the above conjecture and proposition corresponds directly to sub-central digraphs of a given n-vertex central digraph, and sub-matrices of a given A_{k^2} satisfying $A^2_{(k-1)^2} = J_{(k-1)^2}$. By inspection, all 4×4 and 9×9 groupoids can be shown to contain at least one sub-groupoid of size 1×1 and 4×4, respectively. Moreover, we find that for $k = 2, 3$ the standard construction contains exactly k sub-groupoids, which is maximum. Although a general proof has not been discovered to show that all central groupoids contain at least one such sub-central groupoid, we present the following proofs of items (a) and (b).

Proof of (a). First, we observe that the creation of a sub-groupoid corresponds to the removal of exactly one idempotent vertex in an n-vertex central digraph. Recall that an idempotent vertex is one that has a directed edge to itself in the digraph. Let v be the vertex to be removed and consider, for example, the following representation of v in the 16×16 case:



From the removal of v, we have the following observations. First, if all of $i_1, i_2, i_3, o_1, o_2, o_3$ are removed, then we have only $(k-1)^2$ vertices remaining, which may yield at most one sub-central digraph. Also, we note that if at least one of the incoming vertices are preserved for the sub-digraph, then all outgoing vertices must be removed, and vice versa. Otherwise, with v removed there would exist some i_j and o_k, $j, k \in \{1, 2, 3\}$, having no path of length two from i_j to o_k. Finally, we note that the removal of any incoming or outgoing vertex, call it p, must coincide with the removal of the entire two-cycle, call it (p, q), containing the vertex. Preserving q while removing p would result in no length 2 path from q to q in the sub-digraph. Consequently, we have three cases.

Case (1): remove v and all its incoming and outgoing vertices (when all i_j's and o_k's pair into two-cycles),

Case (2): remove either all the incoming or all the outgoing vertices along with their respective two-cycle pairs, assuming that these two processes do not yield two distinct sub-groupoids.

In either (1) or (2), the removal of a particular idempotent v in the $n \times n$ central groupoid yields at most 1 sub-groupoid, resulting in a maximum of k sub-central groupoids. We claim that the following, case (3), is impossible.

Case (3): remove either all the incoming or all the outgoing vertices and their respective two-cycle pairs, each process resulting in its own disjoint sub-groupoid. Consider the following 16×16 construction in matrix form, where vertex 1 is the idempotent vertex to be removed and vertices 2, 3, and 4 are the out-going vertices from 1. Let matrix B be a 9×9 sub-groupoid. Up to permutation, we may assume that A has the following top block row structure.

$$A_{1,1}A_{1,2}A_{1,3}A_{1,4} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The remaining blocks of A may be specified as follows:

$$For\ 2 \leq i,j \leq 4, A_{i,j} = \begin{pmatrix} * & 0 & 0 & 0 \\ * & & & \\ * & & B_{i-1,j-1} & \\ * & & & \end{pmatrix}$$

$$A_{2,1} = A_{3,1} = A_{4,1} = \begin{pmatrix} * & 1 & 1 & 1 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}$$

We have the above 3×3 block of zeros due to the fact that the sub-groupoid will already produce the sub-matrix $J_{(k-1)^2}$ when any rows and columns from the sub-groupoid are multiplied. For example, the product $r_6 \times r_6$ would equal 2 if entry $a_{6,2}$ were a 1.

Without loss of generality and up to permutation similarity, we may assume that a central groupoid with an embedded subgroupoid can always be expressed in this form. We claim that having a 1 in positions $a_{5,1}, a_{9,1}$, and $a_{13,1}$ would correspond to case (1) above (all incoming and outgoing vertices to and from vertex 1 would be removed). Therefore, we wish to consider the case where removing vertex 1 and its incoming vertices versus its outgoing vertices (along with their two-cycle pairs) corresponds to removing two disjoint sets of vertices. Up to permutation similarity, we may therefore assume:

$$A_{2,1} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

However, we will soon find a contradiction. No doubt, the removal of vertex 1 and the outgoing vertices from 1 (along with their two-cycle pairs) will yield the sub-groupoid B. However, if we consider the removal of vertex 1 and the incoming vertices from 1 (along with their two-cycle pairs) we find that no sub-groupoid can remain. First, we note that there must be a 1 in either position $a_{5,9}$ or $a_{5,13}$ to complete the row sum of row 5 since a 1 in position $a_{5,5}$ would violate the trace constraint on A. Up to permutation similarity, we assume $a_{5,13} = 1$, which also implies $a_{13,1} = 1$ (consider the multiplication $r_5 \times c_1$). Therefore, we find that the removal of vertex 1 and its incoming vertices implies the removal of vertices 1, 6 and 13, which in turn implies the removal vertices 2, 3, 4, 5, and 9. Once vertices 1 and 13 are removed, columns 2, 3, and 4 are left with a column sum of at most two for a sub-matrix, so they must be removed. Moreover, vertices 5 and 9 are left with a row sum of only 1, so they must be removed. Consequently, we are left with only $(k-1)^2 - 1$ vertices, so a sub-groupoid of size $(k-1)^2 \times (k-1)^2$ cannot be created.

Note that the opposite case, when the removal of a vertex and its incoming vertices does yield a sub-groupoid, follows a similar proof to show that the removal of a vertex and its outgoing vertices cannot yield another disjoint sub-groupoid. The above is easily extended to $n \times n$ matrices having $n > 16$. □

Proof of (b). Essentially, we claim that for the standard $n \times n$ central groupoid, call it A, there exists a one-to-one correspondence between a sub-central groupoid of size $(k-1)^2 \times (k-1)^2$ and a unique choice of idempotent indices (recall that an index i is idempotent if the $a_{i,i}$ entry in A is 1) in the standard construction. Given that A contains exactly k idempotent indices and that $(k-1)$ must be chosen to be part of the sub-groupoid, there are ${}_kC_{k-1} = k$ unique choices.

We consider A in the standard matrix form. In other words, the first row of A contains k 1's followed by $k^2 - k$ 0's, the second row contains k 0's followed by k 1's followed by $k^2 - 2k$ 0's, and so on until the kth row contains $k^2 - k$ 0's followed by k 1's. Finally, all consecutive blocks of k rows are complete copies of the first k rows. Let $I_i = \{i_1, i_2, \ldots, i_k\}$ be the set of all idempotent indices in A, and consider any choice of one member of I_i that is not chosen to be a part of the sub-groupoid, call it i_j. We claim that the remaining indices that must be chosen to create a sub-central groupoid are completely determined by this choice of i_j. Let I_A be the set of all indices of A labeled 1 to n in increasing order from left to right in the matrix A, which is the same as the set of indices labeled $i_{x,y}$ where $x, y \in \{1, 2, \ldots, k\}$. We claim that if i_j is removed, then the only possible sub-groupoid that can be constructed contains indices $I_A - (i_{x,j} \bigcup i_{j,y})$ for all values of x, y. In other words, for the standard 16×16 central groupoid in matrix form, we have $I_i = \{1, 6, 11, 16\}$. If the third idempotent index is to be removed from consideration, namely vertex 11, then the only possible set of indices that can create a sub-central groupoid is $\{1, 2, \ldots, 16\} - (\{3, 7, 11, 15\} \bigcup \{9, 10, 11, 12\}) = \{1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16\}$. In other words, given the matrix

$$\begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{3,3} & A_{4,4} \end{pmatrix},$$

the removal of the third idempotent index corresponds to the removal of all $A_{3,y}$ and $A_{x,3}$, as well as the removal of the third row and column of the remaining blocks.

To show why such a selection is unique, first note that in standard matrix form, no more than $(k-1)$ indices can be chosen from any block of k equivalent columns or rows (or else the sub-matrix would contain some row or column with sum equal to k, which is impossible for a $(k-1)^2 \times (k-1)^2$ central groupoid). Furthermore, we claim that it is not possible to select an index from each of the k sets of equivalent rows or columns. If an index was chosen from each of the k disjoint sets, then it would be necessary to choose an index corresponding to one of the first k rows, call it row a, $1 \leq a \leq k$. Consequently, $(k-2)$ remaining columns would need to be chosen in the ath disjoint set (to construct a row sum of $(k-1)$ in the submatrix along the ath row). However, if we now consider the ath block row, we see that $(k-2)(k-2)$ columns still need to be chosen to satisfy row sums in the sub-matrix, which is impossible seeing that we have already chosen $k + (k-2)$ rows. Note that the number of required indices minus the number of indices already chosen is

$$\begin{aligned}
&= (k^2 - 2k + 1) - (2k - 2) \\
&= k^2 - 4k + 3 \\
&< k^2 - 4k + 4 \\
&= (k-2)^2
\end{aligned}$$

which is the number of indices required to satify row sums of submatrix in ath block row of A.

Therefore, we have shown that to create a sub-groupoid from the standard central groupoid in matrix form, exactly $(k-1)$ indices must be chosen from exactly $(k-1)$ blocks. In other words, an entire block column will be removed from consideration in the sub-groupoid, call it block column C_a. We also note that if the ath block column is removed from consideration, then any selection of the ath row in any block row will result in a row sum equal to zero in some row of the sub-matrix. Therefore, choosing $(k-1)$ of the k block columns corresponds directly to a unique sub-groupoid and the removal of a unique idempotent index (in this case, the ath idempotent index). □

We conjecture that the above occurs if and only if a central groupoid is permutationally similar to the standard construction.

The proof of such a conjecture could drastically increase the ability to create or simply count the number of unique $k^2 \times k^2$ central groupoids given a complete characterization of all $(k-1)^2 \times (k-1)^2$ central groupoids. As presented in [7], central groupoids of highest possible ranks for a given size n can be created by determined expansions on a sub-groupoid. In matrix form, the problem may be represented as matrix completions of the following form:

$$A_{k^2} = \begin{pmatrix} A_{(k-1)^2} & * \\ * & * \end{pmatrix}$$

Our conjecture suggests that all possible groupoids can be represented in and created through this form.

For a final word on the proposition that all central groupoids contain at least one sub-central groupoid of size $(k-1)^2 \times (k-1)^2$, we propose the following:

Proposition 5.2 *Let rm (cm) be the set of integers corresponding to the row (column) multiplicities vector of a central groupoid A_{k^2}. If the central groupoid A_{k^2} contains a sub-central groupoid $A_{(k-1)^2}$, then $rm \bigcup cm$ must contain at least one of k and $(k-1)$.*

Proof. Consider A_{k^2} in matrix form. Perform permutations to A_{k^2} yielding the standard top row block form and let the vertices of $A_{(k-1)^2}$ be the vertices a_{i+j}, $i = \{k+2, 2k+2, \ldots, (k-1)k+2\}, j = \{0, 1, \ldots, k-2\}$. As described earlier in this section, we are left with equivalent columns $2, 3, \ldots, k$, implying that a multiplicity of at least $k-1$ must exist. □

Although such a proposition cannot provide any insight into whether or not all central groupoids contain a sub-central groupoid of any particular size, it does provide a method for finding a counter-example. In other words, if a central groupoid did not have a set of at least $(k-1)$ equivalent rows or columns, it could not contain a sub-central groupoid, thus disproving the original conjecture.

6 Classifying all Knuthian Matrices

Although it may be impossible or at least very difficult to determine for a given k a complete classification of $\mathcal{A}_n$ when $n = k^2$, for certain subsets of $\mathcal{A}_n$ a much more complete classification is in fact possible. In this section, we will define a subset of $\mathcal{A}_n$ as the Knuthian set (denoted by $\mathcal{K}_n$), give a formula for determining the number of non-isomorphic members of $\mathcal{K}_n$ and therefore a lower bound for the size of $\mathcal{A}_n$, determine all ranks that members of $\mathcal{K}_n$ have, answer some open questions about $\mathcal{A}_n$ as it is applied to $\mathcal{K}_n$, and suggest a possible way to extend $\mathcal{K}_n$ to include more members of $\mathcal{A}_n$.

In [6], the author defines a "natural central groupoid" on $S \times S$, for some set S, to have multiplication defined as:

$$(x_1, x_2) \cdot (y_1, y_2) = (x_2, y_1)$$

He then proves that this "multiplication" satisfies the identity necessary for a central groupoid. He then goes on to describe a method for extending this construction, namely by requiring S to contain the element 0 and have a binary operation ($\circ$) such that $x \circ 0 = 0$ and $x \circ z = y$ has a unique solution whenever y and x are elements of S. He then defines D to be the directed graph on the vertices of $S \times S$ having an arc $(x_1, x_2) \to (y_1, y_2)$ if and only if

$$x_2 = y_1 \text{ and } y_1 \neq 0$$

or

$$x_1 \circ x_2 = y_1 \text{ and } y_2 = 0.$$

He goes on to prove that D is in fact a central digraph, and the central groupoid that results from this construction has multiplication rule:

$$(x_1, x_2) \cdot (y_1, y_2) = \begin{cases} (x_2, y_1) & \text{if } y_1 \neq 0 \text{ and } y_2 \neq 0 \\ (x_2, z_2) & \text{if } x_2 \circ z_2 = y_1 \neq 0 \text{ and } y_2 = 0 \\ (x_1 \circ x_2, 0) & \text{if } y_1 = 0 \end{cases}$$

If in this construction we define $x \circ y = y$ for all $x, y \in S$, then it is not hard to see that this creates a natural central groupoid. However, $\circ$ can be defined in many other ways which yield a new central groupoid structure. If D is a directed graph constructed in the above manner with $S = \{0, 1, \dots, k-1\}$, then we can relabel each ordered pair in D with a single number in the following way:

$$(x, y) \text{ becomes } kx + y$$

This will allow us to construct an adjacency matrix from D.

Definition 6.1 *Let D be a directed graph formed from $S = \{0, 1, \dots, k-1\}$ with the above construction, and let T be the multiplication table for the binary operation $\circ$ that defines arcs in D. Then define $A(T)$ to be the adjacency matrix formed by relabeling the points of D in the above manner.*

This allows us to define a subset of $\mathcal{A}_n$ in the following way.

Definition 6.2 *For all $n = k^2$ $(k \in \mathbb{N})$, define $\mathcal{K}_n$ to be the set of all non-permutationally similar matrices $A(T)$ for which T is a multiplication table for the binary operation $\circ$ in $S = \{0, 1, \dots k-1\}$ with the two above rules.*

It is not hard to see that if T creates a natural central groupoid, then $A(T)$ is the standard matrix. Further, it is proven in [6] that any binary operation $\circ$ is isomorphic to a specific operation $*$ such that $0 * x = x$ for all $x \in S$ so it can be assumed that $0 \circ x = x$ for all $x \in S$. Also in [6] it is shown that two matrices $A(T_1)$ and $A(T_2)$ are permutationally similar if and only if the binary operation in T_1 is isomorphic to the binary operation in T_2. So to determine if $A(T_1)$ is permutation similar $A(T_2)$, one must find a way to determine if the binary operation corresponding to T_1 is isomorphic to the binary operation corresponding to T_2. In [6] it is also proven that any isomorphism of $\circ$ can be assumed to fix 0, so it makes sense to consider only the rows and columns of T which do not contain 0.

Definition 6.3 *For any multiplication table T corresponding to a binary operation $\circ$ in a set $S = \{0, 1, \dots, k-1\}$, define $\widehat{T} \in M_{(k-1)\times(k-1)}$ as the multiplication matrix of T, given by $\widehat{T}_{i,j} = i \circ j$. Further define $\mathcal{T}_k$ to be the set of all $(k-1) \times (k-1)$ multiplication matrices of multiplication tables corresponding to $\mathcal{K}_{k^2}$.*

We now show a way to determine if two matrices $\widehat{T}_1$ and $\widehat{T}_2$ in $\mathcal{T}_k$ correspond to isomorphic binary operations.

Proposition 6.4 *The symmetric group on $k-1$ elements S_{k-1} performs a group action on $\mathcal{T}_k$ given by, if $\sigma \in S_{k-1}$ and $\widehat{T}$ is in $\mathcal{T}_k$:*

$$\sigma(\widehat{T}) = P_\sigma \bar{\sigma}(\widehat{T}) P_\sigma^{-1}$$

where P_σ is the $(k-1) \times (k-1)$ permutation matrix that by left multiplication sends row x to row $\sigma(x)$, and where $\bar{\sigma}(\widehat{T})_{i,j} = \sigma(\widehat{T}_{i,j})$. Further, if $\circ$ and $\star$ are the binary operations associated with $\widehat{T}_1$ and $\widehat{T}_2$ respectively, $\circ$ is isomorphic to $\star$ if and only if $\sigma(\widehat{T}_1) = \widehat{T}_2$.

Proof. First, to prove that this is in fact a group action, it must be shown that:

(a) $id(\widehat{T}) = \widehat{T}$ if id is the identity in S_{k-1}

(b) $\sigma(\tau(\widehat{T})) = (\sigma\tau)(\widehat{T})$ for all $\sigma, \tau \in S_{k-1}$

Since $P_{id} = I$ where I is the identity matrix in M_{k-1}, and $\bar{id}(\widehat{T}) = \widehat{T}$, it follows that

$$id(\widehat{T}) = P_{id}\bar{id}(\widehat{T})P_{id}^{-1} = I(\widehat{T})I = \widehat{T},$$

thus proving (a). To prove (b) first note that $P_\sigma P_\tau = P_{\sigma\tau}$ and that $\bar{\sigma}(\bar{\tau}(\widehat{T})) = \bar{\sigma\tau}(\widehat{T})$ for all $\widehat{T}$ (these two facts follow almost directly from definition). Further, let $B = P_x\bar{x}(A)P_x^{-1}$. B will first replace $A_{i,j}$ with $x(A_{i,j})$ and then send it to the $(x(i), x(j))$ position. Therefore, $x(A_{i,j}) = B_{x(i),x(j)}$. Let $C = \bar{x}(P_xAP_x^{-1})$. C will have every entry $A_{i,j}$ first moved to the $(x(i), x(j))$ position and then replaced with $x(A_{i,j})$. So $C_{x(i),x(j)} = x(A_{i,j})$. But then clearly $B = C$; hence $P_x\bar{x}(A)P_x^{-1} = \bar{x}(P_xAP_x^{-1})$ for any choice of x. Therefore:

$$\sigma(\tau(\widehat{T})) = P_\sigma\bar{\sigma}(\tau(\widehat{T}))P_\sigma^{-1}$$

$$\sigma(\tau(\widehat{T})) = P_\sigma\bar{\sigma}(P_\tau\bar{\tau}(\widehat{T})P_\tau^{-1})P_\sigma^{-1}$$

$$\sigma(\tau(\widehat{T})) = P_\sigma\bar{\sigma}(\bar{\tau}(P_\tau\widehat{T}P_\tau^{-1})P_\sigma^{-1}$$

$$\sigma(\tau(\widehat{T})) = P_\sigma(\bar{\sigma\tau})(P_\tau\widehat{T}P_\tau^{-1})P_\sigma^{-1}$$

$$\sigma(\tau(\widehat{T})) = P_\sigma P_\tau(\bar{\sigma\tau})(\widehat{T})P_\tau^{-1}P_\sigma^{-1}$$

$$\sigma(\tau(\widehat{T})) = P_{\sigma\tau}(\bar{\sigma\tau})(\widehat{T})P_{\sigma\tau}^{-1} = (\sigma\tau)(\widehat{T}))$$

This proves (b), showing that there is in fact a group action. To show the second part of the theorem, just note that if $\circ$ and $\star$ are isomorphic, then there is some permutation of S, call it f, such that:

$$f(x \circ y) = f(x) \star f(y)$$

Therefore, if $\widehat{T}$ is the matrix associated with $\circ$, and $\widehat{U}$ is the matrix associated with $\star$, then $f(\widehat{T}_{i,j}) = \widehat{U}_{f(i),f(j)}$. But, as seen above, this is exactly what happens if $\widehat{U} = P_f\bar{f}(\widehat{T})P_f^{-1}$. Therefore $\widehat{U} = f(\widehat{T})$. This argument clearly holds in either direction, proving the if and only if part, thus proving the whole proposition. □

An example might be useful to help show what this means. Let:

$$\widehat{T} = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix}$$

and let $\sigma = (1, 2)$, ie, the permutation that sends 1 to 2, 2 to 1 and fixes 3. Now, performing the group action yields:

$$\sigma(\widehat{T}) = \begin{pmatrix} 3 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix}$$

Consider a particular value in $\widehat{T}$, say $\widehat{T}_{2,3}$. $\widehat{T}_{2,3} = 1$, which corresponds in multiplication terms to $2 \circ 3 = 1$. If $\circ$ is the binary operation associated with $\widehat{T}$ and $\star$ is associated with $\sigma(\widehat{T})$, $\circ$ is isomorphic to $\star$ by σ, namely, $\sigma(2) \star \sigma(3) = \sigma(2 \circ 3) = \sigma(1)$. In other words, $1 \star 3 = 2$, and indeed $\sigma(\widehat{T}_{1,3}) = 2$, and this will hold for any choice of $\widehat{T}_{i,j}$.

Now that it has been shown that the symmetric group performs a group action on $\mathcal{T}_k$, to count the number of distinct matrices in $\mathcal{K}_n$ one can use the Burnside Theorem to count the number of matrices in $\mathcal{T}_k$ that are fixed under each permutation in S_{k-1}. This Theorem tells us that

$$|\mathcal{K}_n| = |\mathcal{T}_k| = \frac{1}{(k-1)!} \sum_{\sigma \in \ S_{k-1}} F(\sigma),$$

where $F(\sigma)$ is the number of $(k-1) \times (k-1)$ matrices that are the multiplication matrix for some multiplication table T and are fixed under σ. So the question is, given a certain permutation, is there a way to determine how many multiplication matrices are fixed under that permutation. First, a total number of different (not necessarily non-isomorphic) multiplication matrices of this form must be found. Since there has to be a unique solution to $x \circ y = z$ for all x, z, all $k-1$ numbers must appear in each row, and each position in the row must be distinct. So there are $(k-1)!$ ways to make each row, and since no row is dependent on any other, there are $(k-1)!^{k-1}$ ways to make these matrices. By definition, each of these is fixed under the identity permutation, so the first term in the summand is $(k-1)!^{k-1}$. However, a different technique is needed to determine the number of fixed points for the general permutation. First, note that any permutation in S_{k-1} can be written as a product of disjoint cycles. Also, because they are disjoint, the order that these cycles appear does not matter, so they may be put in increasing order according to size. Further, any multiplication matrix $\widehat{T}$ that is fixed under a given permutation $\sigma = (x_{11}, \ldots, x_{1n_1})(x_{21}, \ldots, x_{2n_2}) \ldots (x_{m1}, \ldots, x_{mn_m})$ can be transformed via the group action to another multiplication matrix $\widehat{T}'$ that is fixed under $\sigma' = (1, \ldots, a_1)(a_1 + 1, \ldots, a_2) \ldots (a_m + 1, \ldots, a_{m+1})$, where the numbers in each cycle are consecutive and the number and size of the cycles in σ' is the same as in σ. This can be done by the permutation that sends

$$x_{11} \Rightarrow 1, x_{12} \Rightarrow 2, \ldots x_{1n_1} \Rightarrow a_1, x_{21} \Rightarrow a_1 + 1, \text{etc} \ldots$$

And similarly, given any $\widehat{U}$ that is fixed under σ', by doing the reverse permutation of the one described above, $\widehat{U}$ can be transformed into some multiplication matrix $\widehat{U}'$ that is fixed under σ. Therefore, there is a one to one correspondence between matrices fixed under either of these permutations. To count the number of matrices fixed under σ, one just has to count the number of matrices fixed under σ', and in general to count the number of matrices fixed under all permutations with n cycles that have x_n elements in each cycle, it suffices to just count the number of matrices fixed under $\sigma' = (1, \dots, a_1)(a_1 + 1, \dots, a_2) \dots (a_{n-1} + 1, \dots, a_n)$, where $a_1 = x_1, a_2 = x_2 + a_1, \dots a_n = x_n + a_{n-1}$, and all numbers in each cycle are consecutive. This will be called writing σ in "consecutive form". For example, to find the number of matrices fixed under $(2, 3)(5, 1, 4)$, this number will be the same as the number of matrices fixed under $(1, 2)(3, 4, 5)$. So in the Burnside sum, if x is the number of matrices fixed under $(1, 2)(3, 4, 5)$, and y is the number of different permutations made up of one 2-cycle and one 3-cycle, the amount added will be xy (note: some of these matrices will be counted two or more times; however, since this sum is supposed to have overlap, this is ok). The question becomes twofold: given a specific permutation σ in consecutive form, how matrices are fixed under σ and how many different permutations have the same "size" as σ (here and throughout, two permutations will be said to have the same size if they have the same number n of cycles and the same number x_n elements in each cycle). For a given matrix size of $n \times n$, we will look at a specific permutation $\sigma \in S_n$ and determine how many multiplication matrices of this size are fixed under all permutations with the same size as σ. First, assume the cycles of σ are in increasing order, and that σ is in consecutive form. Let $\sigma = (1, \dots, a_1)(a_1 + 1, \dots, a_2) \dots (a_{k-1} + 1, \dots, a_k)$, where the j_{th} cycle has size x_j. Now try to find a multiplication matrix $\widehat{T}$ such that $\widehat{T}$ is fixed under σ. Note that since $\widehat{T}$ is a multiplication matrix it has certain properties: namely that for any row r of $\widehat{T}$, the numbers 1 through n must appear exactly once in row r, and for any $1 \leq i, j \leq n$ that $\widehat{T}_{i,j} = i \circ j$ where $\circ$ is the binary operation associated with $\widehat{T}$. Since $\widehat{T}$ is fixed under σ, $\sigma(\widehat{T}_{i,j}) = \widehat{T}_{\sigma(i),\sigma(j)}$ for all $1 \leq i, j \leq n$, meaning that $\sigma(i) \circ \sigma(j) = \sigma(i \circ j)$ as well. In particular, since the permutation we are considering is not the identity, there is at least one cycle of size 2 or greater. Therefore, $a_1 > 1$; this means that $2 \circ x = \sigma(1) \circ \sigma(\sigma^{-1}(x)) = \sigma((1 \circ \sigma^{-1}(x))$ for all $1 \leq x \leq n$, so this means that row 2 of $\widehat{T}$ is completely determined by row 1. Further, by this same argument, rows 2 through a_1 will be determined by row 1, and similarly all rows of cycle j will be completely determined by the first row of that cycle. So to determine all the different ways in which $\widehat{T}$ can be made, it suffices to determine

(a) The number of different ways to arrange the first row of each cycle,

(b) The number of different ways to arrange the rows which are not moved by σ (ie all rows r such that $a_k < r \leq n$).

First, we will determine the number of different ways to arrange row 1 and then generalize this to the first row of each cycle. First note that if cycle 1 has size x_1, then $\sigma^{x_1}(1) = 1$ and x_1 is the smallest exponent for which this occurs. Therefore for all y such that $1 \leq y \leq n$, if $1 \circ y = z$ then

$$1 \circ \sigma^{x_1}(y) = \sigma^{x_1}(1) \circ \sigma^{x_1}(y) = \sigma^{x_1}(1 \circ y) = \sigma^{x_1}(z).$$

In addition, for any positive integer m, $\sigma^{mx_1}(1) = 1$, so similarly $1 \circ \sigma^{mx_1}(y) = \sigma^{mx_1}(z)$. If y occurs in a cycle of size x_j then clearly $\sigma^{x_j x_1}(y) = y$, so there is always a solution in positive integers m for $\sigma^{mx_1}(y) = y$. Call the smallest such solution $m_{x_1}(j)$. Therefore, for all

$0 \leq m < m_{x_1}(y)$, $\sigma^{mx_1}(y)$ and therefore $1 \circ \sigma^{mx_1}(y)$ will be different, thus meaning each value of $\sigma^{mx_1}(z)$ must be distinct. These $m_{x_1}(y)$ positions of row 1 are completely fixed by the initial value of $1 \circ y$. Further, if $z = 1 \circ y$, z must also satisfy that $\sigma^{m_{x_1}(y)x_1}(z) = z$. Therefore, $m_{x_1}(z) = m_{x_1}(y)$. Call the set of all a such that $1 \leq a \leq n$ for which $m_{x_1}(a) = m_{x_1}(y)$ the M-class of y for x_1 and denote it by $\mathcal{M}_{x_1}(y)$. Clearly z can be a solution to $1 \circ y = z$ if and only if $z \in \mathcal{M}_{x_1}(y)$. If $M = |\mathcal{M}_{x_1}(y)|$, and each choice of a value of $1 \circ a$ ($a \in \mathcal{M}_{x_1}(y)$) fixes $m_{x_1}(y)$ values of those M positions, there are M choices for the first position, $(M - m_{x_1}(y))$ for the next, etc. Now for every a is in $\mathcal{M}_{x_1}(y)$, there are precisely $m_{x_1}(y)$ in the orbit of a given by: $\sigma^{mx_1}(a)$ for $0 \leq m < m_{x_1}(y)$. It is not difficult to see that each of these orbits are disjoint, and since each has $m_{x_1}(y)$ elements in it, $M = Nm_{x_1}(y)$ for some integer N. Therefore, in these M positions there are

$$(M)(M-m_{x_1}(y))(M-2m_{x_1}(y))\ldots = (Nm_{x_1}(y))(N-1)(m_{x_1}(y))\ldots(1)(m_{x_1}(y)) = (N!)(m_{x_1}(y))^N$$

ways to arrange these M positions. Therefore, to find the total number of ways of arranging the entire first row, it suffices to find the number of different values for $m_{x_1}(a)$ for all a such that $1 \leq a \leq n$ and to find how many a there are that have each as its value. Fortunately there is a simple way to compute $m_{x_1}(a)$. Simply note that if a is in an orbit with size x_j any solution to $\sigma^{m_{x_1}(a)x_1}(a) = a$ must have x_j divide $m_{x_1}(a)x_1$, and this must be the smallest multiple of x_1 for which this occurs, so $m_{x_1}(a)x_1 = lcm(x_1, x_j)$, so

$$m_{x_1}(a) = \frac{lcm(x_1, x_j)}{x_1},$$

where lcm means least common multiple. Therefore it makes sense to compute the set

$$L_{x_1} = \{l \mid l = \frac{lcm(x_1, x_j)}{x_1} \text{ for } 1 \leq j \leq k\}.$$

Now if $l \geq 2$, to determine the total number M of values that have $l = \frac{lcm(x_1,x_j)}{x_1}$ (which we will denote by M_l) it is necessary to first find the set

$$J_l = \{j \mid \frac{lcm(x_1, x_j)}{x_1} = l\}.$$

Since there are exactly x_j elements in the j_{th} cycle, the following sum can be computed:

$$M_l = \sum_{j \in J_l} x_j$$

Now, to compute M_1, we must remember that there are exactly p elements such that $\sigma(a) = a$, where $p = n - a_k$. Though these elements are not in any cycle and have not been counted yet, they will clearly have $m_{x_1}(a) = 1$ as well as those in x_j for $j \in J_1$. Therefore,

$$M_1 = p + \sum_{j \in J_1} x_j.$$

Therefore, since this covers every possible position in row 1, we can compute the number of ways of arranging row 1, which we will denote as $r(1)$, as the product of each of these sections, or as

$$r(1) = \prod_{l\in\ L_{x_1}} (N!)(l^N) = \prod_{l\in\ L_{x_1}} (\frac{M_l}{l})!(l^{\frac{M_l}{l}}).$$

An example might be useful here. Consider $n = 23$ and

$$\sigma = (1,2)(3,4,5)(6,7,8,9,10,11)(12,13,14,15,16,17,18,19).$$

Here, $x_1 = 2, x_2 = 3, x_3 = 6, x_4 = 8$ and also $a_1 = 2, a_2 = 5, a_3 = 11, a_4 = 19$ and $p = 4$. From this information, it is easy to compute the following:

$$L_{x_1} = \{1,3,4\}$$

$$J_1 = \{1\}\ J_3 = \{2,3\}\ J_4 = \{4\}$$

$$M_1 = 4 + (2) = 6\ M_3 = (3+6) = 9\ M_4 = 8$$

$$r(1) = \left(\frac{6}{1}\right)!\left(1^{\frac{6}{1}}\right)\left(\frac{9}{3}\right)!\left(3^{\frac{9}{3}}\right)\left(\frac{8}{4}\right)!\left(4^{\frac{8}{4}}\right) = 3,732,480$$

In principle the same method can be used to determine the ways of arranging the first row of the j^{th} cycle by simply defining all of the above sets and functions by replacing x_1 with the specific x_j, leading to the more general formula

$$r(j) = \prod_{l\in\ L_{x_j}} \left(\frac{M_l}{l}\right)!\left(l^{\frac{M_l}{l}}\right).$$

Therefore, the ways in which the first a_k rows of the matrix can be made is simply the product of each of these individual $r's$. If we denote the multiset $\mathcal{S}$ (keeping all multiple values) by

$$\mathcal{S} = \{x_j \mid 1 \le j \le k\},$$

and if we denote the number of ways the first a_k rows can be arranged as $R(\mathcal{S})$, then

$$R(\mathcal{S}) = \prod_{j=1}^{k} r(j).$$

We can further denote the set of distinct elements of $\mathcal{S}$ as $\mathcal{S}_d$ and denote its elements as $y_1, \ldots, y_m$. Now it remains to determine how many different arrangements can be made from the last p rows, the rows where the value for the row is fixed by σ (referred to here as the fixed rows). However, without a lot of effort, it can be seen that the ways each of these fixed rows can be arranged has the following similar definitions:

$$L_f = \mathcal{S}_d$$

$$J_l = \{j \mid x_j = l\}$$

$$M_l = \sum_{j \in \ J_l} x_j$$

$$r_f = p! \prod_{l \in \ L_{x_1}} \left(\frac{M_l}{l}\right)! \left(l^{\frac{M_l}{l}}\right)$$

And since there are p such rows, the total number of ways of arranging these p rows is $(r_f)^p$, so the total number of ways of arranging $\widehat{T}$ and therefore the number of matrices fixed under σ is $R(\mathcal{S})(r_f)^p$. Hence, this is also the amount of matrices fixed under any permutation with the same size as σ, ie, with the same set $\mathcal{S}$. We denote this number as $\mathcal{P}_\mathcal{S}$ This is the number of ways of arranging the numbers 1 through n in the first x_1 positions (the first cycle) the remaining $(n - x_1)$ numbers in the next x_2 positions, etc, corresponding to

$$\mathcal{P}_\mathcal{S} = (nPx_1)((n - x_1)Px_2) \ldots ((n - x_1 - x_2 - \ldots - x_{k-1})Px_k),$$

$$\mathcal{P}_\mathcal{S} = \left(\frac{n!}{(n - x_1)!}\right) \left(\frac{(n - x_1)!}{(n - x_1 - x_2)!}\right) \cdots \left(\frac{(n - x_1 - x_2 - \ldots - x_{k-1})!}{(n - x_1 - x_2 - \ldots - x_k)!}\right),$$

$$\mathcal{P}_\mathcal{S} = \frac{n!}{(n - x_1 - x_2 - \ldots - x_k)!} = \frac{n!}{(n - a_k)!} = \frac{n!}{p!}.$$

However, this counts many permutations twice, since, for example, it counts $(1, 2)$ and $(2, 1)$ as different permutations, and in general it counts each cycle of size x_j exactly x_j times, so it needs to be divided by the product of all of the x_j, or, in a way that is equivalent, if y_i (from $\mathcal{S}_d$) has multiplicity λ_i then

$$\mathcal{P}_\mathcal{S} = \frac{n!}{p! \prod_{i=1}^{m} (y_i^{\lambda_i})}.$$

Further, this counts two cycles of the same size in different orders differently, for example $(1, 2)(3, 4)$ as different from $(3, 4)(1, 2)$, so this also needs to be divided by the number of ways of arranging cycles of the same size, namely $\lambda_i!$, so the finished product is (now in correct form)

$$\mathcal{P}_\mathcal{S} = \frac{n!}{p! \prod_{i=1}^{m} (y_i^{\lambda_i})(\lambda_i!)}.$$

However, further contemplation of the denominator leads to the conclusion that it is exactly the same as r_f. Therefore, it follows that

$$\mathcal{P}_\mathcal{S} = \frac{n!}{r_f}$$

This leads to the finished theorem:

Theorem 6.5 *The total number of Knuthian matrices of size n (where n is a perfect square), or $|\mathcal{K}_n|$, is given by the following formula:*

$$|\mathcal{K}_n| = (\sqrt{n}-1)!^{\sqrt{n}-2} + \sum_{k=1}^{\lfloor\frac{\sqrt{n}-1}{2}\rfloor} \sum_{\mathcal{S}\in\ \mathfrak{S}_k} R(\mathcal{S})(r_f)^{p-1}$$

where $\mathfrak{S}_k = \{\{x_1, x_2, \ldots, x_k\} \mid 2 \le x_1 \le x_2 \le \ldots \le x_k\}$ and all other notation is as defined above.

Proof. Since matrices of size n correspond to multiplication tables of size $\sqrt{n}$ they therefore multiplication matrices of size $\sqrt{n}-1$. It is already been found that for a given value of $\mathcal{S}$, the number of matrices that are fixed under all permutations of size $\mathcal{S}$ is $\mathcal{P}_{\mathcal{S}}R(\mathcal{S})(r_f)^p$ and since $\mathcal{P}_{\mathcal{S}} = \frac{(\sqrt{n}-1)!}{r_f}$ (for this specific case) it follows that this number is equivalent to $(\sqrt{n}-1)!R(\mathcal{S})(r_f)^{p-1}$. This number must be summed over all possible sizes of permutations. The set of all permutation sizes with exactly k cycles is given by $\mathfrak{S}_k$. Further, since each cycle must have at least two elements, the maximum number of cycles of permutations is the size of the multiplication matrix in question divided by 2 (rounded down if necessary). So this sum must be done from $k=1$ to $k = \lfloor\sqrt{n}-1\rfloor$. Further, this sum covers all matrices fixed under some permutation that is not the identity. The total number of matrices fixed under the identity is given above, namely $(\sqrt{n}-1)!^{\sqrt{n}-1}$. So the total number of fixed points in the Burnside sum, which we will denote as $\mathfrak{F}$ is given by

$$\mathfrak{F} = (\sqrt{n}-1)!^{\sqrt{n}-1} + \sum_{k=1}^{\lfloor\frac{\sqrt{n}-1}{2}\rfloor} \sum_{\mathcal{S}\in\ \mathfrak{S}_k} (\sqrt{n}-1)!R(\mathcal{S})(r_f)^{p-1}.$$

And since $|\mathcal{K}_n| = \frac{\mathfrak{F}}{(\sqrt{n}-1)!}$, it is clear that this reduces to the theorem. □

Though this enumeration can always be done in principle, it gets very large rather quick. A table of values of $|\mathcal{K}_n|$ for small n is included in the Appendix. One use of this theorem is it verifies our belief that complete classification for $\mathcal{A}_n$ is probably impossible; $\mathcal{A}_{25}$ alone must contain at least $14,022$ members! One might wonder what kinds of ranks are possible to construct given this method. Fortunately, there is a simple way to determine this.

Theorem 6.6 *For any integer k, if $n = k^2$, the maximum rank of a matrix in the set $\mathcal{K}_n$ is $2k-2$. Further, for all integers r such that $k \le r \le 2k-2$, matrices exist in $\mathcal{K}_n$ with rank r.*

Proof. This theorem is stated in [6] but the author does not supply a proof. So we will supply one. It is fairly clear from the construction of matrices in $\mathcal{K}_n$ that entries of any matrix K in $\mathcal{K}_n$ differ from the standard matrix can be found in columns $k+1, 2k+1, \ldots, (k-1)k+1$. Further, every row that is different from the standard matrix will contain $(k-1)$ $1's$ in one block of columns (of size k), and the other 1 in another block. If we label the blocks

$0, 1, \ldots, k-1$, and we consider the labeling of row r as an ordered pair given by (i, j) (using the labelling given in the original construction), it has the extra 1 in the block $i \circ j$ and in fact in position $(i \circ j, 0)$ by definition. Therefore, any row vector in K can be written as an ordered pair $[x, y]$, with its $(k-1)$ $1's$ in the x^{th} block, and its lone 1 in the y^{th} block. The standard row vectors are therefore written as $[0, 0], [1, 1]$, etc. This notation allows us to add two rows in a very straightforward manner:

$$[x_1, y_1] + [x_2, y_2] = [(x_1, x_2), (y_1, y_2)]$$

What this means is that adding the row with $(k-1)$ $1's$ in the x_1 block with one with $(k-1)$ $1's$ in the x_2 block, and now it has $(k-1)$ $1's$ in both blocks (if both are the same, then it has $(k-1)$ $2's$ in that block, but this means the same thing since it is the same as 2 sets of $1's$). The same meaning is given to the block with the lone 1. Further, subtraction can be defined similarly, by working backwards. For example:

$$[(a, b), (c, d)] - [a, d] = [b, c](since)[a, d] + [b, c] = [(a, b), (c, d)]$$

Now consider the $k-2$ rows vectors:

$$[1, 2], [1, 3], \ldots, [1, k-1]$$

It is not hard to see that these form a linearly independent set when taken together with the standard row vectors. If it were not, then $[1, y]$ would some combination of positive and negative row vectors, at least some must be of the form $[1, x]$ (for any $x \neq y$) since clearly it cannot be made just from the standard row vectors. But this also requires the subtraction (or addition) of some other row vector that ends in x. However, the only choice is $[x, x]$ but to subtract (or add) that requires another row vector that starts with x. But again, the only choice of a row vector that ends in x is also $[x, x]$, so this addition and subtraction cannot be done. To show that these $k-2$ linearly independent row vectors, along with the k standard row vectors, span the rest of the possibilities, it suffices to show that any $[x, y]$ (x, y both less than $k-1$) can be created by this set. But note that the simple formula

$$[1, y] - [1, x] + [x, x] = [1 - 1 + x, y - x + x] = [x, y]$$

generates all of these ordered pairs. Therefore, any other row of K is a linear combination of these $2k-2$ vectors, so can have at most rank $2k-2$. For any choice of r between k and $2k-2$, to create a matrix of rank r, simply create the multiplication matrix $\widehat{T}$ by first setting $\widehat{T}_{i,j} = j$. Then, in rows 1 through $r-k$ of $\widehat{T}$, simply set $\widehat{T}_{(r-k),1} = (r-k)$ and $\widehat{T}_{(r-k),(r-k)} = 1$. This will preserve the rules for $\widehat{T}$ and will cause the corresponding matrix $K = A(T)$ to have one rank added in each of the blocks 1 through $r-k$, and each of these will be independent, as shown above, so the total rank of K is $r-k+k$ or r. □

One other interesting property that the class $\mathcal{K}_n$ has is that some open questions about $\mathcal{A}_n$ in general can be answered about $\mathcal{K}_n$ specifically. Two are given below.

Proposition 6.7 *Every $K \in \mathcal{K}_n$ can be obtained from the standard matrix by a finite number of switches. In fact, if $k^2 = n$, a maximum of $(k-1)(k-2)$ switches will suffice.*

Proposition 6.8 *Every $K \in \mathcal{K}_n$, if $k^2 = n$, contains in it a sub-matrix L that is $(k-1)^2 \times (k-1)^2$ and is in $\mathcal{K}_{(k-1)^2}$. This matrix, in fact, is the standard matrix in $\mathcal{K}_{(k-1)^2}$.*

Proof. To prove both of these propositions, again note that the only entries in which K differs from the standard matrix are in columns $k+1, 2k+1, \ldots, (k-1)k+1$. To prove the first, consider the places where a given K differs from the standard matrix, and consider the labelling of the points originally given. Consider ordered pairs of the form $(1, y)$ first, corresponding to the first block of K. They will map to $(y, 1), (y, 2), \ldots, (y, k)$, just like the standard matrix does. The only change is that $(1, y)$ also maps to $(1 \circ y, 0)$ which may or may not be different from $(y, 0)$. So the only points that need to be switched are where $(1, y)$ maps to and the points that map to $(y, 0)$ for all $1 \leq y \leq k$. Now note that no matter what the binary operation $\circ$ is, all points of the form $(y, 0)$ map to the points $(0,0), (0,1) \ldots, (0, k)$ since $y \circ 0 = 0$, so all points of this form have the same out-neighborhood. Further, all points of the form $(1, y)$ are mapped to by the points $(0, 1) \ldots (k, 1)$, so all of these points have the same in-neighborhood. What this means is any two points of the form $(1, y_1)$ and $(1, y_2)$, such that $(1, y_1) \to (0, x_1)$ and $(1, y_2) \to (0, x_2)$ can switch so that $(1, y_1) \to (0, x_2)$ and $(1, y_2) \to (0, x_1)$, since $(1, y_1), (1, y_2)$ share the same in-neighborhood and $(0, x_1), (0, x_2)$ share the same out-neighborhood. Now consider the first row of $\widehat{T}$, the multiplication matrix that corresponds to K. Let $\sigma \in S_{k-1}$ be the permutation that sends each point y to $1 \circ y$. Further, it is a known rule that any permutation in S_{k-1} can be written as the product of at most $k-2$ (not necessarily distinct) transpositions. For each transposition of $\sigma = (a_1, a_2)$ simply choose the two points $(a_1, 0)$ and $(a_2, 0)$ and switch the paths from the two points of the form $(1, y)$ that go to either. By the above comments this can always be done, and by the definition of σ it is clear this will make it so every point $(1, y)$ that used to map to $(y, 0)$ now maps to $(x \circ y, 0)$. Further this can be done to every one of the $(k-1)$ rows of $\widehat{T}$, taking at most $(k-2)$ switches in each row. Therefore, every matrix K can be made from the identity matrix in this way, with no more than $(k-1)(k-2)$ switches. To prove the second proposition, again relabel the points in K to correspond with the original construction and remove any point that is either of the form $(x, 0)$ or $(0, x)$. It is clear from the definition that this still is a central groupoid, since by definition the product of any two points not containing a zero in them also does not contain a zero in it. Further, these $(k-1)^2$ points will all satisfy the rule that there is a path $(x_1, x_2) \to (y_1, y_2)$ if and only if $x_2 = y_1$ (since all points with a zero are removed, the second condition does not apply). This corresponds directly to the definition of a natural central groupoid of size $(k-1)^2$, which is of course the standard matrix in $\mathcal{K}_{(k-1)^2}$. □

The construction of $\mathcal{K}_n$ is one that has several nice properties. However, it by itself is incomplete. It is natural to wonder if it can be extended to create more matrices in $\mathcal{A}_n$. It can. Instead of the condition that $(x, y) \to (x \circ y, 0)$, instead use $(x, y) \to (x \circ y, f(x))$, where f is some function from $\{0, 1, \ldots, \sqrt{n}-1\}$ to itself. The full rule would be that there is a arc $(x_1, x_2) \rightarrow (y_1, y_2)$ if and only if

$$x_2 = y_1 \text{ and } y_1 \neq f(x_1)$$

or

$$x_1 \circ x_2 = y_1 \text{ and } y_2 = f(x_1).$$

This construction clearly contains as a subclass the construction for $\mathcal{K}_n$ simply by setting $f(x) = 0$ for all x. However, while only 3 matrices of $\mathcal{A}_9$ are in $\mathcal{K}_9$, all 6 matrices of $\mathcal{A}_9$ can be made by this construction (a list of the multiplication tables for these 6 matrices is included in the Appendix). It does not appear that this holds for all n, but this is still an open question. This certainly allows us to broaden our understanding of $\mathcal{A}_n$ in general, as well as getting a more precise lower bound for its size. One problem with this construction is that one has to be careful that there is always a unique length two path between any two points. For that to happen it appears that f must satisfy these two identities:

(a) For all x, y: $y \circ f(x) = (x \circ y) \circ f(x)$

(b) If $f(x \circ y) \neq f(y)$ then $y \circ f(x) = (x \circ y) \circ f(x) = f(x)$

Whether these two conditions are sufficient, or even necessary, is not completely known. Further work in this particular direction appears difficult, because of the lack of rules that $\circ$ must follow; however, future analysis will determine whether any more can be done with this construction.

7 Computer Implementation

Throughout our analysis, central groupoids (central digraphs, matrices A that satisfy $A^2 = J$) have been considered to be different if and only if they are not similar up to permutation similarity. Theoretically, a computer may be programmed to consider all possible permutations of a matrix to determine if in fact two matrices are permutationally similar. However, even modern day computers require an extensive amount of run-time to consider all possible permutations of, say, a 16×16 matrix (requiring 16! permutations). A more refined algorithm was used throughout our analysis, facilitating our ability to determine whether or not two matrices are permutationally similar. The following is a brief summary of the computer programs (in both MatLab and C/C++) used in our analysis of central groupoids.

permutations.m :

We define the notation $X \sim_p Y$ to denote that the matrix X is permutationally similar to the matrix Y. Given two matrices A_1 and A_2, the following are invariant under permutation similarity:

(a) size (both matrices must be $m \times n$),

(b) rank (both matrices must have rank r),

(c) row and column multiplicities (if $A_1 \sim_p A_2$ and A_1 has row multiplicity vector r_1 and column multiplicity vector c_1, then A_2 must have multiplicity vectors r_2 and c_2 such that $r_1 = r_2$ and $c_1 = c_2$).

Furthermore, we consider only matrices A_1 and A_2 that satisfy $A_1^2 = A_2^2 = J$. Both invariants (a) and (b) can be determined by inspection of the matrix or by reserved MatLab commands. Given two matrices A_1 and A_2 such that size(A_1) = size$(A_2) = n \times n$, rank(A_1) = rank(A_2), and $A_1^2 = A_2^2 = J$, the row and column multiplicities invariant was used to minimize the number of permutations necessary to determine if the two matrices are indeed permutationally similar.

The following algorithm was implemented in MatLab:

(1) permute matrices A_1 and A_2 into DRM (defined below) form, producing $A_1^{(D)}$ and $A_2^{(D)}$,

(2) consider all permutations of indices within each block of equivalent rows,

(3) if two or more blocks of equivalent rows have the same cardinality, consider all permutations of such blocks,

(4) permute A_1 only according to the above considerations and compare each with A_2,

(5) we claim $A_1 \sim_p A_2$ if and only if one of the considered permutations, P_i, yields $P_i * A_1^{(D)} * P_i^t = A_2^{(D)}$.

We define DRM (Decreasing Row Multiplicity) form to be a matrix having the property that if row i and row j are equivalent, then either $i = j+1$, $i = j-1$, or all rows between i and j are equivalent to both i and j. Furthermore, sets of equivalent rows appear in decreasing size from the top to the bottom of the matrix. In other words, if a set of 3 equivalent rows and a set of 4 equivalent rows appear in the matrix, then the set of 4 rows (rows a, b, c, and d) and 3 rows (rows x, y, and z) have $a,b,c,d < x,y,z$. Consider A_2, see Appendix A, as an example of a 16×16 matrix with rank 6, row multiplicity vector $\{4,3,3,2,1,1,1,1\}$, column multiplicity vector $\{4,3,3,3,1,1,1\}$, and having $A_2^2 = J$ with its corresponding DRM form $A_2^{(D)}$. Note that $A_2 \sim_p A_2^{(D)}$ and that $A_2^{(D)}$ contains sets of equivalent rows in decreasing size from the top to the bottom of the matrix. The number of permutations required to determine whether the above matrix is permutationally similar to another matrix with same size, rank, and row and column multiplicities is, $(4!)^2 * (3!)^2 * (2!)^2 = 82,944 < 20,922,789,888,000 = 16!$.

matrices_through_algebra.m :

A specific class of central groupoids of any given size can be found using algebraic methods, see [6]. For the 16×16 case, analysis has found that there are 44 different matrices in this class, which has been verified by this computer program. Essentially, an exhaustive approach was used to determine all possible permutationally different central groupoids that are completions of the following matrix:

$$A_{1,1} = A_{2,1} = A_{3,1} = A_{4,1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_{1,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A_{1,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A_{1,4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$A_{2,2} = A_{3,2} = A_{4,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 1 & 1 & 1 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix},$$

$$A_{2,3} = A_{3,3} = A_{4,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 1 & 1 & 1 \\ * & 0 & 0 & 0 \end{pmatrix},$$

$$A_{2,4} = A_{3,4} = A_{4,4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 1 & 1 & 1 \end{pmatrix}.$$

As shown by previous analysis, theory and computer implementation have verified that there are 44 permutationally different completions.

test_new_matrix.m :

Distinct central groupoids of any given size can be found through a variety of methods. However, utilizing these different methods results in an extensive amount of overlap (two permutationally similar matrices can be found using two different methods). Therefore, this program was created to store all known matrices with their critical properties (rank, multiplicities, etc.), as well as provide a means to test permutation similarity with any (possibly) new matrix. The program takes the new matrix as input and determines if the matrix is permutationally similar to some already known central groupoid.

buildmatrix.cc

To facilitate the process of creating central groupoids through simple addition and removal of 1's from a zero-one matrix, this program was often used. Given any initial zero-one matrix $A = J - B$, with B a zero-one matrix, the program has the following menu of options:

(1) add to a position

(2) remove from a position

(3) see positions allowed for addition

(4) see current A^2 matrix

(5) reprint current matrix A

Options (1) and (2) directly change the current matrix stored in memory, while the remaining options allow the user to view the progress towards the creation of a central groupoid. Option (3) displays all allowable positions for addition of a 1 to the matrix by restricting row and column multiplicities and determining if a 1 in position $a_{i,j}$ would create an entry $a_{s,t}^{(2)}$ in A^2 equal to c, with $c > 1$.

switches.m, unique_switches.m

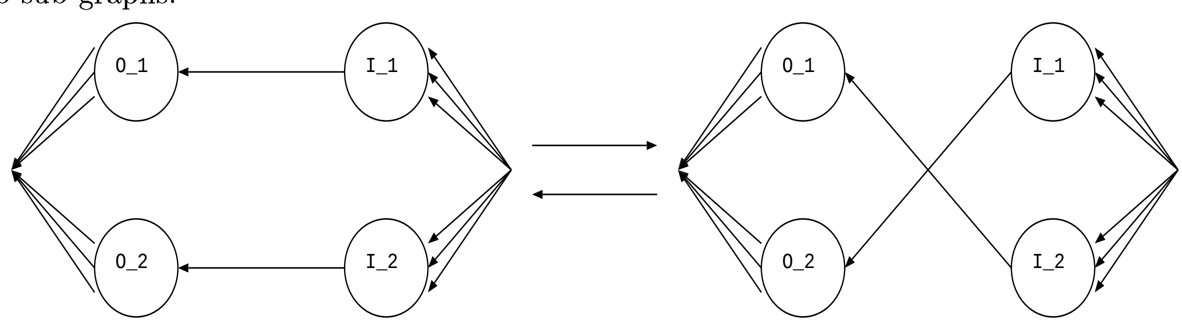
Considering the proposition that all central groupoids of size $n \times n$ can be constructed by a series of switches performed on the standard construction of A_n, these two programs can be used to perform switches on any given matrix A. To perform a switch (otherwise known as an interchange transformation), we first consider all pairs of vertices that have either the same inneighborhood or the same outneighborhood, see [4] and references. In matrix terms, equal inneighborhoods and outneighborhoods correspond to equivalent columns and rows, respectively. Let I be the set of all inneighborhood pairs and let O be the set of all outneighborhood pairs. We wish to consider all selections of one element from I and one element from 0 that, when we considering the submatrix found by using the pair from I as the row indices and the pair from O as the column indices yields either:

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} or \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

In other words, for equivalent columns I_1 and I_2 and equivalent rows O_1 and O_2, we consider the submatrix:

$$\begin{pmatrix} a_{I_1,O_1} & a_{I_1,O_2} \\ a_{I_2,O_1} & a_{I_2,O_2} \end{pmatrix}$$

and determine if it equals either of the above. In graphical terms, we note that an interchange transformation corresponds to a switch from one to the other of the following two sub-graphs:



The program switch.m determines all valid switches that can be performed on a given matrix A. However, since two or more valid switches may create permutationally similar matrices, the program unique_switches.m determines a complete set of matrices that are permutationally unique to one another (using permutation.m).

sub_groupoid.m

Determines how many sub-central groupoids of size $m \times m$ are embedded inside of an $n \times n$ central groupoid. First, we determine all possible choices of $\sqrt{m}$ idempotent indices there are from the original $\sqrt{n}$ idempotent indices. Then, we consider all possible choices of $(m-\sqrt{m})/2$ two-cycle pairs from the original $(n-\sqrt{n})/2$ two-cycle pairs. We claim that all possible sub-central groupoids can be represented as some set of $\sqrt{m}$ idempotent indices and $(m-\sqrt{m})/2$ two-cycle pairs.

8 Other Observations

One possible approach to construct $A \in \mathcal{A}_n$ is to find suitable permutation matrices $P_1, \ldots, P_k$ so that $A = P_1 + \cdots + P_k$, where $k^2 = n$, as shown in the following.

Proposition 8.1 *Let $n = k^2$. Every $A \in \mathcal{A}_n$ can be written as $A = P_1 + \ldots + P_k$ such that P_iP_j and P_rP_s are disjoint permutation matrices for any distinct $(i,j) \neq (r,s)$ with $1 \leq i,j,r,s \leq k$.*

Proof. By Corollary 1.2.5 in [2], A is the sum of k permutation matrices. Since $A^2 = J_n$, the condition on P_iP_j follows. □

It would be desirable to impose more structures on $P_1, \ldots, P_k$. When $n = 9$, we may assume that $P_1^2 = I$.

Proposition 8.2 *For all $A \in \mathcal{A}_9$, we assume the top 3 rows to be of standard form and that the following permutation matrix may be extracted as P_1, where $P_1^2 = I$.*

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix}.$$

Proof. From Proposition 2.5, it follows that three of the vertices of $G(A)$ are idempotent and the other six are partitioned into three two-cycles. We may label the idempotent vertices v_1, v_5 and v_9, while choosing the two-cycles to be (v_2, v_4), (v_3, v_7) and (v_6, v_8). □

A word of $W(A, A^t)$ of length m is defined to be a product of m matrices $X_1, \ldots, X_m$ such that $X_j \in \{A, A^t\}$ for all j. The following proposition is new.

Proposition 8.3 *Suppose $A \in \mathcal{A}_n$, and $W(A, A^t)$ is a word of length m not of the form $(AA^t)^{m/2}$ or $(A^tA)^{m/2}$. Then $W(AA^t)$ has eigenvalues $k^m, 0, \ldots, 0$.*

Proof. If $A^2 = J$ or $(A^t)^2 = J$ appears in $W(A, A^t)$, then $W(A, A^t) = k^{m-2}J$, and we are done. If not, $W(A, A^t)$ must be of the form $(AA^t)^{(m-1)/2}A$ or $(A^tA)^{(m-1)/2}A^t$ by our assumption. In both cases, we can shift the first letter to the last letter in the word to obtain a new word with the same eigenvalues. The result now follows from the first case. □

An open problem in this area asks what are the possible eigenvalues of AA^t.

If A has rank $k = \sqrt{n}$, then AA^t has eigenvalue n with multiplicity k, and 0 with multiplicity $n - k$.

We have complete results for $k = 2, 3$ and partial results for general k when A has rank $k + 1$.

Recall that the permanent of $A = (a_{ij})$ is denoted and defined by $\text{per}\,(A) = \sum_\sigma \prod_{j=1}^n a_{j\sigma j}$, where σ ranges through all possible permutation of $(1, \ldots, n)$. The permanent of of a $(0, 1)$ matrix A can be viewed as the number of permutation matrices P such that $A - P$ is nonnegative. This is useful topic in combinatorial analysis; e.g., see [Chapter 7][2]. In connection to our study, we have the following.

Proposition 8.4 *Let $A \in \mathcal{A}_n$, where $n = k^2$ with $k \in \mathbf{Z}^+$. Then* $\text{per}\,(A) \leq (k!)^k$*, where the equality holds if and only if* $\text{rank}\,(A) = k$.

Proof. See [1] and Theorem 7.4.7 in [2]. □

Remark Note that the study of rank, permanent, and eigenvalues of AA^t, etc. are not affected if we replace A by PAQ for some permutation matrices P and Q.

One might wonder whether there are is a relationship between the rank of $A \in \mathcal{A}_n$ and $\text{per}\,(A)$. For example, if $A, B \in \mathcal{A}_n$ satisfy $\text{rank}\,(A) \leq \text{rank}\,(B)$, does it follow that $\text{per}\,(A) \geq \text{per}\,(B)$? The result is true for $n = 9$, but not for $n = 16$ as shown by the following example: $\text{rank}\,(M_{98}) = 7$ and $\text{rank}\,(M_{100}) = 8$, but $\text{per}\,(M_{98}) = 41472$ and $\text{per}\,(M_{100}) = 43776$ (see Appendix A, Table A).

9 Appendix A: Matrices

$$A_1 = \begin{pmatrix}
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&1&0&1&1&0\\
0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&1\\
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&1&0&1&1&0\\
0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&1\\
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&1&0&1&1&0\\
0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&1
\end{pmatrix},$$

$$A_2 = \begin{pmatrix}
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&1&1&1\\
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1
\end{pmatrix}$$

$$A_2^{(D)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, B_4 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$

$$B_5 = \begin{pmatrix} 1&1&1&0&0&0&0&0&0 \\ 0&0&0&1&1&1&0&0&0 \\ 0&0&0&0&0&0&1&1&1 \\ 1&1&0&0&0&0&0&0&1 \\ 0&0&0&1&1&1&0&0&0 \\ 0&0&1&0&0&0&1&1&0 \\ 1&0&1&0&1&0&0&0&0 \\ 0&1&0&1&0&1&0&0&0 \\ 0&0&0&0&0&0&1&1&1 \end{pmatrix}, B_6 = \begin{pmatrix} 1&1&1&0&0&0&0&0&0 \\ 0&0&0&1&1&1&0&0&0 \\ 0&0&0&0&0&0&1&1&1 \\ 1&1&0&0&0&1&0&0&0 \\ 0&0&1&1&1&0&0&0&0 \\ 0&0&0&0&0&0&1&1&1 \\ 1&0&1&0&0&0&0&1&0 \\ 0&0&0&1&1&1&0&0&0 \\ 0&1&0&0&0&0&1&0&1 \end{pmatrix}$$

Values of $\mathcal{K}_n$ for small n:

n	$\mathcal{K}_n$
1	1
4	1
9	3
16	44
25	14022
36	207392556
49	$6!^5 + 95967624$
64	$7!^6 + 16909969741152$

Multiplication Tables (with functions) for all of $\mathcal{A}_9$:

$$T_{B_1} = \begin{array}{c|ccc|c} \circ & 0 & 1 & 2 & f(x) \\ \hline 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 2 & 0 \\ 2 & 0 & 1 & 2 & 0 \end{array} \quad T_{B_2} = \begin{array}{c|ccc|c} \circ & 0 & 1 & 2 & f(x) \\ \hline 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 1 & 0 \\ 2 & 0 & 2 & 1 & 0 \end{array} \quad T_{B_3} = \begin{array}{c|ccc|c} \circ & 0 & 1 & 2 & f(x) \\ \hline 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 2 & 0 \\ 2 & 0 & 2 & 1 & 0 \end{array}$$

$$T_{B_4} = \begin{array}{c|ccc|c} \circ & 0 & 1 & 2 & f(x) \\ \hline 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 2 & 0 \\ 2 & 1 & 0 & 2 & 1 \end{array} \quad T_{B_5} = \begin{array}{c|ccc|c} \circ & 0 & 1 & 2 & f(x) \\ \hline 0 & 0 & 1 & 2 & 0 \\ 1 & 2 & 1 & 0 & 2 \\ 2 & 1 & 0 & 2 & 1 \end{array} \quad T_{B_6} = \begin{array}{c|ccc|c} \circ & 0 & 1 & 2 & f(x) \\ \hline 0 & 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 & 2 \\ 2 & 2 & 1 & 0 & 1 \end{array}$$

Over the course of our analysis, over 100 central groupoids of size 16×16 were created and used. The following tables display some useful data related to these matrices. Let M_i be the set of matrices obtained by a variety of methods, including all Knuthian matrices (note that $K_i \subset M_i$). Let s be the number of sub-central groupoids of size 9×9.

TABLE A: Data for M_i

i	Rank	Row Multiplicities, Column Multiplicities	eig(AA^t)	per(A)	s
1	4	{4, 4, 4, 4}, {4, 4, 4, 4}	{16, 16, 16, 16, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}	331776	4
2	5	{4, 4, 3, 3, 1, 1}, {4, 4, 3, 3, 1, 1}	{16, 16, 16, 13.29, 2.71, 0, ..., 0}	207360	3
3	5	{4, 4, 3, 3, 1, 1}, {4, 4, 3, 3, 1, 1}	{16, 16, 16, 13.29, 2.71, 0, ..., 0}	207360	3
4	5	{4, 4, 3, 3, 1, 1}, {4, 4, 3, 3, 1, 1}	{16, 16, 16, 13.29, 2.71, 0, ..., 0}	207360	2
5	5	{4, 4, 3, 3, 1, 1}, {4, 4, 3, 3, 1, 1}	{16, 16, 16, 13.29, 2.71, 0, ..., 0}	207360	2
6	5	{4, 4, 3, 3, 1, 1}, {4, 4, 2, 2, 2, 2}	{16, 16, 16, 12, 4, 0, ..., 0}	165888	2
7	5	{4, 4, 3, 3, 1, 1}, {4, 4, 2, 2, 2, 2}	{16, 16, 16, 12, 4, 0, ..., 0}	165888	2
8	5	{4, 4, 2, 2, 2, 2}, {4, 4, 3, 3, 1, 1}	{16, 16, 16, 12, 4, 0, ..., 0}	165888	2
9	5	{4, 4, 2, 2, 2, 2}, {4, 4, 3, 3, 1, 1}	{16, 16, 16, 12, 4, 0, ..., 0}	165888	2
10	5	{4, 4, 2, 2, 2, 2}, {4, 4, 2, 2, 2, 2}	{16, 16, 16, 8, 8, 0, ..., 0}	165888	2
11	6	{4, 3, 3, 3, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 14.08, 14.08, 1.92, 1.92, 0, ..., 0}	145152	2
12	6	{4, 3, 3, 3, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 14.08, 14.08, 1.92, 1.92, 0, ..., 0}	145152	2
13	6	{4, 3, 3, 3, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 14.08, 14.08, 1.92, 1.92, 0, ..., 0}	145152	1
14	6	{4, 3, 3, 2, 1, 1, 1, 1}, {4, 3, 3, 2, 1, 1, 1, 1}	{16,16, 14.66, 11.90, 3.34, 2.10, 0, ..., 0}	131328	3
15	6	{4, 3, 3, 2, 1, 1, 1, 1}, {4, 3, 3, 2, 1, 1, 1, 1}	{16,16, 14.66, 11.90, 3.34, 2.10, 0, ..., 0}	131328	3
16	6	{4, 3, 3, 3, 1, 1, 1}, {4, 3, 3, 2, 1, 1, 1, 1}	{16, 16, 14.56, 12.36, 3.64, 1.44, 0, ..., 0}	124416	3
17	6	{4, 3, 3, 2, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 14.56, 12.36, 3.64, 1.44, 0, ..., 0}	124416	3
18	6	{4, 3, 3, 3, 1, 1, 1}, {4, 3, 3, 2, 1, 1, 1, 1}	{16, 16, 14.56, 12.36, 3.64, 1.44, 0, ..., 0}	124416	2
19	6	{4, 3, 3, 3, 1, 1, 1}, {4, 3, 3, 2, 1, 1, 1, 1}	{16, 16, 14.56, 12.36, 3.64, 1.44, 0, ..., 0}	124416	2
20	6	{4, 3, 3, 2, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 14.56, 12.36, 3.64, 1.44, 0, ..., 0}	124416	2
21	6	{4, 3, 3, 2, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 14.56, 12.36, 3.64, 1.44, 0, ..., 0}	124416	2
22	6	{4, 3, 3, 3, 1, 1, 1}, {4, 3, 3, 2, 1, 1, 1, 1}	{16, 16, 14.56, 12.36, 3.64, 1.44, 0, ..., 0}	124416	1
23	6	{4, 3, 3, 3, 1, 1, 1}, {4, 3, 3, 2, 1, 1, 1, 1}	{16, 16, 14.56, 12.36, 3.64, 1.44, 0, ..., 0}	124416	1
24	6	{4, 3, 3, 3, 1, 1, 1}, {4, 3, 3, 2, 1, 1, 1, 1}	{16, 16, 14.56, 12.36, 3.64, 1.44, 0, ..., 0}	124416	1
25	6	{4, 3, 3, 3, 1, 1, 1}, {4, 3, 3, 2, 1, 1, 1, 1}	{16, 16, 14.56, 12.36, 3.64, 1.44, 0, ..., 0}	124416	1
26	6	{4, 3, 3, 2, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 14.56, 12.36, 3.64, 1.44, 0, ..., 0}	124416	1
27	6	{4, 3, 3, 2, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 14.56, 12.36, 3.64, 1.44, 0, ..., 0}	124416	1
28	6	{4, 3, 3, 2, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 14.56, 12.36, 3.64, 1.44, 0, ..., 0}	124416	1
29	6	{4, 3, 3, 2, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 14.56, 12.36, 3.64, 1.44, 0, ..., 0}	124416	1
30	6	{4, 3, 3, 3, 1, 1, 1}, {4, 3, 2, 2, 2, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	2
31	6	{4, 3, 3, 3, 1, 1, 1}, {4, 3, 2, 2, 2, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	2
32	6	{4, 3, 3, 3, 1, 1, 1}, {4, 3, 2, 2, 2, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	2
33	6	{4, 3, 2, 2, 2, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	2
34	6	{4, 3, 2, 2, 2, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	2
35	6	{4, 3, 2, 2, 2, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	2
36	6	{4, 3, 3, 3, 1, 1, 1}, {4, 3, 2, 2, 2, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	1
37	6	{4, 3, 3, 3, 1, 1, 1}, {4, 3, 2, 2, 2, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	1
38	6	{4, 3, 3, 3, 1, 1, 1}, {4, 3, 2, 2, 2, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	1
39	6	{4, 3, 3, 3, 1, 1, 1}, {4, 3, 2, 2, 2, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	1
40	6	{4, 3, 3, 3, 1, 1, 1}, {4, 3, 2, 2, 2, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	1
41	6	{4, 3, 3, 3, 1, 1, 1}, {4, 3, 2, 2, 2, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	1
42	6	{4, 3, 3, 3, 1, 1, 1}, {4, 3, 2, 2, 2, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	1
43	6	{4, 3, 3, 3, 1, 1, 1}, {4, 3, 2, 2, 2, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	1
44	6	{4, 3, 3, 3, 1, 1, 1}, {4, 3, 2, 2, 2, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	1
45	6	{4, 3, 2, 2, 2, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	1
46	6	{4, 3, 2, 2, 2, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	1
47	6	{4, 3, 2, 2, 2, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	1
48	6	{4, 3, 2, 2, 2, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	1
49	6	{4, 3, 2, 2, 2, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	1
50	6	{4, 3, 2, 2, 2, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	1
51	6	{4, 3, 2, 2, 2, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	1
52	6	{4, 3, 2, 2, 2, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	1
53	6	{4, 3, 2, 2, 2, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 14.20, 12.20, 3.80, 1.80, 0, ..., 0}	103680	1
54	6	{4, 3, 3, 3, 1, 1, 1}, {4, 2, 2, 2, 2, 1, 1, 1, 1}	{16, 16, 13.29, 12, 4, 2.71, 0, ..., 0}	82944	1
55	6	{4, 3, 3, 3, 1, 1, 1}, {4, 2, 2, 2, 2, 1, 1, 1, 1}	{16, 16, 13.29, 12, 4, 2.71, 0, ..., 0}	82944	1
56	6	{4, 3, 3, 3, 1, 1, 1}, {4, 2, 2, 2, 2, 1, 1, 1, 1}	{16, 16, 13.29, 12, 4, 2.71, 0, ..., 0}	82944	1
57	6	{4, 3, 3, 3, 1, 1, 1}, {4, 2, 2, 2, 2, 1, 1, 1, 1}	{16, 16, 13.29, 12, 4, 2.71, 0, ..., 0}	82944	1
58	6	{4, 3, 3, 3, 1, 1, 1}, {4, 2, 2, 2, 2, 1, 1, 1, 1}	{16, 16, 13.29, 12, 4, 2.71, 0, ..., 0}	82944	1
59	6	{4, 3, 3, 3, 1, 1, 1}, {4, 2, 2, 2, 2, 1, 1, 1, 1}	{16, 16, 13.29, 12, 4, 2.71, 0, ..., 0}	82944	1
60	6	{4, 2, 2, 2, 2, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 13.29, 12, 4, 2.71, 0, ..., 0}	82944	1
61	6	{4, 2, 2, 2, 2, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 13.29, 12, 4, 2.71, 0, ..., 0}	82944	1
62	6	{4, 2, 2, 2, 2, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 13.29, 12, 4, 2.71, 0, ..., 0}	82944	1
63	6	{4, 2, 2, 2, 2, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 13.29, 12, 4, 2.71, 0, ..., 0}	82944	1
64	6	{4, 2, 2, 2, 2, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 13.29, 12, 4, 2.71, 0, ..., 0}	82944	1
65	6	{4, 2, 2, 2, 2, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 13.29, 12, 4, 2.71, 0, ..., 0}	82944	1
66	6	{4, 3, 3, 3, 1, 1, 1}, {4, 2, 2, 2, 1, 1, 1, 1, 1, 1}	{16, 16, 12.36, 12.36, 3.64, 3.64, 0, ..., 0}	82944	1
67	6	{4, 3, 3, 3, 1, 1, 1}, {4, 2, 2, 2, 1, 1, 1, 1, 1, 1}	{16, 16, 12.36, 12.36, 3.64, 3.64, 0, ..., 0}	82944	1
68	6	{4, 3, 3, 3, 1, 1, 1}, {4, 2, 2, 2, 1, 1, 1, 1, 1, 1}	{16, 16, 12.36, 12.36, 3.64, 3.64, 0, ..., 0}	82944	1
69	6	{4, 3, 3, 3, 1, 1, 1}, {4, 2, 2, 2, 1, 1, 1, 1, 1, 1}	{16, 16, 12.36, 12.36, 3.64, 3.64, 0, ..., 0}	82944	1
70	6	{4, 3, 3, 3, 1, 1, 1}, {4, 2, 2, 2, 1, 1, 1, 1, 1, 1}	{16, 16, 12.36, 12.36, 3.64, 3.64, 0, ..., 0}	82944	1
71	6	{4, 3, 3, 3, 1, 1, 1}, {4, 2, 2, 2, 1, 1, 1, 1, 1, 1}	{16, 16, 12.36, 12.36, 3.64, 3.64, 0, ..., 0}	82944	1
72	6	{4, 3, 3, 3, 1, 1, 1}, {4, 2, 2, 2, 1, 1, 1, 1, 1, 1}	{16, 16, 12.36, 12.36, 3.64, 3.64, 0, ..., 0}	82944	1
73	6	{4, 3, 3, 3, 1, 1, 1}, {4, 2, 2, 2, 1, 1, 1, 1, 1, 1}	{16, 16, 12.36, 12.36, 3.64, 3.64, 0, ..., 0}	82944	1
74	6	{4, 3, 3, 3, 1, 1, 1}, {4, 2, 2, 2, 1, 1, 1, 1, 1, 1}	{16, 16, 12.36, 12.36, 3.64, 3.64, 0, ..., 0}	82944	1
75	6	{4, 3, 3, 3, 1, 1, 1}, {4, 2, 2, 2, 1, 1, 1, 1, 1, 1}	{16, 16, 12.36, 12.36, 3.64, 3.64, 0, ..., 0}	82944	1
76	6	{4, 2, 2, 2, 1, 1, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 12.36, 12.36, 3.64, 3.64, 0, ..., 0}	82944	1
77	6	{4, 2, 2, 2, 1, 1, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 12.36, 12.36, 3.64, 3.64, 0, ..., 0}	82944	1
78	6	{4, 2, 2, 2, 1, 1, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 12.36, 12.36, 3.64, 3.64, 0, ..., 0}	82944	1
79	6	{4, 2, 2, 2, 1, 1, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 12.36, 12.36, 3.64, 3.64, 0, ..., 0}	82944	1
80	6	{4, 2, 2, 2, 1, 1, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 12.36, 12.36, 3.64, 3.64, 0, ..., 0}	82944	1
81	6	{4, 2, 2, 2, 1, 1, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 12.36, 12.36, 3.64, 3.64, 0, ..., 0}	82944	1
82	6	{4, 2, 2, 2, 1, 1, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 12.36, 12.36, 3.64, 3.64, 0, ..., 0}	82944	1
83	6	{4, 2, 2, 2, 1, 1, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 12.36, 12.36, 3.64, 3.64, 0, ..., 0}	82944	1
84	6	{4, 2, 2, 2, 1, 1, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 12.36, 12.36, 3.64, 3.64, 0, ..., 0}	82944	1
85	6	{4, 2, 2, 2, 1, 1, 1, 1, 1, 1}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 12.36, 12.36, 3.64, 3.64, 0, ..., 0}	82944	1
86	6	{4, 3, 3, 3, 1, 1, 1}, {4, 2, 2, 2, 2, 2, 2}	{16, 16, 13.29, 13.29, 2.71, 2.71, 0, ..., 0}	82944	1
87	6	{4, 2, 2, 2, 2, 2, 2}, {4, 3, 3, 3, 1, 1, 1}	{16, 16, 13.29, 13.29, 2.71, 2.71, 0, ..., 0}	82944	1
88	6	{4, 2, 2, 2, 2, 2, 2}, {4, 3, 3, 2, 1, 1, 1, 1}	{16, 16, 14, 10, 6, 2, 0, ..., 0}	82944	1
89	6	{4, 3, 3, 2, 1, 1, 1, 1}, {4, 2, 2, 2, 2, 2, 2}	{16, 16, 14, 10, 6, 2, 0, ..., 0}	82944	1
90	6	{4, 3, 3, 2, 1, 1, 1, 1}, {4, 2, 2, 2, 2, 1, 1, 1, 1}	{16, 16, 14, 10, 4, 4, 0, ..., 0}	82944	2
91	6	{4, 2, 2, 2, 2, 1, 1, 1, 1}, {4, 3, 3, 2, 1, 1, 1, 1}	{16, 16, 14, 10, 4, 4, 0, ..., 0}	82944	2
92	6	{3, 3, 3, 3, 1, 1, 1, 1}, {2, 2, 2, 2, 2, 2, 2, 2}	{16, 16, 12, 12, 4, 4, 0, ..., 0}	82944	1
93	6	{2, 2, 2, 2, 2, 2, 2, 2}, {3, 3, 3, 3, 1, 1, 1, 1}	{16, 16, 12, 12, 4, 4, 0, ..., 0}	82944	1

94	7	{3, 3, 3, 1, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 1, 1, 1, 1, 1, 1, 1}	{16, 14.66, 14.66, 10.47, 3.34, 3.34, 1.53, 0, . . ., 0}	83808	3
95	7	{3, 3, 3, 1, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 1, 1, 1, 1, 1, 1, 1}	{16, 14.66, 14.66, 10.47, 3.34, 3.34, 1.53, 0, . . ., 0}	83808	3
96	7	{3, 3, 3, 1, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 2, 1, 1, 1, 1, 1}	{16, 14.71, 14.56, 10.92, 4.23, 2.14, 1.44, 0, . . ., 0}	80352	3
97	7	{3, 3, 3, 2, 1, 1, 1, 1, 1}, {3, 3, 3, 1, 1, 1, 1, 1, 1, 1}	{16, 14.71, 14.56, 10.92, 4.23, 2.14, 1.44, 0, . . ., 0}	80352	3
98	7	{3, 3, 2, 2, 2, 1, 1, 1, 1}, {3, 2, 2, 2, 2, 2, 1, 1, 1}	{16, 14.59, 13.35, 11.08, 4.38, 2.55, 2.04, 0, . . ., 0}	41472	1
99	7	{3, 2, 2, 2, 2, 2, 1, 1, 1}, {3, 3, 2, 2, 2, 1, 1, 1, 1}	{16, 14.59, 13.35, 11.08, 4.38, 2.55, 2.04, 0, . . ., 0}	41472	1
100	8	{3, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1}	{16, 14.63, 11.79, 9.98, 5.31, 3.19, 1.96, 1.15, 0, . . ., 0}	43776	1
101	8	{2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1}, {3, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1}	{16, 14.63, 11.79, 9.98, 5.31, 3.19, 1.96, 1.15, 0, . . ., 0}	43776	1

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