Schur Multiplicative Maps on Matrices *

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Abstract

The structure of Schur multiplicative maps on matrices over a field is studied. The result is then used to characterize Schur multiplicative maps f satisfying $f(S) \subseteq S$ for different subsets S of matrices including the set of rank k matrices, the set of singular matrices, and the set of invertible matrices. Characterizations are also obtained for maps on matrices such that $\Gamma(f(A)) = \Gamma(A)$ for various functions Γ including the rank function, the determinant function, and the elementary symmetric functions of the eigenvalues. These results include analogs of the theorems of Frobenius and Dieudonné on linear maps preserving the determinant functions and linear maps preserving the set of singular matrices, respectively.

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1 Introduction

Let $M_{m,n}$ be the set of $m \times n$ matrices over a field $\mathbf{F}$ with at least three elements. Define the *Schur product* (also known as *Hadamard product* or *entrywise product*) of $A = [a_{ij}], B = [b_{ij}] \in M_{m,n}$ by $A \circ B = [a_{ij}b_{ij}]$. A map $f : M_{m,n} \to M_{m,n}$ is *Schur multiplicative* if

$$f(A \circ B) = f(A) \circ f(B) \qquad \text{for all } A, B \in M_{m,n}.$$

The study of Schur product is related to many pure and applied areas; see [8]. There has been considerable interest in studying linear maps, additive maps, and multiplicative maps f on matrices with some special properties such as $f(S) \subseteq S$ for a certain subset of matrices, or $\Gamma(f(A)) = \Gamma(A)$ for a given function Γ on matrices; for example see [7, 10, 11, 13, 15] and their references. In this paper, we study Schur multiplicative maps on matrices with some of these special properties.

In Section 2, we consider general Schur multiplicative maps $f : M_{m,n} \to M_{m,n}$. In particular, it is shown that under some mild assumptions on the Schur multiplicative map f has the form

(†) $[a_{ij}] \mapsto \mathcal{P}[f_{ij}(a_{ij})]$, *where* $f_{ij} : \mathbf{F} \to \mathbf{F}$ *satisfies* $f_{ij}(0) = 0$ *for each* (i, j) *pair, and* $\mathcal{P}(X) \in M_{m,n}$ *is obtained from* X *by permuting its entries in a fixed pattern.*

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The result is then used to study Schur multiplicative maps which map rank k matrices to rank k matrices for a given value k. In particular, our results include the characterization of those Schur multiplicative maps that preserve the rank function, and those Schur multiplicative maps that map the set of singular (respectively, invertible) square matrices to itself. In Section 4, we study Schur multiplicative maps on square matrices which preserve functions related to eigenvalues including the determinant function and the spectrum. We also characterize maps on matrices of the form (†) that preserve some special sets and functions. These results include analogs of the theorems of Frobenius and Dieudonné on linear maps preserving the determinant functions and linear maps preserving the set of singular matrices, respectively.

In our discussion, let $J_{m,n}$ denote the $m \times n$ matrix with all entries equal to 1, and let $0_{m,n}$ be the $m \times n$ matrix with all entries equal to 0. Denote by $\mathcal{B} = \{E_{11}, E_{12}, \dots, E_{m,n}\}$ the standard basis for $M_{m,n}$. When $m = n$, we use the notation $M_n, J_n, 0_n$, etc. The set of nonzero elements in $\mathbf{F}$ is denoted by $\mathbf{F}^*$.

A square matrix is a *monomial matrix* if each row and each column has exactly one nonzero entry. A monomial matrix is a *permutation matrix* if all the nonzero entries equal to the unity in $\mathbf{F}$.

2 Schur Multiplicative Maps

The structure of a Schur multiplicative map $f : M_{m,n} \to M_{m,n}$ can be quite arbitrary if one does not impose any additional assumptions on f. In general, one can define $f(A) = [f_{ij}(A)]$, where $f_{ij} : M_{m,n} \to \mathbf{F}$ is any Schur multiplicative map. For example, one can define $f(A) = B$ for a fixed matrix B satisfying $B \circ B = B$; another example is to define $f(A) = J_{m,n}$ if $a_{11} \neq 0$ and $f(A) = E_{11}$ otherwise. On the other hand, if one imposes some mild conditions on a Schur multiplicative map, then its structure will be more tractable as shown in the following.

Theorem 2.1 *Let $f : M_{m,n} \to M_{m,n}$. The following conditions are equivalent.*

(A1) *f is Schur multiplicative, $f(0_{m,n}) = 0_{m,n}$, and $f(E_{ij}) \neq 0_{m,n}$ for each (i, j) pair.*

(A2) *f is Schur multiplicative and $f^{-1}[\{0_{m,n}\}] = \{0_{m,n}\}$.*

(A3) *There is a mapping $\mathcal{P} : M_{m,n} \to M_{m,n}$ such that $\mathcal{P}(A)$ is obtained from A by permuting its entries in a fixed pattern, and a family of multiplicative maps $f_{ij} : \mathbf{F} \to \mathbf{F}$ satisfying $f_{ij}^{-1}[\{0\}] = \{0\}$ such that*

$$f([a_{ij}]) = \mathcal{P}\left([f_{ij}(a_{ij})]\right).$$

Proof. Note that a matrix $X \in M_{m,n}$ satisfies $X \circ X = X$ if and only if all the entries of X belong to $\{0, 1\}$.

Assume that (A1) holds. Suppose there is X with nonzero (i, j) entry such that $f(X) = 0_{m,n}$. Then $f(E_{ij}) = f(E_{ij} \circ X/x_{ij}) = f(E_{ij}/x_{ij}) \circ f(X) = 0_{m,n}$, which is a contradiction. Thus, $f^{-1}[\{0_{m,n}\}] = \{0_{m,n}\}$. We see that (A2) holds.

Suppose (A2) holds. Consider $X \in \mathcal{B} = \{E_{ij} : 1 \leq i \leq m,\ 1 \leq j \leq n\}$. Then $f(X) = f(X) \circ f(X)$. So, all entries of $f(X)$ lie in $\{0, 1\}$, and $f(X) \neq 0$ by assumption (A2). For any $X, Y \in \mathcal{B}$ with $X \neq Y$, we have $f(X) \circ f(Y) = f(0_{m,n}) = 0_{m,n}$. Thus, $f(X)$ and $f(Y)$ have nonzero entries in different positions. As a result, for each $X \in \mathcal{B}$, $f(X)$ has exactly one non-zero entry. Thus, $f(\mathcal{B}) = \mathcal{B}$.

We can apply a map $\mathcal{P} : M_{m,n} \to M_{m,n}$ such that $\mathcal{P}(A)$ is obtained from A by a fixed permutation of the entries of A so that $\mathcal{P}(f(E_{ij})) = E_{ij}$ for all (i, j). It remains to show that there are $f_{ij} : \mathbf{F} \to \mathbf{F}$ such that $\mathcal{P}(f(A)) = [f_{ij}(a_{ij})]$ for any $A = [a_{ij}]$.

Replace f by the map $A \mapsto \mathcal{P}^{-1}(f(A))$, where $\mathcal{P}^{-1}(\mathcal{P}(X)) = X$ for all matrices X. If we can prove the conclusion for the modified map, then the same conclusion will be valid for the original map. So, we assume that $\mathcal{P}$ is the identity map, i.e., $f(E_{ij}) = E_{ij}$ for all (i, j) pairs. Now, fix an (i, j) pair. For any $a \in \mathbf{F}$, $f(aE_{ij}) = f(aE_{ij}) \circ f(E_{ij}) = bE_{ij}$ for some $b \in \mathbf{F}$. Define $f_{ij} : \mathbf{F} \to \mathbf{F}$ such that $f(aE_{ij}) = f_{ij}(a)E_{ij}$. Since $f^{-1}[\{0_{m,n}\}] = \{0_{m,n}\}$, $f_{ij}(x) = 0$ if and only if $x = 0$. Also, for any $a, b \in \mathbf{F}$,

$$f_{ij}(ab)E_{ij} = f(abE_{ij}) = f(aE_{ij}) \circ f(bE_{ij}) = f_{ij}(a)f_{ij}(b)E_{ij}.$$

Suppose $A = [a_{ij}]$ and $f(A) = [b_{ij}]$. Then

$$b_{ij}E_{ij} = E_{ij} \circ f(A) = f(E_{ij} \circ A) = f_{ij}(a_{ij})E_{ij}.$$

Thus, we see that $f(A) = [f_{ij}(a_{ij})]$, and the conclusion holds.

The implication (A3) $\Rightarrow$ (A1) is clear. □

Corollary 2.2 *Let $f : M_{m,n} \to M_{m,n}$. The following are equivalent.*

(A4) *f is Schur multiplicative and injective.*

(A5) *Condition* (A3) *in Theorem 2.1 holds with the additional assumption that f_{ij} is injective for each (i, j) pair.*

Proof. Suppose f is Schur multiplicative and injective. Since $f(0_{m,n}) = f(0_{m,n}) \circ f(0_{m,n})$, all entries of $f(0_{m,n})$ lie in $\{0, 1\}$. Let S be the set of (i, j) pairs such that the (i, j) entry of $f(0_{m,n})$ equals 1. Then for any $X \in M_{m,n}$, we have

$$f(0_{m,n}) = f(X \circ 0_{m,n}) = f(X) \circ f(0_{m,n}).$$

Hence the (i, j) entry of $f(X)$ equals 1 for each $(i, j) \in S$.

For $(i, j) \neq (p, q)$, we have $f(E_{ij}) \neq f(E_{pq})$ and $f(E_{ij} \circ E_{pq}) = f(0_{m,n})$. Thus, $f(E_{ij})$ and $f(E_{pq})$ cannot have a common nonzero entry at the (r, s) position if $(r, s) \notin S$. Because f is injective, $f(E_{ij}) \neq f(0_{m,n})$. Thus, every $f(E_{ij})$ has at least one nonzero entry at a position $(r, s) \notin S$. Since E_{ij} and $E_{p,q}$ cannot have nonzero entry at any (r, s) position with $(r, s) \notin S$, we need at least mn pairs of $(r, s) \notin S$ to accommodate the nonzero entries of $f(E_{ij})$. Hence, we conclude that $S = \emptyset$, i.e., $f(0_{m,n}) = 0_{m,n}$, and each $f(E_{ij})$ has exactly one nonzero entry equal to 1. So, condition (A1) of Theorem 2.1 holds and f has the form described in (A3). Since f is injective, for $x \neq y$ in $\mathbf{F}$ we have seen that $f(xE_{ij}) \neq f(yE_{ij})$ and hence $f_{ij}(x) \neq f_{ij}(y)$. So, f_{ij} is injective for each (i, j) pair.

The implication (A5) $\Rightarrow$ (A4) is clear. □

Remark 2.3 As we will see in the subsequent discussion, in the study of preserver problems we can sometimes assume only

(A0) $f : M_{m,n} \to M_{m,n}$ is Schur multiplicative and $f(0_{m,n}) = 0_{m,n}$,

together with some preserving property to conclude that f has the form (†) with some additional nice structure. In some problems, we believe that one can even remove the assumption that $f(0_{m,n}) = 0_{m,n}$ in (A0). On the other hand, we will see that the assumption (A1) or (A0) are indispensable in certain problems.

Note also that our result and proof are valid if $\mathbf{F}$ is replaced by an integral domain $\mathbf{D}$.

3 Rank preservers

Linear maps, additive maps, and multiplicative maps on matrices mapping the set of rank-k matrices to itself have been studied by many researchers; e.g., see [2, 3, 13, 15] and their references. In this section, we characterize Schur multiplicative maps that map the set of rank-k matrices to itself. We begin with rank one preservers.

Theorem 3.1 *Suppose $f : M_{m,n} \to M_{m,n}$ is a Schur multiplicative map. Then $f(0_{m,n}) = 0_{m,n}$ and f maps rank one matrices to rank one matrices if and only if there exist permutation matrices $P \in M_m$ and $Q \in M_n$, and a multiplicative map $\tau : \mathbf{F} \to \mathbf{F}$ satisfying $\tau(\mathbf{F}^*) \subseteq \mathbf{F}^*$ such that*

(a) *f has the form $[a_{ij}] \mapsto P[\tau(a_{ij})]Q$, or*

(b) *$m = n$ and f has the form $[a_{ij}] \mapsto P[\tau(a_{ij})]^t Q$.*

Proof. First we consider the implication ($\Leftarrow$). Note that $A \in M_{m,n}$ has rank one if and only if there are $x_1, \dots, x_m, y_1, \dots, y_n \in \mathbf{F}$ such that $A = [a_{ij}] = [x_1, \dots, x_m]^t[y_1, \dots, y_n]$. Thus, for any injective multiplicative map $\tau : \mathbf{F} \to \mathbf{F}$, we have

$$[\tau(a_{ij})] = [\tau(x_i)\tau(y_j)] = [\tau(x_1), \dots, \tau(x_m)]^t[\tau(y_1), \dots, \tau(y_n)]$$

with rank one. By this observation, the implication ($\Leftarrow$) is clear.

Next, we consider the converse. By the given assumption, f satisfies condition (A1) in Theorem 2.1 and hence its conclusion. Thus, f has the form (†). Without loss of generality, we may assume that $m \le n$. The case $n < m$ can be proved by similar arguments. Since $f(X)$ has rank one for $X = \sum_{j=1}^n E_{1j}$, we see that the nonzero entries of $f(X)$ lie in the same row, or in the same column if $m = n$. We may assume that the former case holds. Otherwise, replace f by a map of the form $A \mapsto f(A)^t$. Note that if we can prove the result for the modified map, the conclusion will be valid for the original map. Then there exist permutation matrices $P \in M_m$ and $Q \in M_n$ so that $f(E_{1j}) = PE_{1j}Q$ for $j = 1, \dots, n$. Replace f by the map $A \mapsto P^t f(A) Q^t$ so that we have $f(E_{1j}) = E_{1j}$ for $j = 1, \dots, n$. Now, consider $f(X)$ for $X = \sum_{i=1}^m E_{i1}$. Since $f(E_{11}) = E_{11}$ and f maps rank one matrices to rank one matrices, we see that $f(X) = X$. There exists a permutation matrix $R \in M_m$ such that $f(E_{i1}) = RE_{i1}$ for $i = 1, \dots, m$. We may replace f by the map $A \mapsto R^t f(A)$, and assume that $f(E_{i1}) = E_{i1}$ for $i = 1, \dots, m$. For any (i, j) with $i \ne 1$ and $j \ne 1$, since $f(X)$ has rank one for $X = E_{11} + E_{1j} + E_{i1} + E_{ij}$, we see that $f(E_{ij}) = E_{ij}$. Furthermore, for any $1 \le j \le n$ and $a \in \mathbf{F}$ the matrix $f(X)$ has rank one for $X = aE_{11} + aE_{1j} + E_{21} + E_{2j}$, we see that $f_{11}(a) = f_{1j}(a)$. Similarly, we can show that $f_{i1}(a) = f_{11}(a)$ for all $a \in \mathbf{F}$. Finally, for any (i, j) with $i \neq 1$ and $j \neq 1$, since $f(X)$ has rank one for $X = E_{11} + E_{1j} + aE_{i1} + aE_{ij}$ with $a \in \mathbf{F}$, $f_{ij}(a) = f_{i1}(a) = f_{11}(a)$ for all $a \in \mathbf{F}$. Our conclusion follows. $\square$

The conclusion of Theorem 3.1 may fail if the Schur multiplicative map does not maps $0_{m,n}$ to itself. For example, we can choose a fixed rank one matrix B satisfying $B \circ B = B$ and define $f(A) = B$ for all $A \in M_{m,n}$. Then f is Schur multiplicative and maps rank one matrices to rank one matrices.

Next, we show that one can get a similar conclusion for maps on matrices of the form (†) even though f_{ij} is not assumed to be multiplicative *a priori.*

Theorem 3.2 *Suppose $f : M_{m,n} \to M_{m,n}$ has the form (†). Then f maps rank one matrices to rank one matrices if and only if there exist invertible monomial matrices $P \in M_m$ and $Q \in M_n$ and a multiplicative map $\tau : \mathbf{F} \to \mathbf{F}$ satisfying $\tau(\mathbf{F}^*) \subseteq \mathbf{F}^*$ such that*

(a) *f has the form $[a_{ij}] \mapsto P[\tau(a_{ij})]Q$, or*

(b) *$m = n$ and f has the form $[a_{ij}] \mapsto P[\tau(a_{ij})]^t Q$.*

Proof. The implication ($\Leftarrow$) can be verified as in the proof of Theorem 3.1.

We consider the converse. Assume that f has the form (†) and maps rank one matrices to rank one matrices. Without loss of generality, we may assume that $m \leq n$. Since $f(X)$ has rank one for $X = \sum_{j=1}^{n} E_{1j}$, we see that the nonzero entries of $f(X)$ lie in the same row, or in the same column if $m = n$. We may assume that the former case holds. Otherwise, replace f by a map of the form $A \mapsto f(A)^t$. Then there exist permutation matrices $P \in M_m$ and $Q \in M_n$ so that $f(E_{1j}) = P f_{1j}(1) E_{1j} Q$ for $j = 1, \ldots, n$. Let $D = \operatorname{diag}(f_{11}(1), f_{12}(1), \ldots, f_{1n}(1))$. Since $f(E_{1j})$ has rank 1, we see that $f_{1j}(1) \neq 0$ for $j = 1, \ldots, n$. Replace f by the map $A \mapsto P^{-1} f(A) Q^{-1} D^{-1}$ so that we have $f(E_{1j}) = E_{1j}$ for $j = 1, \ldots, n$. Now, consider $f(X)$ for $X = \sum_{i=1}^{m} E_{i1}$. Since $f(E_{11}) = E_{11}$ and f maps rank one matrices to rank one matrices, there exists an invertible monomial matrix $R \in M_m$ such that $f(E_{i1}) = RE_{i1}$ for $i = 1, \ldots, m$. We may replace f by the map $A \mapsto R^{-1} f(A)$, and assume that $f(E_{i1}) = E_{i1}$ for $i = 1, \ldots, m$. For any (i, j) with $i \neq 1$ and $j \neq 1$, since $f(X)$ has rank one for $X = E_{11} + E_{1j} + E_{i1} + E_{ij}$, we see that $f(E_{ij}) = E_{ij}$.

Note that for any (i, j) pair and any nonzero $a \in \mathbf{F}$, $f(aE_{ij}) = f_{ij}(a)E_{ij}$ has rank one, and thus $f_{ij}(a) \neq 0$. Furthermore, for any $1 \leq j \leq n$ and any $a \in \mathbf{F}$ the matrix $f(X)$ has rank one for $X = aE_{11} + aE_{1j} + E_{21} + E_{2j}$, we see that $f_{11}(a) = f_{1j}(a)$. Similarly, we can show that $f_{i1}(a) = f_{11}(a)$ for all $a \in \mathbf{F}$. Finally, for any (i, j) with $i \neq 1$ and $j \neq 1$, since $f(X)$ has rank one for $X = E_{11} + E_{1j} + aE_{i1} + aE_{ij}$ with $a \in \mathbf{F}$, $f_{ij}(a) = f_{i1}(a) = f_{11}(a)$ for all $a \in \mathbf{F}$.

Let $f_{11} = \tau$. For any $a, b \in \mathbf{F}$, let $X = E_{11} + aE_{12} + bE_{21} + abE_{22}$. Since $f(X)$ has rank one, we see that $\tau(ab) = \tau(a)\tau(b)$. So, τ is multiplicative. $\square$

Next, we characterize maps $f : M_{m,n} \to M_{m,n}$ of the form (†) which map the set of rank k matrices to itself for $1 < k < \min\{m, n\}$. It turns out that such maps will preserve the ranks of all matrices, and have very nice structure. The result will be used to characterize Schur multiplicative maps which preserve rank k matrices in Corollary 3.4

Theorem 3.3 *Let $1 < k < \min\{m, n\}$. Suppose $f : M_{m,n} \to M_{m,n}$ has the form (†). The following are equivalent.*

(a) $\operatorname{rank}(f(A)) = \operatorname{rank}(A)$ *for all* $A \in M_{m,n}$.

(b) f *maps rank* k *matrices to rank* k *matrices.*

(c) *There are invertible monomial matrices* $P \in M_m$ *and* $Q \in M_n$, *and a field monomorphism* $\tau : \mathbf{F} \to \mathbf{F}$ *such that one of the following holds.*

(c.i) f *has the form* $A \mapsto P[\tau(a_{i,j})]Q$.

(c.ii) $m = n$ *and* f *has the form* $A \mapsto P[\tau(a_{i,j})]^t Q$.

Note that for rank preservers f, we have $f(0_{m,n}) = 0_{m,n}$. Thus, one may further relax the assumption that $f_{ij}(0) = 0$ for all (i,j) pairs in (†), and conclude that conditions (b) and (c) are equivalent.

Proof. The implications (c) ⇒ (a) ⇒ (b) are clear. We focus on the proof of (b) ⇒ (c). Without loss of generality, we may assume that $m \le n$. The proof for the case $n < m$ is similar. We divide the proof into several assertions.

Assertion 1 There is a diagonal matrix $D \in M_m$ and permutation matrices $P \in M_m$ and $Q \in M_n$ such that $f(E_{jj}) = PDE_{jj}Q$ for $j = 1, \ldots, m$.

Consider $\mathcal{D} = \{E_{jj} : 1 \le j \le m\}$. If X is a sum of k matrices in $\mathcal{D}$, then $k = \operatorname{rank}(X) = \operatorname{rank}(f(X))$. So, $f(X)$ must have k nonzero entries lying on k distinct rows and k distinct columns. Thus, the m non-zero entries of $f(\sum_{j=1}^m E_{jj})$ lie on m different rows and m different columns. Hence, there are permutation matrices $P \in M_m$ and $Q \in M_n$ such that $f(E_{jj}) = Pf_{jj}(1)E_{jj}Q$ for $j = 1, \ldots, m$. Let $D = \operatorname{diag}(f_{11}(1), \ldots, f_{mm}(1))$. Then we get the desired conclusion.

By Assertion 1, we may replace f by the map $A \mapsto D^{-1}P^t f(A) Q^t$ and assume that $f(E_{jj}) = E_{jj}$ for $j = 1, \ldots, m$. *We will make this assumption in the rest of the proof.*

Assertion 2 For any (i,j) pair, $f_{ij}(\mathbf{F}^*) \subseteq \mathbf{F}^*$.

Let $a \in \mathbf{F}^*$, and let $X = aE_{ij} + \sum_{s \in S} E_{ss}$ for a subset S of $\{1, \ldots, m\} \setminus \{i,j\}$ with $k-1$ elements. Since $f(X)$ has rank k, we see that $f_{ij}(a) \ne 0$.

Assertion 3 For any $1 \le i < j \le m$, we have $f(E_{ij} + E_{ji}) = b_{ij}E_{ij} + b_{ij}^{-1}E_{ji}$ for some $b_{ij} \in \mathbf{F}^*$.

For simplicity, assume that $(i,j) = (1,2)$, and $X = E_{12} + E_{21}$. If $Y = X + \sum_{j=3}^k E_{jj}$, then $f(Y)$ has rank k. So, $f(X) = f_{12}(1)E_{pq} + f_{21}(1)E_{rs}$ for some $p \ne q$ and $r \ne s$. If $Y = \sum_{j=1}^{k+1} E_{jj} + X$, then $f(Y)$ has rank k. Thus, $p, q, r, s, \in \{1, \ldots, k+1\}$; otherwise, the leading $(k+1) \times (k+1)$ matrix of $f(Y)$ will be invertible so that $f(Y)$ has rank larger than k. Furthermore, we must have $(p,q) = (s,r)$ and $f(X) = bE_{pq} + b^{-1}E_{qp}$ for some $b \in \mathbf{F}$ with $1 \le p < q \le k+1$; otherwise, $f(Y)$ has rank larger than k. Now, for any $s \in \{3, \ldots, k+1\}$, we have $k = \operatorname{rank}(Z) = \operatorname{rank}(f(Z))$ for any $Z \in \{Y - E_{ss} - E_{11}, Y - E_{ss} - E_{22}\}$. It follows that $p, q \notin \{3, \ldots, k+1\}$, i.e., $\{p,q\} = \{1,2\}$. So, $f(X) = bE_{12} + b^{-1}E_{21}$ as asserted.

Assertion 4 There is an invertible diagonal matrix $D \in M_m$ such that one of the following holds.

(i) $f(E_{ij}) = D^{-1}E_{ij}(D \oplus I_{n-m})$ for all $1 \le i \le m$ and $1 \le j \le m$.

(ii) $f(E_{ij}) = D^{-1}E_{ji}(D \oplus I_{n-m})$ for all $1 \le i, j \le m$.

By Assertion 3, $f(E_{ij}+E_{ji}) = b_{ij}E_{ij} + b_{ij}^{-1}E_{ij}$ for all (i,j) pairs with $1 \le i,j \le m$. Let $D^{-1} = \operatorname{diag}(1, b_{21}, b_{31}, \ldots, b_{m1})$. Then $f(X) = D^{-1}X(D \oplus I_{n-m})$ for $X = E_{1j} + E_{j1}$ for $j = 2, \ldots, m$. Replace f by the map $A \mapsto Df(A)(D^{-1} \oplus I_{n-m})$. Then

$$\text{(i)' } f(E_{12}) = E_{12}, \qquad \text{or} \qquad \text{(ii)' } f(E_{12}) = E_{21}.$$

Assume (i)' holds. We prove that (i) holds accordingly as follows. First, consider $f(E_{1j})$ for $j = 3, \ldots, m$. Consider $A = E_{11} + E_{12} + E_{1j} + E_{22} + E_{2j} + E_{j2} + E_{jj} + \sum_{s\in S} E_{ss}$, where S is a subset of $\{3, \ldots, n\} \setminus \{j\}$ with $k-2$ elements. Then $k = \operatorname{rank}(A) = \operatorname{rank}(f(A))$. If $f(E_{1j}) = E_{j1}$, then $f(A)$ have rank $k+1$, which is a contradiction. Thus, $f(X) = X$ for $X \in \{E_{1j}, E_{j1}\}$.

Now, suppose $1 \notin \{i,j\}$. Let $A = E_{11} + E_{1i} + E_{1j} + E_{i1} + E_{ii} + E_{ij} + E_{jj} + \sum_{s\in S} E_{ss}$, where S is a subset of $\{2, \ldots, n\} \setminus \{i,j\}$ with $k-2$ elements. Then $k = \operatorname{rank}(A)$. If $f(E_{ij}) = b_{ij}^{-1}E_{ji}$, then $\operatorname{rank}(f(A)) = k+1$, which is a contradiction. So, $f(E_{ij}) = b_{ij}E_{ij}$. If $b_{ij} \ne 1$, consider $B = E_{11} + E_{1i} + E_{1j} + E_{i1} + E_{ii} + E_{ij} + E_{j1} + \sum_{s\in S} E_{ss}$, where S is a subset of $\{2, \ldots, n\} \setminus \{i,j\}$ with $k-2$ elements. Then $\operatorname{rank}(B) = k < k+1 = \operatorname{rank}(f(B))$, which is a contradiction. So, we conclude that $f(X) = X$ for $X \in \{E_{ij}, E_{ji}\}$. Our proof of (i) is complete.

If condition (ii)' holds, we can prove (ii) by a similar argument.

Assertion 5 Suppose $m < n$. Then condition (ii) in Assertion 4 cannot hold, and there is an invertible monomial matrix $Q \in M_n$ such that $f(E_{ij}) = E_{ij}Q$ for any $1 \le i \le m$ and $1 \le j \le n$.

To prove the above assertion, note that if $r > m$ then $f(E_{1r}) = E_{pq}$ for some $q > m$ because $f(E_{ij}) = E_{ij}$ for $1 \le i,j \le m$. If $p \ne 1$, then for $A = E_{1r} + E_{11} + \sum_{s\in S} E_{ss}$, where S is a subset of $\{2, \ldots, n\} \setminus \{p\}$ with $k-1$ elements, we see that $f(A)$ has $k+1$ linear independent rows and thus $\operatorname{rank}(f(A)) = k+1$ and $\operatorname{rank}(A) = k$, which is a contradiction. So, there are $b_{1j} \in \mathbf{F}^*$ for $j = m+1, \ldots, n$ such that $\{f(E_{1r}) : m < r \le n\} = \{b_{1r}E_{1r} : m < r \le n\}$. We may assume that $f(E_{1r}) = E_{1r}$ for all $m < r \le n$. Otherwise, replace f by a map of the form $A \mapsto f(A)Q$, where $Q \in M_n$ is a monomial matrix of the form $I_m \oplus \tilde{Q}$ with $\tilde{Q} \in M_{n-m}$ is an invertible monomial matrix.

To see that condition (ii) cannot hold, consider $A = E_{1,m+1} + E_{12} + E_{23} + \cdots + E_{k,k+1}$. Then there is $b \in \mathbf{F}^*$ such that $f(A) = bE_{1,m+1} + E_{21} + E_{32} + \cdots + E_{k+1,k}$ has rank $k+1$ while $\operatorname{rank}(A) = k$, which is a contradiction. So, at this point, we have $f(X) = X$ for $X = E_{ij}$ for $1 \le i,j \le m$ and $X \in \{E_{1r} : m < r \le n\}$.

Now, for E_{ij} with $i > 1$ and $j > m$, consider $A = E_{1i} + E_{1j} + E_{i1} + E_{ij} + \sum_{s\in S} E_{ss}$, where S is a subset of $\{2, \ldots, n\} \setminus \{i\}$ with $k-2$ elements. Since $k = \operatorname{rank}(A) = \operatorname{rank}(f(A))$, we conclude that $f(E_{ij}) = E_{ij}$.

By the above discussion, we may further replace f by a map of the form $A \mapsto Pf(A)Q$ for some suitable invertible monomial matrices $P \in M_m$ and $Q \in M_n$ so that the resulting map satisfies

(1) $f(E_{ij}) = E_{ij}$ for all (i,j) pairs, or (2) $m = n$ and $f(E_{ij}) = E_{ji}$ for all (i,j) pairs.

Assertion 6 There is a field monomorphism $\tau : \mathbf{F} \to \mathbf{F}$ such that $f_{ij} = \tau$ for every (i,j) pair.

First, for any $a \in \mathbf{F}$, consider $A = aE_{11} + aE_{1j} + E_{21} + E_{2j} + \sum_{s\in S} E_{ss}$, where S is a subset of $\{3, \ldots, m\}$ with $k-1$ elements. Since $k = \operatorname{rank}(A) = \operatorname{rank}(f(A))$, we have $f_{11}(a) = f_{1j}(a)$. Hence $f_{1j} = f_{11}$ for $j = 2, \ldots, n$. Similarly, we can show that $f_{ii} = f_{ij}$ for any $j \in \{1, \ldots, n\} \setminus \{i\}$.

Next, for any $a \in \mathbf{F}$ consider $A = aE_{11} + aE_{j1} + E_{21} + E_{j2} + \sum_{s\in S} E_{ss}$, where S is a subset of $\{3, \dots, m\}$ with $k-1$ elements. Since $k = \operatorname{rank}(A) = \operatorname{rank}(f(A))$, we have $f_{11}(a) = f_{j1}(a)$. Hence $f_{j1} = f_{11}$ for $j = 2, \dots, m$. Similarly, we can show that $f_{ii} = f_{ri}$ for any $r \in \{1, \dots, m\} \setminus \{i\}$.

By the arguments in the above two paragraphs, we conclude that there is $\tau : \mathbf{F} \to \mathbf{F}$ such that $f_{ij} = \tau$ for all (i, j) pairs.

Suppose $\tau(a) = \tau(b)$ for some $a \neq b$ in $\mathbf{F}$. Let $A = E_{11} + aE_{12} + E_{21} + bE_{22} + \sum_{j=3}^{k+1} E_{jj}$. Then $\operatorname{rank}(A) = k > k-1 = \operatorname{rank}(f(A))$, which is a contradiction. So, τ is injective. Now, let $A = E_{11} + aE_{12} + bE_{21} + abE_{22} + \sum_{j=3}^{k+1} E_{jj}$. Then $k = \operatorname{rank}(A) = \operatorname{rank}(f(A))$ implies that $\tau(ab) = \tau(a)\tau(b)$ for all $a, b \in \mathbf{F}$.

Finally, let

$$A = E_{11} + aE_{12} + (a+b)E_{13} + E_{21} + bE_{23} + E_{32} + E_{33} + \sum_{s\in S} E_{ss}$$

for some subset S of $\{4, \dots, n\}$ with $k-2$ elements. Then $k = \operatorname{rank}(A) = \operatorname{rank}(f(A))$. Since

$$f(A) = E_{11} + \tau(a)E_{12} + \tau(a+b)E_{13} + E_{21} + \tau(b)E_{23} + E_{32} + E_{33} + \sum_{s\in S} E_{ss},$$

this implies $\tau(a+b) - \tau(b) = \tau(a)$, or equivalently $\tau(a+b) = \tau(a) + \tau(b)$. Thus, τ is also additive, and the result follows. $\square$

Corollary 3.4 *Let* $2 < k < \min\{m, n\}$ *and* $f : M_{m,n} \to M_{m,n}$.

(1) *If f is Schur multiplicative, then* (b) *and* (c) *in Theorem* 3.3 *are equivalent with the additional requirement in condition* (c) *that P and Q are permutation matrices.*

(2) *If f is Schur multiplicative and has the form* (†) (*or satisfies any of the conditions* (A1) – (A3) *in Theorem* 2.1), *then conditions* (a) – (c) *in Theorem* 3.3 *are equivalent with the additional requirement in condition* (c) *that P and Q are permutation matrices.*

Proof. Suppose f is Schur multiplicative. Clearly, (c) $\Rightarrow$ (b) $\Rightarrow$ (a).

If (b) holds, then condition (A1) in Theorem 2.1 holds, and hence f has the form (†). We can then apply Theorem 3.3 to get condition (c) for some invertible monomial matrices P and Q. Now, if $X \circ X = X$, i.e., X has entries in $\{0, 1\}$, then so is $f(X)$. Thus, we see that P and Q can be chosen to be permutation matrices in condition (c).

If f is Schur multiplicative and has the form (†), we can apply Theorem 3.3 and the argument in the last paragraph to get the conclusion. $\square$

The conclusion in Corollary 3.4 (2) is not valid if we just assume that f is Schur multiplicative and $f(0_{m,n}) = 0_{m,n}$. For instance, one can define f by $f(0_{m,n}) = 0_{m,n}$ and $f(A) = B$ for all other A, where $B \in M_{m,n}$ is any rank k satisfying $B \circ B = B$. Then f maps all rank k matrices to a rank k matrix, but f does not have the structure described in Theorem 3.3 (c).

One can examine the proof and see that condition (b) in Theorem 3.3 (and also Corollary 3.4 can be replaced by any one of the following conditions.

(b.1) *$f(A)$ has rank at most k whenever $A \in M_{m,n}$ has rank k.*

(b.2) $f(A)$ *has rank at most* k *whenever* $A \in M_{m,n}$ *has rank at most* k.

In particular, the conclusion holds for those functions which map singular matrices to singular matrices when $m = n$. This can be viewed as an analog of the linear preserver result of Dieudonné [4].

Next, we consider preservers of full rank matrices.

Theorem 3.5 *Suppose* $2 \leq m \leq n$ *and* $f : M_{m,n} \to M_{m,n}$ *has the form* (†). *If* f *maps rank* m *matrices to rank* m *matrices, then there exist invertible monomial matrices* $P \in M_m$ *and* $Q \in M_n$, *and maps* $f_{ij} : \mathbf{F} \to \mathbf{F}$ *such that* $f_{ij}(\mathbf{F}^*) \subseteq \mathbf{F}^*$ *for all* (i,j) *pairs and one of the following holds:*

(a) f *has the form* $[a_{ij}] \mapsto P[f_{ij}(a_{ij})]Q$.

(b) $m = n$ *and* f *has the form* $[a_{ij}] \mapsto P[f_{ij}(a_{ij})]^t Q$.

If one of the f_{ij} *is surjective, then there is an injective multiplicative map* $\tau : \mathbf{F} \to \mathbf{F}$ *such that* $f_{ij} = \tau$ *for all* (i,j) *pairs; furthermore, if* $m \geq 3$, *then* τ *is a field automorphism.*

Proof. We divide the proof into several assertions.

Assertion 1 There are invertible monomial matrices $P \in M_m$ and $Q \in M_n$ such that $f(X) = PXQ$ for $X \in \{E_{jj} : 1 \leq j \leq m\}$.

To prove the assertion, let $X = \sum_{j=1}^m E_{jj}$. Since rank $(f(X)) =$ rank $(X) = m$, we see that $f(X)$ has nonzero entries on m distinct rows and columns. So, there are permutation matrices $P \in M_m$ and $Q \in M_n$ such that $f(E_{jj}) = Pf_{jj}(1)E_{jj}Q$ for $j = 1, \ldots, m$. We may replace f by the map $A \mapsto P^{-1}f(A)Q^{-1}$ for suitable invertible monomial matrices $P \in M_m$ and $Q \in M_n$ so that $f(E_{jj}) = E_{jj}$ for $j = 1, \ldots, m$.

Assertion 2 Assume $m < n$. There are invertible monomial matrices $P \in M_m$ and $Q \in M_n$ such that $f(X) = PXQ$ for $X \in \{E_{jj} : 1 \leq j \leq m\} \cup \{E_{1j} : m < j \leq n\}$. Moreover, $f(E_{ij}) = f_{ij}(1)PE_{ij}Q$.

By Assertion 1, we may assume that $f(E_{jj}) = E_{jj}$ for $j = 1, \ldots, m$. For any $r > m$, consider $X = E_{1r} + \sum_{s=2}^m E_{ss}$. Assume that $f(E_{1r}) = f_{1r}(1)E_{pq}$. Since rank $(f(X)) =$ rank $(X) = m$ and $f(E_{jj}) = E_{jj}$, it is impossible to have $p > 1$ or $q \leq m$. It follows that $f(E_{1r}) = f_{1r}(1)E_{1q}$ for some $q > m$. Thus, we may further modify f by a map of the form $A \mapsto f(A)(I_m \oplus Q)$ for a suitable invertible monomial matrix $Q \in M_{n-m}$ so that

$$f(X) = X \quad \text{for} \quad X \in \{E_{jj} : 1 \leq j \leq m\} \cup \{E_{1j} : m < j \leq n\}. \tag{3.1}$$

If $n = m+1$, consider $X = E_{i,m+1} + \sum_{j \neq i} E_{jj}$. Since rank $(f(X)) =$ rank $(X) = m$, we see that $f(E_{i,m+1}) = f_{i,m+1}(1)E_{i,m+1}$ for all $i > 1$. Now, suppose $n > m+1$. Consider $f(E_{2j})$ for $j > m$. Let $X = E_{1r} + E_{2j} + \sum_{s=3}^m E_{ss}$ with $r > m$ and $r \neq j$. Assume that $f(E_{2j}) = f_{2j}(1)E_{pq}$. Since rank $(f(X)) =$ rank $(X) = m$ and and condition (3.1) holds, we see that $p = 2$, $q \neq r$, and $q > m$. Because the argument holds for all $r > m$ with $r \neq j$, we conclude that $f(E_{2j}) = f_{2j}(1)E_{2j}$. Using the same argument, we can prove that $f(E_{ij}) = f_{ij}(1)E_{ij}$ for all $i > 1$ and $j > m$ as asserted.

Next, we turn to $f(E_{ij})$ for $1 < i \leq m$, $1 \leq j \leq m$ with $i \neq j$. The result is clear if $m = 2$. Assume $m \geq 3$ and $f(E_{ij}) = f_{ij}(1)E_{pq}$. Let $X = E_{ij} + E_{j,m+1} + \sum_{s \in S} E_{ss}$, where $S = \{1, 2, ...m\} \setminus \{i, j\}$. By the conclusion above $f(E_{j,m+1}) = f_{j,m+1}(1)E_{j,m+1}$. Since $m =$ rank $(f(X)) =$ rank (X) and (3.1) holds, we see that $(p,q) = (i,j)$.

Assertion 3 Assume that $m = n$. Then there are invertible monomial matrices $P, Q \in M_n$ such that (i) $f(E_{ij}) = Pf_{ij}(1)E_{ij}Q$ for all (i,j) pairs, or (ii) $f(E_{ij}) = Pf_{ij}(1)E_{ji}Q$ for all (i,j) pairs.

By Assertion 1, we may assume that $f(E_{jj}) = E_{jj}$ for all j. Consider $X = E_{ij} + E_{ji} + \sum_{s\notin\{i,j\}} E_{ss}$. Since $m = \text{rank}\,(X) = \text{rank}\,(f(X))$, we see that either

(i)' $f(E_{ij}) = f_{ij}(1)E_{ij}$ and $f(E_{ji}) = f_{ji}(1)E_{ji}$, or
(ii)' $f(E_{ij}) = f_{ij}(1)E_{ji}$ and $f(E_{ji}) = f_{ji}(1)E_{ij}$.

Assume $f(E_{12}) = f_{12}(1)E_{12}$; otherwise replace f by the map $A \mapsto f(A)^t$. We will prove that conclusion (i) holds. To this end, let $X = E_{12} + E_{2j} + E_{j1} + \sum_{s\in S} E_{ss}$, where $S = \{3, \dots, m\} \setminus \{j\}$. Then $m = \text{rank}\,(A) = \text{rank}\,(f(A))$. If $f(E_{j1}) = f_{j1}(1)E_{1j}$, then $f(A)$ will only have $m-1$ nonzero columns so $\text{rank}\,(f(A)) < m$, which is a contradiction. Thus, condition (i) holds for (i,j) pairs with $i = 1$ or $j = 1$. Now, for $X = E_{ij}$ with $1 \notin \{i,j\}$ and $i \neq j$, consider $X = E_{1i} + E_{ij} + E_{j1} + \sum_{s\in S} E_{ss}$, where $S = \{2, \dots, m\} \setminus \{i,j\}$ with $m-3$ elements. Since $m = \text{rank}\,(A) = \text{rank}\,(f(A))$, we see that condition (i) holds.

By Assertions 2 and 3, we get the first conclusion of the theorem, namely, f has the form $[a_{ij}] \mapsto P[f_{ij}(a_{ij})]Q$ or $m = n$ and f has the form $[a_{ij}] \mapsto P[f_{ij}(a_{ij})]^tQ$, We finish the proof by establishing the following.

Assertion 4 Suppose there is (p,q) such that f_{pq} is surjective. Then $f_{ij} = f_{pq}$ for each (i,j) pair, and f_{pq} is injective multiplicative. Furthermore, if $m \geq 3$ then f_{pq} is a field isomorphism.

Assume condition (a) holds. (If (b) holds, replace f by the map $A \mapsto f(A)^t$ and apply a similar argument.) We may further assume that $P = I_m$ and $Q = I_n$ in condition (a); otherwise, replace f by the map $A \mapsto P^{-1}f(A)Q^{-1}$. Moreover, we assume that $(p,q) = (1,1)$, i.e., f_{11} is a surjective map. Otherwise, we may find a pair of permutation matrices $R \in M_m$ and $S \in M_n$ such that $RE_{11}S = E_{pq}$, and replace the map f by the map $A \mapsto R^t f(RAS)S^t$. Furthermore, we may replace f by the map $A \mapsto f(A)/f_{22}(1)$ and assume that $f_{22}(1) = 1$. Let $D_1 = \text{diag}\,(f_{12}(1), 1, f_{32}(1), \dots, f_{m2}(1))$ and $D_2 = \text{diag}\,(f_{21}(1), 1, f_{23}(1), \dots, f_{2n}(1))$. We may replace f by the map $A \mapsto D_1^{-1}f(A)D_2^{-1}$ and assume that

$$f(X) = X \qquad \text{for} \qquad X \in \{E_{i2} : 1 \leq i \leq m\} \cup \{E_{2j} : 1 \leq j \leq n\}.$$

We claim that $f_{i1} = f_{11}$ for all $i > 1$. To see this, let $a \in \mathbf{F}$ and let $\{s_3, \dots, s_m\} = \{1, \dots, m\} \setminus \{1, i\}$. If $b \neq a$, then $Y = bE_{11} + aE_{i1} + E_{21} + E_{i2} + \sum_{k=3}^m E_{s_k,k}$ has rank m and so has $f(Y) = f_{11}(b)E_{11} + f_{i1}(a)E_{i1} + E_{21} + E_{i2} + \sum_{k=3}^m f_{s_k,k}(1)E_{s_k,k}$. It follows that $f_{11}(b) \neq f_{i1}(a)$ whenever $b \neq a$. Since f_{11} is surjective, $f_{i1}(a)$ is in the range of f_{11}. Thus, $f_{11}(a) = f_{i1}(a)$.

Next, we show that $f_{1j} = f_{11}$ for all $j > 1$. To see this, let $a \in \mathbf{F}$ and let $\{s_3, \dots, s_m\}$ be an $m-2$ element subset of $\{1, \dots, n\} \setminus \{1, j\}$. If $b \neq a$, then $Y = bE_{11} + aE_{1j} + E_{21} + E_{2j} + \sum_{j=3}^m E_{j,s_j}$ has rank m and so has $f(Y) = f_{11}(b)E_{11} + f_{1j}(a)E_{1j} + E_{21} + E_{2j} + \sum_{k=3}^m f_{k,s_k}(1)E_{k,s_k}$. It follows that $f_{11}(b) \neq f_{1j}(a)$ whenever $b \neq a$. Since f_{11} is surjective, $f_{1j}(a)$ is in the range of f_{11}. Thus, $f_{11}(a) = f_{1j}(a)$.

Now, consider f_{ij} with $i, j > 1$. Let $a \in \mathbf{F}$, $\{r_3, \dots, r_m\} = \{1, \dots, m\} \setminus \{1, i\}$, and $\{s_3, \dots, s_m\}$ be an $m-2$ element subset of $\{1, \dots, n\} \setminus \{1, j\}$. If $b \neq a$, then $Z = bE_{11} + bE_{i1} + bE_{1j} + aE_{ij} + \sum_{k=3}^m E_{r_k,s_k}$ has rank m and so has $f(Z) = f_{11}(b)E_{11} + f_{11}(b)E_{i1} + f_{11}(b)E_{1j} + f_{ij}(a)E_{ij} + \sum_{k=3}^m f_{r_k,s_k}(1)E_{r_k,s_k}$. It follows that

$$f_{11}(b) \neq f_{ij}(a) \qquad \text{whenever } b \neq a. \tag{3.2}$$

Since f_{11} is surjective, $f_{ij}(a)$ is in the range of f_{11}. Thus, $f_{11}(a) = f_{ij}(a)$.

At this point, we may assume that $f_{ij} = f_{11} = \tau$ for all (i, j) pairs, with $\tau(0) = 0$ and $\tau(1) = f_{11}(1) = 1$.

Now, we show that τ is multiplicative. Let $a, b \in \mathbf{F}$. If $a = 0$ or $b = 0$, then $\tau(ab) = 0 = \tau(a)\tau(b)$. If $ab \neq 0$, then for any $c \neq ab$, the matrix $X = cE_{11} + aE_{12} + bE_{21} + \sum_{j=2}^{m} E_{jj}$ has rank m, and so is $f(X) = \tau(c)E_{11} + \tau(a)E_{12} + \tau(b)E_{21} + \sum_{j=2}^{n} E_{jj}$. Thus, $\tau(c) \neq \tau(a)\tau(b)$. Since τ is surjective, $\tau(a)\tau(b)$ is in the range of τ. Thus, $\tau(ab) = \tau(a)\tau(b)$. Note that for $b \neq a$, we have $\tau(b) = f_{11}(b) \neq f_{ij}(a) = \tau(a)$ by (3.2). Thus, τ is injective.

Finally, suppose $m \geq 3$. Let $a, b \in \mathbf{F}$. If $a = 0$ or $b = 0$, then $\tau(a + b) = \tau(a) + \tau(b)$. Suppose $ab \neq 0$. Let $c \neq a + b$. $X = E_{11} + aE_{12} + cE_{13} + E_{21} + bE_{23} + E_{32} + E_{33} + \sum_{s=4}^{m} E_{ss}$. Then X has rank m and so is $f(X) = E_{11} + \tau(a)E_{12} + \tau(c)E_{13} + E_{21} + \tau(b)E_{23} + E_{32} + E_{33} + \sum_{s=4}^{m} E_{ss}$. Thus, $\tau(c) - \tau(b) \neq \tau(a)$, or equivalently $\tau(c) \neq \tau(a) + \tau(b)$. Since τ is surjective, $\tau(a) + \tau(b)$ is in the range of τ. Thus, $\tau(a + b) = \tau(a) + \tau(b)$. $\square$

Corollary 3.6 *Suppose f is Schur multiplicative and has the form (†). Then the conclusion of Theorem 3.5 holds with the additional restriction that f_{ij} is multiplicative for each (i, j) pair, P and Q are permutation matrices.*

Clearly, the conclusion of Corollary 3.6 holds if f is Schur multiplicative and satisfies any of the conditions (A1) – (A3) in Theorem 2.1. However, the conclusion is no longer valid if we just assume that f is Schur multiplicative. For instance, one can define f such that $f(0_{m,n}) = 0_{m,n}$ and $f(A) = B$ for all other A, where B is any rank m matrix satisfying $B \circ B = B$.

4 Preservers of determinant and other functions of eigenvalues

Linear maps, additive maps and multiplicative maps of determinants and functions of eigenvalues have been studied by researchers; see [5, 6, 16]. In this section, we study Schur multiplicative maps preserving determinant and related functions. In most cases, we obtain results for maps of the form (†), and then use them to study Schur multiplicative maps. We begin with an analog of the result of Frobenius [6] on linear preservers of the determinant function.

Theorem 4.1 *Suppose $f : M_n \to M_n$ has the form (†). Then f satisfies $\det(f(A)) = \det(A)$ for all A if and only if there are monomial matrices P and Q satisfying $\det(PQ) = 1$ such that f has the form*

$$A \mapsto PAQ \qquad \text{or} \qquad A \mapsto PA^tQ.$$

Proof. The ($\Leftarrow$) is clear. We consider the converse. Note that $f(I_n) = P$ is a monomial matrix with determinant 1. Replacing f by the mapping $A \mapsto P^{-1}f(A)$, we may assume that $f(I_n) = I_n$. Furthermore, we may replace f by a map of the form $A \mapsto Q^t f(A)Q$ for a suitable permutation matrix Q, so that $f(E_{jj}) = E_{jj}$ for $j = 1, \ldots, n$. Now, for any $a \in \mathbf{F}$,

$$a = \det(I_n + (a - 1)E_{jj}) = \det(f(I_n + (a - 1)E_{jj})) = f_{jj}(a).$$

So, $f_{jj}(a) = a$ for all $a \in \mathbf{F}$.

Next, we show that $f(E_{ij} + E_{ji}) = \gamma_{ij} E_{ij} + \gamma_{ij}^{-1} E_{ji}$ for all $1 \le i < j \le n$. For simplicity, assume that $(i,j) = (1,2)$. Then $-1 = \det(X) = \det(f(X))$ for $X = E_{12} + E_{21} + \sum_{j=3}^n E_{jj}$. Thus, $f(E_{12} + E_{21}) = \gamma_{12} E_{12} + \gamma_{12}^{-1} E_{21}$. Let $D = \mathrm{diag}\,(1, \gamma_{12}, \ldots, \gamma_{1n})$. We may replace f by the map $A \mapsto D^{-1} f(A) D$, and assume that $f(X) = X$ for all $X = E_{1j} + E_{j1}$ with $2 \le j \le n$.

Now, for $X_a = aE_{12} + E_{21} + \sum_{j=3}^n E_{jj}$, we have $-a = \det(X_a) = \det(f(X_a))$. We see that one of the following holds.

(i) $f(E_{12}) = E_{12}$, $f(E_{21}) = E_{21}$, and $f_{12}(a) = a$ for all $a \in \mathbf{F}$, or
(ii) $f(E_{12}) = E_{21}$, $f(E_{21}) = E_{12}$, and $f_{21}(a) = a$ for all $a \in \mathbf{F}$.

We may assume that (i) holds. Otherwise, replace f by the map $A \mapsto f(A)^t$.

We are done if $n = 2$. Assume that $n \ge 3$. For $j \ge 3$, let $S = \{3, \ldots, n\} \setminus \{j\}$, and $X_j = E_{12} + E_{1j} + E_{2j} + E_{j1} + E_{j2} + \sum_{s \in S} E_{ss}$. Then X_j has determinant 1 and so does $f(X_j) = E_{12} + E_{1j} + \gamma_{2j} E_{2j} + E_{j1} + \gamma_{2j}^{-1} E_{j2} + \sum_{s \in S} E_{ss}$. It follows that $\gamma_{2j} = 1$. Next, note that $X_a = E_{12} + E_{1j} + aE_{2j} + E_{j1} + E_{j2} + \sum_{s \in S} E_{ss}$ has determinant a and so does $f(X_a)$. We conclude that $f(aE_{2j}) = aE_{2j}$ and $f(E_{j2}) = E_{j2}$. Since $\det(f(Y_a)) = \det(Y_a)$ for $Y_a = E_{21} + E_{1j} + E_{2j} + E_{j1} + aE_{j2} + \sum_{s \in S} E_{ss}$, we see that $f(aE_{j2}) = aE_{j2}$. Now, note that $Z_a = aE_{1j} + E_{21} + E_{j2} + \sum_{s \in S} E_{ss}$ has determinant a and so does $f(Z_a)$. We conclude that $f(aE_{1j}) = aE_{1j}$ and $f(E_{j1}) = E_{j1}$. Furthermore, $U_a = aE_{j1} + E_{12} + E_{2j} + \sum_{s \in S} E_{ss}$ has determinant a and so does $f(U_a)$. We conclude that $f(aE_{j1}) = aE_{j1}$.

We are done if $n \le 3$. Otherwise, consider $j \ge 4$ and $S = \{1, 4, \ldots, n\} \setminus \{j\}$. Let $X_a = E_{23} + aE_{3j} + E_{j2} + \sum_{s \in S} E_{jj}$. Using the fact that $a = \det(X_a) = \det(f(X_a))$, we conclude that $f(aE_{3j}) = aE_{3j}$. Using the matrix $Y_a = E_{32} + aE_{j3} + E_{2j} + \sum_{s \in S} E_{jj}$, we see that $f(aE_{j3}) = aE_{j3}$.

We can repeat the above argument until we conclude that $f(aE_{ij}) = aE_{ij}$ for all (i,j) pairs. $\square$

If f is Schur multiplicative of the form (†), then one easily show that conclusion of Theorem 4.1 holds with the additional restriction that P and Q are permutation matrices. In the following, we show that one can obtain the same conclusion for Schur multiplicative maps $f : M_n \to M_n$ such that $f(0_n) = 0_n$ and $\det(f(A)) = \det(A)$ for all $A \in M_n$, if $n \ge 3$. For $n = 2$, one can define f by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & bc \\ 1 & d \end{pmatrix}.$$

Then f will be Schur multiplicative and preserves determinant. [In fact, it preserves all eigenvalues.]

Theorem 4.2 *Suppose $n \ge 3$ and $f : M_n \to M_n$ is Schur multiplicative such that $f(0_n) = 0_n$ and $\det(f(A)) = \det(A)$ for all $A \in M_n$. Then there are permutation matrices $P, Q \in M_n$ such that f has the form*

$$A \mapsto PAQ \qquad \text{or} \qquad A \mapsto PA^tQ.$$

Proof. Suppose f satisfy the hypothesis of the theorem. We will show that f also satisfies condition (A1) of Theorem 2.1. So, f has the form (†), and we can apply Theorem 4.1 to get the desired conclusion. We divide our proof into several assertions.

Assertion 1 The function f maps the set S_n of permutation matrices back to itself bijectively.

Suppose $R \in S_n$. Let $S = E_{12} + E_{23} + \ldots + E_{n,n-1} + E_{n1}$ be the basic circulant, and let $\mathcal{T} = \{R, RS, RS^2, \ldots RS^{n-1})$. For any two distinct $X, Y \in \mathcal{T}$,

(1) $X \circ X = X$ implies $f(X) \circ f(X) = f(X)$ so that all entries of $f(X)$ belongs to $\{0, 1\}$;

(2) $X \circ Y = 0_n$ implies $f(X) \circ f(Y) = f(X \circ Y) = f(0_n) = 0_n$ so that the nonzero entries of $f(X)$ must be disjoint from the nonzero entries of $f(Y)$.

(3) $\pm 1 = \det(X) = \det(f(X))$ so that $f(X)$ has at least n nonzero entries on distinct rows and columns.

Consequently, $f(X)$ must have exactly n nonzero entries, and therefore each $f(X)$ is a permutation matrix. In particular $f(R)$ is a permutation matrix. For two distinct permutation matrices R_1, R_2, the condition $f(R_1) = f(R_2)$ implies that $1 = \det(f(R_1) \circ f(R_2)) = \det(R_1 \circ R_2) = 0$, a contradiction. Therefore $f(S_n) = S_n$.

Assertion 2 Two matrices $R_1, R_2 \in S_n$ satisfy $R_1 \circ R_2 = 0_n$ if and only if $f(R_1) \circ f(R_2) = 0_n$.

Recall that a permutation matrix is a derangement if none all its diagonal entries are zero, and there is a fixed number, say, d_n, for $n \times n$ derangements by basic combinatorial theory. Now consider $\mathcal{N}(R) = \{X : X \in S_n, R \circ X = 0_n\} = \{X : RX^t \text{ is a derangement}\}$. Then $\mathcal{N}(R)$ has d_n elements and so has $\mathcal{N}(f(R))$. For $X \in \mathcal{N}(R)$, we have $0_n = R \circ X$, so that $0_n = f(R \circ X) = f(R) \circ f(X)$, and then $f(X) \in \mathcal{N}(f(R))$. So $f(\mathcal{N}(R)) \subseteq \mathcal{N}(f(R))$. Since each set has d_n elements, the sets must be equal. Then for any permutation matrix Z, $Z \notin \mathcal{N}(R)$ if and only if $f(Z) \notin \mathcal{N}(f(R))$. So for any permutations R_1, R_2, we have $R_1 \circ R_2 = 0_n$ if and only if $f(R_1) \circ f(R_2) = 0_n$.

Assertion 3 The function f satisfies condition (A1) in Theorem 3.1.

Note that for $n \geq 3$, each E_{ij} can be written as $E_{ij} = R_1 \circ R_2$ for a pair of permutations R_1, R_2. Therefore $f(E_{ij}) = f(R_1) \circ f(R_2) \neq 0_n$. So f satisfies (A1), and by Theorem 2.1, f is of the form (†), so we can apply Theorem 4.1 and conclude that f has the asserted form. $\square$

We believe that in the theorem one may even remove the assumption that $f(0_n) = 0_n$ to get the same conclusion. It would be nice to prove or disprove this.

If $\mathbf{F}$ is algebraically closed, then $\det(A)$ is the product of the eigenvalues of A. Researchers have studied linear preservers of other elementary symmetric functions of the eigenvalues; see [1, 9, 12, 14]. Denote by $E_k(A)$ the kth elementary symmetric function of the eigenvalues of $A \in M_n$. Then $E_n(A) = \det(A)$, and $E_k(A)$ equals the sum of the $k \times k$ principal minors of A. Defining $E_k(A)$ as the sum of the $k \times k$ minors of A, we can study $E_k(A)$ even if $\mathbf{F}$ is not algebraically closed. We have the following result.

Theorem 4.3 *Suppose $f : M_n \to M_n$ has the form (†) and $3 \leq k < n$. Then $E_k(A) = E_k(f(A))$ for all A if and only if there is an invertible monomial matrix P and a scalar $\gamma \in \mathbf{F}$ satisfying $\gamma^k = 1$ such that f has the form*

$$A \mapsto \gamma PAP^{-1} \qquad \text{or} \qquad A \mapsto \gamma PA^tP^{-1}.$$

If f is Schur multiplicative of the form (†), then the above conclusion holds with the additional restriction that the matrix P is a permutation matrix and $\mu = 1$.

Proof. The implication ($\Leftarrow$) is clear. For the converse, consider any k element subset S of $\{1, \ldots, n\}$, and $X = \sum_{j \in S} E_{jj}$. Since $1 = E_k(X) = E_k(f(X))$ we see that $f(X)$ is the sum of k

nonzero matrices in $\{\gamma E_{jj} : 1 \le j \le n, \gamma \in \mathbf{F}\}$ with distinct j. Thus, there exists a permutation matrix $P \in M_n$ such that $f(E_{jj}) = \gamma_j P E_{jj} P^t$ for $j = 1, \dots, n$. Since

$$1 = E_k(X) = E_k(f(X)) = \prod_{s \in S} \gamma_s$$

for any k element subset S of $\{1, \dots, n\}$, we see that $\gamma_1 = \cdots = \gamma_n = \gamma$ such that $\gamma^k = 1$.

Replace f by the map $A \mapsto P^t f(A) P/\gamma$ so that we may assume that $f(E_{jj}) = E_{jj}$ for $j = 1, \dots, n$. Now, for any $a \in \mathbf{F}$,

$$a = E_k\left(aE_{11} + \sum_{j=2}^{k} E_{jj}\right) = E_k\left(f\left(aE_{11} + \sum_{j=2}^{k} E_{jj}\right)\right) = f_{11}(a).$$

So, $f_{11}(x) = x$ for all $x \in \mathbf{F}$. Similarly, we can show that $f_{jj}(x) = x$ for all $x \in \mathbf{F}$.

Next, we show that $f(E_{ij} + E_{ji}) = \gamma_{ij} E_{ij} + \gamma_{ij}^{-1} E_{ji}$ for all (i,j) pairs. For simplicity, assume that $(i,j) = (1,2)$. For any subset S of $\{3, \dots, n\}$ with $k-2$ elements, consider $X_S = E_{12} + E_{21} + \sum_{j \in S} E_{jj}$. We have $-1 = E_k(X_S) = E_k(f(X_S))$. Thus, $f(E_{12} + E_{21}) = \gamma_{12} E_{pq} + \gamma_{21} E_{rs}$ satisfies $\{p, q, r, s\} \cap S = \emptyset$. As this is true for all subsets S of $\{3, \dots, n\}$ with $k-2$ elements, we conclude that $f(E_{12} + E_{21}) = \gamma_{12} E_{12} + \gamma_{21} E_{21}$. Since -1 $= E_k(X) = E_k(f(X))$ for $X = E_{12} + E_{21} + \sum_{j=3}^{k} E_{jj}$, we see that $\gamma_{12}\gamma_{21} = 1$.

Let $D = \operatorname{diag}(1, \gamma_{12}, \gamma_{13}, \dots, \gamma_{1n})$. Replace f by the map $A \mapsto Df(A)D^{-1}$ so that we can assume that $f(X) = X$ for $X \in \{E_{jj} : 1 \le j \le n\} \cup \{E_{1j} + E_{j1} : 2 \le j \le n\}$. Now, consider $X_a = aE_{12} + E_{21} + \sum_{j=3}^{k} E_{jj}$ with $a \in \mathbf{F}$. Since $E_k(X_a) = E_k(f(X_a))$, one of the following holds.

(i) $f(E_{12}) = E_{12}$, $f(E_{21}) = E_{21}$, and $f_{12}(a) = a$ for all $a \in \mathbf{F}$, or
(ii) $f(E_{12}) = E_{21}$, $f(E_{21}) = E_{12}$, and $f_{12}(a) = a$ for all $a \in \mathbf{F}$.

We may assume that (i) holds. Otherwise, replace f by the mapping $A \mapsto f(A)^t$.

For $j \ge 3$, let S be an $k-3$ element subset of $\{3, \dots, n\} \setminus \{j\}$, and $X_j = E_{12} + E_{1j} + E_{2j} + E_{j1} + E_{j2} + \sum_{s \in S} E_{ss}$. Then $1 = E_k(X_j) = E_k(f(X_j))$ where $f(X_j) = E_{12} + E_{1j} + \gamma_{2j} E_{2j} + E_{j1} + \gamma_{2j}^{-1} E_{j2} + \sum_{s \in S} E_{ss}$. It follows that $\gamma_{2j} = 1$. Next, note that

$$a = E_k(X_a) = E_k(f(X_a)) \qquad \text{for } X_a = E_{12} + E_{1j} + aE_{2j} + E_{j1} + E_{j2} + \sum_{s \in S} E_{ss}.$$

We conclude that $f(aE_{2j}) = aE_{2j}$ and $f(E_{j2}) = E_{j2}$. It follows that $f_{2j}(a) = a$. Since

$$a = E_k(f(Y_a)) = E_k(Y_a) \qquad \text{for } Y_a = E_{21} + E_{1j} + E_{2j} + E_{j1} + aE_{j2} + \sum_{s \in S} E_{ss},$$

we see that $f(aE_{j2}) = aE_{j2}$. Now, note that

$$a = E_k(Z_a) = E_k(f(Z_a)) \qquad \text{for } Z_a = aE_{1j} + E_{21} + E_{j2} + \sum_{s \in S} E_{ss}.$$

We conclude that $f(aE_{1j}) = aE_{1j}$ and $f(E_{j1}) = E_{j1}$. Furthermore,

$$a = E_k(U_a) = E_k(f(U_a)) \qquad \text{for } U_a = aE_{j1} + E_{12} + E_{2j} + \sum_{s \in S} E_{ss}.$$

We conclude that $f(aE_{j1}) = aE_{j1}$.

We are done if $n \leq 3$. Otherwise, consider $j \geq 4$ and let S be a $k-3$ element subset of $\{1, 4, \ldots, n\} \setminus \{j\}$. Note that

$$a = E_k(X_a) = E_k(f(X_a)) \qquad \text{for } X_a = E_{23} + aE_{3j} + E_{j2} + \sum_{s \in S} E_{ss}.$$

We conclude that $f(aE_{3j}) = aE_{3j}$. Next note that

$$a = E_k(Y_a) = E_k(f(Y_a)) \qquad \text{for } Y_a = E_{32} + aE_{j3} + E_{2j} + \sum_{s \in S} E_{ss},$$

We conclude that $f(aE_{j3}) = aE_{j3}$.

We can repeat the above argument until we conclude that $f(aE_{ij}) = aE_{ij}$ for all (i, j) pairs.

The proof for Schur multiplicative maps satisfying (A1) is similar and simpler. $\square$

Note that $E_1(A)$ is the trace function. It is easy to prove that a map $f : M_n \to M_n$ has the form (†) and satisfies $E_1(A) = E_1(f(A))$ for all $A \in M_n$ if and only if $\{f(E_{jj}) : 1 \leq j \leq n\} = \{E_{jj} : 1 \leq j \leq n\}$ and, for each $j \in \{1, \ldots, n\}$ we have $f_{jj}(x) = x$ for all $x \in \mathbf{F}$. Moreover, a Schur multiplicative map $f : M_n \to M_n$ satisfying (A1) also satisfies $E_1(A) = E_1(f(A))$ for all $A \in M_n$ if and only if f has the form described in Theorem 2.1 with the additional condition that $\{f(E_{jj}) : 1 \leq j \leq n\} = \{E_{jj} : 1 \leq j \leq n\}$ and, for each $j \in \{1, \ldots, n\}$ we have $f_{jj}(x) = x$ for all $x \in \mathbf{F}$.

For $E_2(A)$, there are a lot of linear preservers; see [9, 14]. In our case, we have the following.

Theorem 4.4 *Suppose $2 < n$ and $f : M_n \to M_n$ has the form (†). Then $E_2(A) = E_2(f(A))$ if and only if*

(a) *there is $\mu = \pm 1$ such that μf_{jj} is the identity map on $\mathbf{F}$ for all j with*

$$\{f(E_{jj}) : 1 \leq j \leq n\} = \{\mu E_{jj} : 1 \leq j \leq n\}, \quad \textit{and}$$

(b) *$f_{ij}(a) f_{ji}(b) = ab$ for any $a, b \in \mathbf{F}$ with*

$$\{f(E_{ij} + E_{ji}) : 1 \leq i < j \leq n\} = \{E_{ij} + E_{ji} : 1 \leq i < j \leq n\}.$$

If f is a Schur multiplicative map of the form (†), then the above conclusion holds with the additional restriction that f_{ij} is multiplicative in condition (b)*, and $\mu = 1$ in condition* (a)*.*

Proof. Note that for $A = (a_{ij})$, we have $E_2(A) = E_2(a_{11}, \ldots, a_{nn}) - \sum_{1 \leq i < j \leq n} a_{ij} a_{ji}$. By this observation, the implication ($\Leftarrow$) follows.

For the converse, note that $a = E_2(aE_{ii} + E_{jj}) = E_2(f(aE_{ii} + E_{jj}))$ for any $a \in \mathbf{F}$ and $i \neq j$. We see that $\{f(E_{jj}) : 1 \leq j \leq n\} = \{E_{jj} : 1 \leq j \leq n\}$ and there is $\mu = \pm 1$ so that μf_{jj} is the identity map on $\mathbf{F}$ for each $j \in \{1, \ldots, n\}$.

Now, $-ab = E_2(aE_{ij} + bE_{ji}) = E_2(f(aE_{ij} + bE_{ji})) = E_2(f_{ij}(a)E_{pq} + f_{ji}(b)E_{rs})$ for some $(p, q), (r, s)$ pairs with $p \neq q$ and $r \neq s$. We see that $(p, q) = (s, r)$ and $f_{ij}(a) f_{ji}(b) = ab$ for all $x \in \mathbf{F}$. The result follows.

One can readily verify the last assertion concerning Schur multiplicative maps. $\square$

By Theorem 4.3, one can easily determine the structures of eigenvalue preservers on M_n with $n \geq 4$. Here we show that they have the same structure as the preservers of the spectrum (not counting multiplicities of the eigenvalues) of matrices. Note that if $\mathbf{F}$ is not algebraically closed, $A \in M_n$ may not have eigenvalues in $\mathbf{F}$. In such case, we may assume that the spectrum of A is the empty set.

Theorem 4.5 *Let $\Gamma(A)$ denote the set of eigenvalues (counting multiplicities) or the spectrum of $A \in M_n$. Suppose $f : M_n \to M_n$ satisfies (†). Then f satisfies $\Gamma(f(A)) = \Gamma(A)$ for all $A \in M_n$ if and only if there is an invertible monomial matrix Q such that f has the form*

$$A \mapsto Q^{-1}AQ \qquad \text{or} \qquad A \mapsto Q^{-1}A^tQ.$$

If f is a Schur multiplicative map satisfying $f(0_n) = 0_n$, then the above conclusion holds with the additional restriction that Q is a permutation matrix.

Proof. The implication ($\Leftarrow$) is clear. We focus on the converse. If f preserves the set of eigenvalues, it will preserve the spectrum. Suppose f preserves the spectrum. We will show that f has the the asserted form.

Let $\mathrm{Spec}\,(A)$ denote the spectrum of $A \in M_n$. Then $\mathrm{Spec}\,(E_{ii}) = \{1, 0\}$. By Theorem 2.1, $f(E_{ii}) = E_{jk}$. Since f preserves the spectrum, we see that $j = k$. Thus $f(E_{ii}) = E_{jj}$, and we can assume that all $f(E_{ii}) = E_{ii}$ by replacing the mapping f with $A \to Qf(A)Q^t$ for a suitable permutation matrix Q. For any a in the field consider aE_{ii} with $\mathrm{Spec}\,(aE_{ii}) = \{a, 0\}$. We have $f(aE_{ii}) = f_{ii}(a)E_{ii}$ with $\mathrm{Spec}\,(f_{ii}(a)E_{ii}) = \{f_{ii}(a), 0\}$, and therefore we conclude that $f_{ii}(a) = a$.

Now for $X = E_{12}+E_{21}+E_{11}+E_{22}$ we have $\mathrm{Spec}\,(X) = \{2, 0\}$. Since we have already concluded that the diagonal entries map to themselves, we see that $f(E_{12} + E_{21}) = \mu_{12}E_{12} + \mu_{12}^{-1}E_{21}$. Using a similar argument, we can prove that $f(E_{ij} + E_{ji}) = \mu_{ij}E_{ij} + \mu_{ij}^{-1}E_{ji}$ with $\mu_{ij} \in \mathbf{F}^*$ for each (i, j) pair with $i < j$. Let $D = \mathrm{diag}\,(1, \mu_{12}, \ldots, \mu_{1n})$. Replace f by the map $A \mapsto Df(A)D^{-1}$ so that we have $f(X) = X$ for $X \in \{E_{jj} : 1 \leq j \leq n\} \cup \{E_{1j} + E_{j1} : 2 \leq j \leq n\}$. We can then consider two cases:

$$\text{(i) } f(E_{12}) = E_{12},\ f(E_{21}) = E_{21}, \quad \text{or} \quad \text{(ii) } f(E_{12}) = E_{21},\ f(E_{21}) = E_{12}$$

We can assume (i) holds; otherwise, replace f with the mapping $A \to f(A)^t$. Observe that $\mathrm{Spec}\,(X_a) = \{1 + a, 1, 0\}$ for $X_a = E_{11} + aE_{12} + E_{21} + aE_{22} + \sum_{i=3}^n E_{ii}$. If $f_{12}(a) \neq a$, then $f(X_a)$ is non-singular, and $0 \notin \mathrm{Spec}\,(f(X_a))$, which is a contradiction. Therefore $f_{12}(a) = a$.

Note that $f(E_{1j}) = E_{1j}$ or E_{j1} for $j > 2$. For $X_j = E_{12} + E_{1j} + E_{j2} + E_{2j}$, $\mathrm{Spec}\,(X_j) = \{0, -1, 1\} = \mathrm{Spec}\,(f(X_j))$. If $f(E_{1j}) = E_{j1}$, then $\mathrm{Spec}\,(f(X_j))$ is a subset of the set of zeros of the polynomial $z(z^3 - z - 1)$, which contains neither 1 nor -1. Therefore $f(E_{1j}) = E_{1j}$ and $f(E_{j1}) = E_{j1}$. For any $a \in \mathbf{F}$, $0 \in \mathrm{Spec}\,(Y_a) = \mathrm{Spec}\,(f(Y_a))$ for

$$Y_a = aE_{11} + E_{j1} + aE_{1j} + \sum_{i=2}^n E_{ii}.$$

Thus, $f(aE_{1j}) = aE_{1j}$. Similarly, $f(aE_{j1}) = aE_{j1}$ for all $a \in \mathbf{F}$.

We are done if $n = 2$. Assume $n \geq 3$. For $j \geq 3$. Since $0 \notin \mathrm{Spec}\,(f(X_j)) = \mathrm{Spec}\,(X_j)$ for $X_j = E_{12} + E_{2j} + E_{j1} + \sum_{s\in S} E_{ss}$ with $S = \{1, \ldots, n\} \setminus \{1, 2, j\}$, we see that $f(E_{2j}) = E_{2j}$, and thus $f(E_{j2}) = E_{j2}$. Assume $n \geq 4$. For $j \geq 4$. Since $0 \notin \mathrm{Spec}\,(f(X_j)) = \mathrm{Spec}\,(X_j)$ for

$X_j = E_{23} + E_{3j} + E_{j2} + \sum_{s\in S} E_{ss}$ with $S = \{1,\dots,n\} \setminus \{2,3,j\}$, we see that $f(E_{3j}) = E_{3j}$, and thus $f(E_{j3}) = E_{j3}$. We can repeat the argument until we conclude that $f(E_{ij}) = E_{ij}$ for each (i,j) pair with $i \neq j$. Moreover, for any (i,j) pair with $i \neq j$, since $0 \in \mathrm{Spec}\,(f(Y)) = \mathrm{Spec}\,(Y)$ for $Y = aE_{ij} + E_{ji} + aE_{ii} + \sum_{s\neq i} E_{ss}$, we see that $f(aE_{ij}) = aE_{ij}$ for any $a \in \mathbf{F}$. Thus, f has the asserted form.

Next, we consider Schur multiplicative maps satisfying $f(0_n) = 0_n$. Consider $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ such that $\mathcal{T}_1 = \{E_{jj} : 1 \leq j \leq n\}$ and $\mathcal{T}_2 = \{E_{ij} + E_{ji} : 1 \leq i,j \leq n\}$. Then $f(X) = f(X \circ X) = f(X) \circ f(X)$, $\{1,0\} = \mathrm{Spec}\,(X) = \mathrm{Spec}\,(f(X))$ for $X \in \mathcal{T}_1$ and $\{1,-1\} \subseteq \mathrm{Spec}\,(X) = \mathrm{Spec}\,(f(X))$ for $X \in \mathcal{T}_2$. Thus, $f(X)$ is nonzero with entries in $\{0,1\}$. For any $X \neq Y$ in $\mathcal{T}$, $0 = f(X \circ Y) = f(X) \circ f(Y)$, i.e., $f(X)$ and $f(Y)$ have nonzero entries in disjoint positions. Clearly, it is impossible to have more than $n(n-1)/2$ matrices with more than one nonzero entries in $f(\mathcal{T})$. Thus, at least n matrices in $\mathcal{T}$ have exactly one nonzero entry. Since none of these matrices have spectrum equal to $\{0\}$, we see that there are exactly n such matrices, which are the matrices in $\mathcal{T}_1$. As a result, we see that $f(\mathcal{T}_1) = \mathcal{T}_1$. Consequently, $f(X)$ has at least two nonzero entries for each $X \in \mathcal{T}$. Since the nonzero entries can only lie in the off-diagonal positions, we see that $f(X)$ has exactly two nonzero entries for each $X \in \mathcal{T}_2$. Since $\{1,-1\} \subseteq \mathrm{Spec}\,(X) = \mathrm{Spec}\,(f(X))$, we see that $f(X) \in \mathcal{T}_2$. Hence $f(\mathcal{T}_2) = \mathcal{T}_2$.

We may assume that $f(X) = X$ for all $X \in \mathcal{T}_1$. Otherwise, we may replace f by a map of the form $A \mapsto Pf(A)P^t$ for a suitable permutation P. Note that for any $1 \leq j < k \leq n$, if $X = E_{jj} + E_{jk} + E_{kj} + E_{kk}$ then $X \circ X = X$ and $2 \in \mathrm{Spec}\,(X)$; moreover, $Y \in \mathcal{T}$ satisfies $\{1,-1\} \subseteq \mathrm{Spec}\,(X \circ Y)$ if and only if $Y = E_{jk} + E_{kj}$. It follows that $f(X) \circ f(X) = f(X)$ and $2 \in \mathrm{Spec}\,(f(X))$; moreover, $Z \in f(\mathcal{T})$ satisfies $\{1,-1\} \subseteq (f(X) \circ Z)$ if and only if $Z = f(E_{jk} + E_{kj})$. Hence $f(E_{jk} + E_{kj}) = E_{jk} + E_{kj}$. As a result, we have $f(X) = X$ for every $X \in \mathcal{T}$.

Next, we show that $f(E_{jk}) \in \{E_{jk}, E_{kj}\}$ for any $1 \leq j < k \leq n$. For simplicity, we assume $(j,k) = (1,2)$. Let $X = E_{12} + E_{23} + E_{31}$. Then $Y \in \mathcal{T}$ satisfies $X \circ Y = 0$ whenever $Y \notin \mathcal{T}_0 = \{E_{12} + E_{21}, E_{13} + E_{31}, E_{23} + E_{32}\}$. Thus, $f(X) \circ f(Y) = f(X \circ Y) = f(0_n) = 0_n$ whenever $Y \in \mathcal{T} \setminus \mathcal{T}_0$. Since $f(X) = f(X \circ X) = f(X) \circ f(X)$ and $1 \in \mathrm{Spec}\,(X) = \mathrm{Spec}\,(f(X))$, we see that $f(X) \in \{X, X^t\}$. Now, for $\hat{X} = E_{12} + E_{21}$, we have $E_{12} = X \circ \hat{X}$, hence $f(E_{12}) = f(X) \circ f(\hat{X}) = f(X) \circ \hat{X} \in \{E_{12}, E_{21}\}$.

By the above arguments, we see that $f(E_{ij}) \neq 0$ for any (i,j) pair. Thus, condition (A1) of Theorem 2.1 holds. Thus, f has the form (†), and hence satisfies the desired conclusion by the first part of our proof. □

Clearly, the conclusions of most of the results in this section are valid if f is Schur multiplicative and satisfies any of the conditions (A1) – (A3) of Theorem 2.1. If $n = 2$, the conclusions of the theorems are not valid if we just assume that f is Schur multiplicative. For example, one may define f by $f(A) = 0$ if A is strictly upper or lower triangular, and $f(A) = A$ otherwise. However, for $n \geq 3$, it is unclear whether the conclusions of the theorems hold if we just assume that f is Schur multiplicative. It would be interesting to prove the results under the weaker assumption, or give examples showing the the results are not true.

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