Preserves of isometries

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Abstract

Let γ be a unimodular complex number, and let k be an integer. Then γA^k is an isometry for any isometry A of a complex Banach space. It is shown that if f is an analytic function on the unit circle sending an isometry to an isometry for any norm, then f has the form $z \mapsto \gamma z^k$ for some unimodular γ and integer k. The same conclusion on f can be deduced if f is merely continuous and preserves the isometries of some special classes of norms on a fixed finite-dimensional complex Banach space. The result is extended to real Banach spaces X with $\dim X \geq 4$, and it is shown that one cannot get the same conclusion on f if $\dim X < 4$. Further extensions of these results are also considered.

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1 Introduction

Suppose A is an isometry of a norm on $\mathbb{C}^n$. Then $A \mapsto \gamma A^k$ is also an isometry for any unimodular complex number γ and integer k. Therefore, this map will send the isometry group $\mathcal{G}$ of the norm back to itself. We are interested in the converse of this statement.

To formulate the converse, let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Since an isometry A of a norm on $\mathbb{C}^n$ is always similar to a diagonal unitary matrix, i.e., $A = SDS^{-1}$ (see, e.g., [1, Theorem 3.6]), one can apply any continuous function $f\colon \mathbb{T} \to \mathbb{T}$ to A via $f(A) = Sf(D)S^{-1}$. We say that the continuous function f preserves the isometries of a given norm if $f(A) \in \mathcal{G}$ whenever $A \in \mathcal{G}$, where $\mathcal{G}$ denotes the isometry group of the norm. We consider the following question.

Characterize continuous functions $f\colon \mathbb{T} \to \mathbb{T}$ that preserve the isometry group for any norm.

We also identify special classes of norms such that a continuous function $f\colon \mathbb{T} \to \mathbb{T}$ preserves its isometries if and only if f preserves all isometries.

More generally, let $B(X)$ denote the set of all bounded linear operators acting on a Banach space X. If X is infinite dimensional and $A \in B(X)$, then one may not be able to define $g(A)$ for a continuous function $g\colon \mathbb{C} \to \mathbb{C}$. Nevertheless, suppose $A \in B(X)$ and there is an invertible $S \in B(X)$ such that $S^{-1}AS = a_1 I_{X_1} \oplus \cdots \oplus a_k I_{X_k}$, where $X = X_1 \oplus \cdots \oplus X_k$. Such an operator A is said to be diagonalizable with finite spectrum (in other words, A is an algebraic operator whose minimal polynomial has only simple roots, see e.g. [7, p. 98]). Then $g(A)$ can be defined as $S(g(a_1)I_{X_1} \oplus \cdots \oplus g(a_k)I_{X_k})S^{-1}$.

It is a well known fact that on every complex Banach space the spectrum of a surjective isometry is contained in the unit circle (cf. [5, p. 80] and use the fact that the boundary of the spectrum is contained in the approximate point spectrum). For the sake of convenience we give a proof of this simple fact.

Lemma 1.1. *Let T be an isometry on a complex Banach space. If T is surjective, then $\operatorname{Sp} T \subseteq \mathbb{T}$, the unit circle. If T is not surjective, then $\operatorname{Sp} T = \overline{\Delta}$, the closed unit disk.*

Proof. Let α be a boundary spectral point of T. Then α belongs to the approximate point spectrum, so there exists a sequence of unit vectors x_n with $(T - \alpha I)x_n \overset{n\to\infty}{\longrightarrow} 0$. By the triangle inequality, $|1 - |\alpha|| = |\|Tx_n\| - \|\alpha x_n\|| \leq \|Tx_n - \alpha x_n\| \overset{n\to\infty}{\longrightarrow} 0$, which implies $|\alpha| = 1$. Hence, the boundary of a compact set, $\operatorname{Sp} T$, is contained in $\mathbb{T}$. □

We will also consider continuous functions $f\colon \mathbb{T} \to \mathbb{T}$ preserving those isometries on infinite dimensional Banach spaces that are diagonalizable with finite spectrum. Note that if X is finite dimensional, then we can identify X with $\mathbb{C}^n$, and every isometry is diagonalizable with finite spectrum. We consider separately the complex Banach spaces and the real Banach spaces.

Once we see that $f(SDS^{-1}) = Sf(D)S^{-1}$, the problem reduces to the study of functions $f\colon \mathbb{T} \to \mathbb{T}$ satisfying certain "nice" conditions. The techniques used in this paper can also be used to study other important functions $f\colon \mathbb{C} \to \mathbb{C}$ that can be characterized using a "preserver" property.

2 Preservers of linear isometries

2.1 Complex Banach spaces

The study of isometries is closely related to the study of extreme points since surjective isometries map extreme points to extreme points.

Proposition 2.1. *Let $m > n \geq 2$ and let $\omega = e^{\frac{2\pi i}{m}}$ be a primitive m-th root of unity. There exists a norm $\|\cdot\|_m$ on $\mathbb{C}^n$ such that the set of extreme points of its unit ball equals*

$$\mathcal{E} = \left\{e^{is}(1, \omega^k, \omega^{2k}, \ldots, \omega^{(n-1)k})^T : s \in \mathbb{R},\ k \in \{0, \ldots, m-1\}\right\}.$$

Moreover,

$$e^{is} \operatorname{diag}(1, \omega^r, \ldots, \omega^{(n-1)r}); \qquad s \in \mathbb{R},\ \ r = 0, \ldots, m-1$$

are the only diagonal matrices in the isometry group $\mathcal{G}$ of $\|\cdot\|_m$.

Proof. Note that the set $\mathcal{E}$ is equilibrated, i.e., closed under multiplication by unimodular complex numbers, contains n linearly independent vectors

$$v_k = (1, \omega^k, \dots, \omega^{(n-1)k})^T; \qquad k = 0, \dots, n-1$$

(they are columns of a Vandermonde matrix), and is completely contained in the Euclidean sphere of radius $\sqrt{n}$ because the ℓ_2 norm of every member of $\mathcal{E}$ equals $\sqrt{n}$. It follows that $\overline{\mathrm{conv}}(\mathcal{E})$, the closed convex hull of $\mathcal{E}$, is again an equilibrated subset inside a closed Euclidean ball of radius $\sqrt{n}$, contains 0 in its interior (by [10, Theorem 6.4]), and its set of extreme points coincides with $\mathcal{E}$ (because the ℓ_2 norm is strictly convex and $\mathcal{E}$ is compact, so $\overline{\mathrm{conv}}(\mathcal{E}) = \mathrm{conv}(\mathcal{E})$). The existence of the desired norm whose closed unit ball equals $\overline{\mathrm{conv}}(\mathcal{E})$ is then guaranteed by [4, Theorem 5.5.8].

Let $\mathcal{G}$ be the isometry group for the norm $\|\cdot\|$. Choose a diagonal isometry $A = \mathrm{diag}(\alpha_1, \dots, \alpha_n) \in \mathcal{G}$. Being a surjective isometry, A leaves the set of extreme points of the norm's unit ball invariant, so we have $Av_0 = e^{is} v_k$ for some integer k and some $s \in \mathbb{R}$. Hence,

$$(\alpha_1, \alpha_2, \dots, \alpha_n) = e^{is}(1, \omega^k, \omega^{2k}, \dots, \omega^{(n-1)k}).$$

Conversely, an easy computation shows that every such diagonal matrix maps the set $\mathcal{E}$ bijectively onto itself, so it also maps the norm's unit ball $\overline{\mathrm{conv}}(\mathcal{E})$ onto itself. Therefore, A is an isometry of $\|\cdot\|_m$. □

Remark 2.2. *Note that for a complex, possibly infinite-dimensional, Banach space $(X, \|\cdot\|)$ with $\dim X \geq 3$ there exists a bounded projection $P \in B(X)$ of rank-two. By identifying its image, $\operatorname{Im} P \subseteq X$, with $\mathbb{C}^2$ we devise an equivalent norm*

$$\|x\|_m' := \|Px\|_m + \|(I-P)x\|$$

where $\|\cdot\|_m$ is a norm on $\mathbb{C}^2$ constructed in Proposition 2.1. To see the equivalence of the two norms use the open mapping theorem and the fact that the identity map $I\colon (X, \|\cdot\|_m') \to (X, \|\cdot\|)$ is bounded.

Theorem 2.3. *Let $f\colon \mathbb{T} \to \mathbb{T}$ be a continuous function. Then the following are equivalent.*

(i) *There is a unimodular γ and an integer k such that $f(z) = \gamma z^k$.*

(ii) *$f(A)$ is an isometry for any diagonalizable isometry A with finite spectrum for any norm on any complex Banach space.*

(iii) *Given a fixed integer $n \geq 2$, $f(A)$ is an isometry for any isometry A on $\mathbb{C}^n$ corresponding to one of the special sequence of norms $(\|\cdot\|_m)_{m>n}$ constructed in Proposition 2.1.*

(iv) *Given a fixed complex Banach space X with $\dim X \geq 3$, $f(A)$ is an isometry for any diagonalizable isometry A with finite spectrum corresponding to one of the sequence of equivalent norms $(\|\cdot\|_m')_{m>2}$ on X constructed in Remark 2.2.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) are clear. It remains to prove (iv) $\Rightarrow$ (iii)' where (iii)' is the same as (iii) except that $n = 2$ and m is suitably chosen, and it remains to prove (iii) $\Rightarrow$ (i).

(iv) $\Rightarrow$ (iii) under $\mathbb{C}^n = \mathbb{C}^2$. By replacing f with $\frac{1}{f(1)} f$ we may assume $f(1) = 1$. Fix an integer $m \geq 3$ such that (iv) holds for a Banach space equipped with the norm $\|\cdot\|'_m$ and consider an isometry $A\colon \mathbb{C}^2 \to \mathbb{C}^2$ of the norm $\|\cdot\|_m$. Then, $\widehat{A} = A \oplus I$ is an isometry of the equivalent norm $\|\cdot\|'_m$ on X and by the assumptions, $f(\widehat{A}) = f(A) \oplus f(1)I = f(A) \oplus I$ is again an isometry of $\|\cdot\|'_m$. Thus, $f(A)\colon \mathbb{C}^2 \to \mathbb{C}^2$ is an isometry with respect to the norm $\|\cdot\|_m$, as claimed.

(iii) $\Rightarrow$ (i). Again we may assume $f(1) = 1$. For any $c \in \mathbb{T}$ and any primitive m-th root of unity ω $(m > n)$, the matrix $A = c\,\mathrm{diag}(1, \omega^k, \omega^{2k}, \dots \omega^{(n-1)k})$ is an isometry for the norm $\|\cdot\|_m$. Then $f(A)$ is also an isometry for $\|\cdot\|_m$ so by Proposition 2.1,

$$f(A) = \mathrm{diag}\big(f(c), f(c\omega^k), \dots, f(c\omega^{(n-1)k})\big) = f(c)\,\mathrm{diag}(1, \omega^{r(c)}, \dots, \omega^{(n-1)r(c)}),$$

where the integers $r(c)$ depend on c and on ω and k. In particular, $f(c\omega^k) = f(c)\omega^{r(c)}$. But $f(c\omega^k)/f(c)$ is a continuous function of c, and the m-th roots of unity form a discrete set, so $\omega^{r(c)}$ must be a constant when ω is kept fixed. Let $w = \omega^k$ be an m-th root of unity (possibly not primitive); since $f(1) = 1$ we have

$$\frac{f(cw)}{f(c)} = \frac{f(1w)}{f(1)} = f(w).$$

This holds for all $c \in \mathbb{T}$ and all roots of unity w. By continuity, $f(\alpha\beta) = f(\alpha)f(\beta)$ for all $\alpha, \beta \in \mathbb{T}$, so f is multiplicative, whence $f(z) = z^k$ for some integer k (see, e.g., [8, Example 1.2.7]). □

If A in $B(X)$ is not diagonalizable with finite spectrum, $f(A)$ may not be well defined for a continuous map f. It is more natural to consider analytic functions g on $A \in B(X)$. By Theorem 2.3, we have the following corollary.

Corollary 2.4. *Let $g\colon \mathbb{T} \to \mathbb{T}$ be an analytic function. Then the following are equivalent.*

(i) *There is a unimodular γ and an integer k such that $g(z) = \gamma z^k$.*

(ii) *$g(A)$ is an isometry for any isometry A for any norm on any complex Banach space.*

(iii) *Given a fixed integer $n \geq 2$, $g(A)$ is an isometry for any isometry A on $\mathbb{C}^n$ corresponding to one of the special sequence of norms $\big(\|\cdot\|_m\big)_{m>n}$ constructed in Proposition 2.1.*

(iv) *Given a fixed complex Banach space X with $\dim X \geq 3$, $g(A)$ is an isometry for any isometry on X corresponding to one of the sequence of equivalent norms $\big(\|\cdot\|'_m\big)_{m>2}$ constructed in Remark 2.2.*

2.2 Real Banach spaces

One may also study continuous maps $g\colon U \to \mathbb{C}$, with domain $U \subseteq \mathbb{C}$, such that for every isometry A on a real finite-dimensional Banach space, the map $g(A)$ is also an isometry. Note that an isometry on a real finite-dimensional Banach space is similar to an orthogonal matrix (see [2, §1] or [1, Theorem 3.6]) and as such may have complex eigenvalues, necessarily of modulus 1. So, it is natural to consider maps g with domain and co-domain on the unit circle $\mathbb{T}$. We continue to consider two problems:

1. Characterize g such that $g(A)$ is an isometry for any norm on a real Banach space.
2. Determine a class of norms on $\mathbb{R}^n$ such that if g preserves the isometry groups of these norms then g preserves surjective isometries for every norm on real Banach spaces.

For Problem 2, let us first note that real Banach spaces behave differently than complex ones. The example below is in sharp contrast to implication (iii) $\Rightarrow$ (i) in Theorem 2.3, with $n \in \{2, 3\}$.

Example 2.5. Consider a map $g \colon \mathbb{T} \to \mathbb{T}$ defined by

$$g(z) = \begin{cases} z^6; & z \text{ belongs to the short arc between } e^{2\pi i/6} \text{ and } e^{-2\pi i/6} \\ 1; & z \text{ belongs to the long arc between } e^{2\pi i/6} \text{ and } e^{-2\pi i/6} \end{cases}.$$

Then g is continuous. We claim that g preserves isometries with respect to every norm $\|\cdot\|$ on real Banach spaces $\mathbb{R}^2$ or $\mathbb{R}^3$.

To see this, recall [2, §1] that every isometry in $M_n(\mathbb{R})$ is similar to an orthogonal matrix. Hence, every isometry $A \in M_2(\mathbb{R})$ has either two real eigenvalues in $\{-1, 1\}$ or a pair of complex conjugate eigenvalues $e^{ir}, e^{-ir} \in \mathbb{T}$ (here, $-\pi \le r \le \pi$). Then the effect of $g(A)$ is either $g(A) = A^6$ if $r \in [-\pi/3, \pi/3]$, or $g(A) = I$ otherwise.

Likewise, every isometry $A \in M_3(\mathbb{R})$ has either three real eigenvalues in $\{-1, 1\}$, or one real eigenvalue in $\{-1, 1\}$ and two complex conjugated eigenvalues. Then the above function will again satisfy $g(A) = A^6$ if A has a pair of complex eigenvalues $e^{ir}, e^{-ir} \in \mathbb{T}$ with $r \in [-\pi/3, \pi/3]$ and $g(A) = I$ otherwise.

Of course, one can extend the above example and consider, for any positive integer k, $g(z) = z^{2k}$, if $z = e^{ir}$ with $r \in [-\pi/k, \pi/k]$ and $g(z) = 1$ otherwise. By the same reasoning, $g(A)$ will be an isometry for any isometry $A \in \{M_2(\mathbb{R}), M_3(\mathbb{R})\}$.

However, on real Banach spaces of dimension greater than 3 such pathologies do not occur.

Definition 2.6. Let $n \ge 4$ and let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$. For every integer $m > 2$ define a norm $\|\cdot\|_m$ on $\mathbb{R}^n$ as the norm whose unit ball has extreme points

$$\begin{cases} \pm\sqrt{2}e_5, \dots, \pm\sqrt{2}e_n, \\ \big(\cos(s + 2k\pi/m), \sin(s + 2k\pi/m), \cos s, \sin s, 0, \dots, 0\big)^T \end{cases} \tag{1}$$

where $s \in [0, 2\pi]$ and $k = 0, 1, \dots, m-1$.

For a real Banach space $(X, \|\cdot\|)$ with $\dim X \ge 4$, fix a bounded projection $P \in B(X)$ of rank-four, identify its image with $\mathbb{R}^4$, and define a norm on X, equivalent to $\|\cdot\|$, by

$$\|x\|_m' = \|Px\|_m + \|(I - P)x\|,$$

where $\|\cdot\|_m$ is the norm on $\mathbb{R}^4$ defined above.

We refer to the proof of Proposition 2.1 for the existence of a norm $\|\cdot\|_m$ whose unit ball has the extreme points listed in (1) and we refer to Remark 2.2 for the fact that $\|\cdot\|'_m$ and $\|\cdot\|$ are equivalent norms.

Theorem 2.7. *Let $f\colon \mathbb{T}\to\mathbb{T}$ be a continuous function. Then the following are equivalent.*

(i) *There is an integer k and a scalar $\gamma\in\{-1,1\}$ such that $f(z)=\gamma z^k$.*

(ii) *$f(A)$ is an isometry for any isometry A on any finite-dimensional real Banach space.*

(iii) *Given a fixed integer $n\geq 4$, $f(A)$ is an isometry for any isometry A on $\mathbb{R}^n$ corresponding to one of the special sequence of norms $\big(\|\cdot\|_m\big)_{m>n}$ defined in Definition 2.6.*

(iv) *Given a fixed real Banach space X with $\dim X\geq 4$, $f(A)$ is an isometry for any isometry on X corresponding to one of the sequence of equivalent norms $\big(\|\cdot\|'_m\big)_{m>2}$ defined in Definition 2.6.*

Proof. The only nontrivial implications are (iv) ⇒ (iii) ⇒ (i). Of these, the proof of (iv) ⇒ (iii) uses the same arguments (with isometry $\widehat{A}=A\oplus I$ of $\|\cdot\|'_m$) as the proof of Theorem 2.3.

Assume (iii) holds. Define unitary matrices

$$\hat{U}=\tfrac{1}{\sqrt{2}}\left(\begin{smallmatrix} i & -i\\ 1 & 1\end{smallmatrix}\right)\in M_2(\mathbb{C})\qquad\text{and}\qquad U=\hat{U}\oplus\hat{U}\oplus I_{n-4}\in M_n(\mathbb{C}).$$

It is straightforward that, for an isometry of the form

$$A=\begin{pmatrix}\cos(s+2k\pi/m) & \sin(s+2k\pi/m)\\ -\sin(s+2k\pi/m) & \cos(s+2k\pi/m)\end{pmatrix}\oplus\begin{pmatrix}\cos s & \sin s\\ -\sin s & \cos s\end{pmatrix}\oplus I_{n-4},$$

we have

$$A=U\,\mathrm{diag}(e^{-is}\omega_m^{-k},e^{is}\omega_m^{k},e^{-is},e^{is},1,\dots,1)U^*;\qquad \omega_m=e^{2\pi i/m}$$

and so

$$f(A)=U\,\mathrm{diag}(f(e^{-is}\omega_m^{-k}),f(e^{is}\omega_m^{k}),f(e^{-is}),f(e^{is}),f(1),\dots,f(1))U^*. \tag{2}$$

Now if, for some (necessarily unimodular) scalars $\beta_j=e^{-ir_j}$, the matrix

$$B=f(A)=U\,\mathrm{diag}(\beta_1,\dots,\beta_n)U^*$$

is real and an isometry for the norm $\|\cdot\|_m$, then the block-diagonal structure of U implies

$$\begin{aligned}B&=\hat{U}\,\mathrm{diag}(\beta_1,\beta_2)\hat{U}^*\oplus\hat{U}\,\mathrm{diag}(\beta_3,\beta_4)\hat{U}^*\oplus\mathrm{diag}(\beta_5,\dots,\beta_n)\\ &=\left(\begin{smallmatrix}\cos t_1 & \sin t_1\\ -\sin t_1 & \cos t_1\end{smallmatrix}\right)\oplus\left(\begin{smallmatrix}\cos t_3 & \sin t_3\\ -\sin t_3 & \cos t_3\end{smallmatrix}\right)\oplus\mathrm{diag}(\beta_5,\dots,\beta_n)\in M_n(\mathbb{R}),\end{aligned}$$

where the last equality follows from the fact that β_1 and β_2 are the eigenvalues of the first diagonal block of B, hence a conjugate pair since this block is a real matrix; likewise for β_3 and β_4.

Being an isometry for the norm $\|\cdot\|_m$, the matrix B must preserve the set of extreme points for the norm's unit ball. An easy calculation reveals that B maps extreme point

$$\big(\cos(s+2j\pi/m),\sin(s+2j\pi/m),\cos s,\sin s,0,\dots,0\big)^T$$

to

$$\big(\cos(s-t_1+2\pi/m)\ ,\ \sin(s-t_1+2\pi/m)\ ,\ \cos(s-t_3)\ ,\ \sin(s-t_3)\ ,0,\ldots,0\big)^T,$$

which is an extreme point if and only if $t_1 \in t_3 + \frac{2\pi}{m}\mathbb{Z}$. Hence, taking $B = f(A)$ we must have from (2) that

$$\begin{aligned}\big(f(e^{-is}\omega_m^{-k})\ ,\ f(e^{is}\omega_m^{k})\ ,\ f(e^{-is})\ ,\ f(e^{is})\ ,\ f(1),\ldots,f(1)\big)&\\ = \big(e^{-it_3}\omega_m^{-\ell}\ ,\ e^{it_3}\omega_m^{\ell}\ ,\ e^{-it_3},\ e^{it_3}\ ,\ \pm 1,\ldots,\pm 1\big)&\end{aligned}$$

for suitable integer ℓ which depends on s, k, and m. We conclude that

$$f(e^{is}\omega_m^k)f(e^{-is}) = \omega_m^{\ell(s,k,m)}$$

for some integer $\ell(s,k,m)$. The function $s \mapsto f(e^{is}\omega_m^k)f(e^{-is})$ is clearly continuous, while on the right-hand side, the set $\{\omega_m^{\ell(s,k,m)} : s \in \mathbb{R}\}$ can take only m different values. Hence, $\omega_m^{\ell(s,k,m)}$ is a constant with respect to s.

By changing f to $-f$, we may assume that $f(1) = 1$. Then, for $s \in \mathbb{R}$,

$$f(e^{is}\omega_m^k)f(e^{-is}) = f(e^{0}\omega_m^k)f(e^{-0}) = f(\omega_m^k).$$

Since this holds for every root of unity ω_m^k, we deduce by continuity that

$$f(zw)f(z^{-1}) = f(w);\quad z, w \in \mathbb{T}. \tag{3}$$

With $w = 1$ we get $f(z)f(z^{-1}) = f(1) = 1$, so (3) implies that the continuous function f is multiplicative, hence a character. The result then follows by [8, Example 1.2.7]. □

3 Preservers of other matrix groups

The main results in Section 2 concern maps that preserve all isometries or isometry groups (of surjective isometries). In applications, it turns out that it is much easier to conclude that f has the desired form provided it preserves some special subsets or subgroups of matrices or operators. Here we give two examples which will be verified later (after Theorem 3.6).

Example 3.1. Let $n \geq 4$ and let $\omega_1 = e^{2\pi i\, a_1/b_1}, \omega_2 = e^{2\pi i\, a_2/b_2} \in \mathbb{T}$. Suppose a cyclic subgroup of orthogonal matrices $\mathcal{G}_{\omega_1,\omega_2} \subseteq M_n(\mathbb{R})$ is generated by a matrix $-A_{\omega_1,\omega_2}$ where

$$A_{\omega_1,\omega_2} = \begin{pmatrix} \cos(2\pi a_1/b_1) & \sin(2\pi a_1/b_1) \\ -\sin(2\pi a_1/b_1) & \cos(2\pi a_1/b_1)\end{pmatrix} \oplus \begin{pmatrix} \cos(2\pi a_2/b_2) & \sin(2\pi a_2/b_2) \\ -\sin(2\pi a_2/b_2) & \cos(2\pi a_2/b_2)\end{pmatrix} \oplus I_{n-4}.$$

Then a continuous function $f\colon \mathbb{T} \to \mathbb{T}$, which satisfies $f(\mathcal{G}_{\omega_1,\omega_2}) \subseteq \mathcal{G}_{\omega_1,\omega_2}$ for all integers a_1, a_2, b_1, b_2 with $b_1, b_2 \geq 3$ odd, has the form $f(z) = z^k$ or $f(z) = -z^k$ for some integer k.

We remark that, with $a_j, b_j \in \mathbb{Z}$ and $b_j \geq 3$ odd, $(-A_{\omega_1,\omega_2})^{b_1 b_2} = -I \in \mathcal{G}_{\omega_1,\omega_2}$. Hence, there exists a norm on $\mathbb{R}^n$ such that $\mathcal{G}_{\omega_1,\omega_2}$ is its isometry group (see [2, Theorem 3.1]).

Example 3.2. Let $\mathcal{G}_n \subseteq M_n(\mathbb{R})$ be the isometry group of the ℓ_p-norm for some fixed $p \neq 2$. If a continuous function $f\colon \mathbb{T} \to \mathbb{T}$ satisfies $f(\mathcal{G}_n) \subseteq \mathcal{G}_n$ for infinitely many integers n, then $f(z) = z^k$ or $f(z) = -z^k$ for some integer k.

Instead of doing a case by case study, we will prove a general theorem (Theorem 3.6) that covers many similar cases. We will also illustrate how to use the theorem to verify the above two examples. We continue with two lemmas.

Recall that an m-th root of unity is any unimodular number $\gamma \in \mathbb{T}$ (possibly $\gamma = 1$) with $\gamma^m = 1$; its order, $\operatorname{ord}(\gamma)$, is the smallest integer $a \geq 1$ such that $\gamma^a = 1$. A root of unity γ is trivial if $\gamma = 1$. Let $\mathbb{T}_T \subseteq \mathbb{T}$ be the torsion subgroup of the circle group $\mathbb{T}$, that is, the subgroup consisting of all roots of unity of finite order. Note that the epimorphism $\mathbb{Q} \to \mathbb{T}_T$, defined by $q \mapsto e^{2\pi i q}$, induces a group isomorphism $\mathbb{Q}/\mathbb{Z} \simeq \mathbb{T}_T$. A subset $\mathfrak{F} \subseteq \mathbb{T}_T$ is multiplicative if $\alpha\beta \in \mathfrak{F}$ for every $\alpha, \beta \in \mathfrak{F}$. The next lemma will be needed in slightly less generality than stated.

Lemma 3.3. *Let $\mathfrak{F} \subseteq \mathbb{T}_T$ be a multiplicative set. Then for every two $\omega_1, \omega_2 \in \mathfrak{F}$ there exists $\omega \in \mathfrak{F}$ such that $\omega_j = \omega^{k_j}$ for some integer k_j, $j = 1, 2$.*

Proof. Choose $\omega_1, \omega_2 \in \mathfrak{F} \setminus \{1\}$ and let $b_j = \operatorname{ord}(\omega_j) \geq 2$, $j = 1, 2$ be their orders. Clearly, $\{\omega_j, \omega_j^2, \ldots, \omega_j^{b_j}\}$ consists of b_j different b_j-th roots of unity. Therefore,

$$\{\omega_j, \omega_j^2, \ldots, \omega_j^{b_j}\} = \{e^{\frac{2\pi i}{b_j}}, e^{\frac{4\pi i}{b_j}}, e^{\frac{6\pi i}{b_j}}, \ldots, e^{\frac{2\pi i b_j}{b_j}}\}.$$

Since $\omega_j^{-1} = \omega_j^{b_j - 1}$ and $\mathfrak{F}$ is multiplicative it follows that $e^{\frac{2\pi i\, k}{b_j}} \in \mathfrak{F}$ for every $k \in \mathbb{Z}$.

There exist integers t_1, t_2 such that $t_1 b_1 + t_2 b_2 = \operatorname{GCD}(b_1, b_2)$, the greatest common divisor. Then $e^{\frac{2\pi i\, t_2}{b_1}}, e^{\frac{2\pi i\, t_1}{b_2}} \in \mathfrak{F}$, so their product

$$\omega := e^{\frac{2\pi i\, t_2}{b_1}} \cdot e^{\frac{2\pi i\, t_1}{b_2}} = e^{\frac{2\pi i\, \operatorname{GCD}(b_1, b_2)}{b_1 b_2}} = e^{\frac{2\pi i}{\operatorname{LCM}(b_1, b_2)}} \in \mathfrak{F},$$

where $\operatorname{ord}(\omega) = \operatorname{LCM}(b_1, b_2)$ is the lowest common multiplier of b_1 and b_2. If $\omega_j = e^{2\pi i m_j / b_j}$ then the desired integers are $k_1 = m_1 b_2 / \operatorname{GCD}(b_1, b_2)$, $k_2 = m_2 b_1 / \operatorname{GCD}(b_1, b_2)$. □

Lemma 3.4. *Let $f \colon \mathbb{T} \to \mathbb{T}$ be a continuous function, $\mathfrak{F} \subseteq \mathbb{T}_T$ an infinite subset, and $m \geq 1$ a fixed integer. If for all $\omega \in \mathfrak{F}$ and $a \in \mathbb{Z}$ there exists an integer $\ell = \ell_{\omega,a}$ such that $\big(f(\omega^a)^m, f(\omega)^m\big) = (\omega^{a\ell}, \omega^{\ell})$, then $f(z) = \gamma z^K$ for some integer K and some m-th root of unity γ.*

Proof. Choose $\omega \in \mathfrak{F} \setminus \{1\}$ and let $a \in \mathbb{Z}$. By the assumptions there exist integers ℓ_1, ℓ_2 such that

$$\big(f(1)^m, f(\omega)^m\big) = (1^{\ell_1}, \omega^{\ell_1}) \quad \text{and} \quad \big(f(\omega^a)^m, f(\omega)^m\big) = (\omega^{a\ell_2}, \omega^{\ell_2})$$

where the first equality follows by choosing $a = \operatorname{ord}(\omega)$. Comparing the first components gives $f(1)^m = 1$ and $f(\omega^a)^m = \omega^{a\ell_2}$, while comparing the second components gives $f(\omega)^m = \omega^{\ell_1} = \omega^{\ell_2}$. Then $f(\omega^a)^m = \omega^{a\ell_2} = (f(\omega)^m)^a = (f(\omega)^a)^m$. From here, given $b \in \mathbb{Z}$ and denoting $x = \omega^b$ we have

$$f\big(x^a\big)^m = \big(f(\omega^{ba})\big)^m = \big(f(\omega)\big)^{bam} = \big(f(\omega^b)\big)^{am} = \big(f(x)^a\big)^m; \qquad a \in \mathbb{Z}.$$

Let us define $g(z) = f(z^a)/f(z)^a$. The function g is continuous and $g(x)^m = 1$ for all $x = \omega^b$ with $\omega \in \mathfrak{F}$ and $b \in \mathbb{Z}$. Since $\mathfrak{F}$ is infinite, it has elements of arbitrarily large order. As such $g(x)^m = 1$ on the dense subset $\{x = \omega^b : \omega \in \mathfrak{F},\ b \in \mathbb{Z}\}$. We conclude that $g(z)^m = 1$ for every $z \in \mathbb{T}$. By continuity, g is a constant function. Let $\gamma = f(1)$, then γ is some m-th root of unity. Since $g(1) = \gamma^{1-a}$, we have $g(z) = \gamma^{1-a}$ for every $z \in \mathbb{T}$. Hence $f(z^a) = \gamma^{1-a} f(z)^a$, which implies $\frac{1}{\gamma} f(z^a) = (\frac{1}{\gamma} f(z))^a$ for every $z \in \mathbb{T}$ and $a \in \mathbb{Z}$. By Lemma 3.3 the restriction $(\frac{1}{\gamma} f)|_{\mathbb{T}_T}$ is multiplicative. Since $\mathbb{T}_T$ is dense in $\mathbb{T}$, the continuity of $\frac{1}{\gamma} f$ implies that $\frac{1}{\gamma} f$ is multiplicative. As such, by [8, Example 1.2.7] there exists an integer K with $f(z) = \gamma z^K$ for every $z \in \mathbb{T}$. □

Note that Lemma 3.4 fails if we relax the assumptions and assume only that on some dense subset of $\mathbb{T} \times \mathbb{T}$ we have $\big(f(\omega_1), f(\omega_2)\big) = (\omega_1^\ell, \omega_2^\ell)$ with $\ell = \ell_{\omega_1,\omega_2} \in \mathbb{Z}$.

Example 3.5. By identifying 0 and 1, the map $f\colon [0,1] \to [0,1]$ defined by $x \mapsto x^2$ induces a continuous map $f_*\colon \mathbb{T} \to \mathbb{T}$ defined by

$$f_*(e^{2\pi i x}) = e^{2\pi i x^2}$$

which is clearly not of the form $z \mapsto \gamma z^K$. Nonetheless, we claim that (i) the set of ordered pairs $\Omega = \{(e^{2\pi i\,(a+b\sqrt{2})}, e^{2\pi i\,(a+c\sqrt{3})}) : a,b,c \in \mathbb{Z}\}$ is dense in $\mathbb{T} \times \mathbb{T}$, and (ii) for every ordered pair $(\omega_1,\omega_2) = (e^{2\pi i\,(a+b\sqrt{2})}, e^{2\pi i\,(a+c\sqrt{3})}) \in \Omega$ we have $(f_*(\omega_1), f_*(\omega_2)) = (\omega_1^{2a}, \omega_2^{2a})$.

Namely, it is easily seen that for every prime p, the algebraic integer $x = a + b\sqrt{p} \in \mathbb{Z}[\sqrt{p}]$ satisfies the equation

$$f(x) = x^2 = 2ax + q; \qquad (q = pb^2 - a^2 \in \mathbb{Z}).$$

Hence $f_*(\omega_1) = f_*(e^{2\pi i\,(a+b\sqrt{2})}) = \omega_1^{2a}$ and similarly, $f_*(\omega_2) = \omega_2^{2a}$, proving (ii). As for item (i), note that $1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}$ are linearly independent over $\mathbb{Q}$ so that, by Kronecker's theorem (see Theorem 442 in [3]), for every $x, y \in [0,1]$ and every $\varepsilon > 0$ there exist an integer a and integers b, c with $|a\frac{1}{\sqrt{2}} + b - \frac{x}{\sqrt{2}}|, |a\frac{1}{\sqrt{3}} + c - \frac{y}{\sqrt{3}}| < \frac{\varepsilon}{\sqrt{6}}$. It follows that $|a + b\sqrt{2} - x|, |a + c\sqrt{3} - y| < \varepsilon$ and (i) easily follows by the uniform continuity of $t \mapsto e^{2\pi i t}$.

Below we state the general result based on the previous lemma. Note that $\mathrm{ord}(A)$, the order of a matrix A, is the minimal positive integer k such that $A^k = I$, the identity. If a matrix has finite order then it is diagonalizable and all its eigenvalues are roots of unity.

Theorem 3.6. *Let $m \geq 1$ be an integer. Let $\mathcal{G}$ be a subset of matrices of finite order (real or complex, of possibly different sizes) with the following property: there exists an infinite subset $\mathfrak{F}$ of $\mathbb{T}_T$ such that for all $\omega \in \mathfrak{F}$ and $a \in \mathbb{Z}$ there exists a matrix in $\mathcal{G}$ that has ω and ω^a as eigenvalues. If $f\colon \mathbb{T} \to \mathbb{T}$ is a continuous function satisfying $f(A)^m = A^{k_A}$ for all $A \in \mathcal{G}$ (here $k_A \in \mathbb{Z}$ may depend on A) then $f(z) = \gamma z^K$ for some m-th root of unity γ and some integer K.*

Proof. Choose any $\omega \in \mathfrak{F}$ and $a \in \mathbb{Z}$. By the assumptions, we can find $A \in \mathcal{G}$ such that ω and ω^a are among its eigenvalues. Write $A = S^{-1}\,\mathrm{diag}(\omega^a, \omega, \dots)S \in \mathcal{G}$. Then

$$S^{-1}\,\mathrm{diag}(f(\omega^a)^m, f(\omega)^m, \dots)S = f(A)^m = A^\ell = S^{-1}\,\mathrm{diag}(\omega^{a\ell}, \omega^\ell, \dots)S$$

for some integer ℓ. In particular,

$$\big(f(\omega^a)^m, f(\omega)^m\big) = (\omega^{a\ell}, \omega^\ell). \tag{4}$$

The result then follows from Lemma 3.4. □

To apply this theorem consider a collection of real or complex bounded matrix groups $\mathcal{G}_k$ (of possibly different sizes) indexed by $k \in \Theta$. Recall that a bounded matrix group in $M_n(\mathbb{C})$ is simultaneously similar to a subgroup of unitary matrices while a bounded matrix group in $M_n(\mathbb{R})$ is simultaneously similar to a subgroup of orthogonal matrices ([1, Theorem 3.6]) and as such, every matrix from a bounded group is diagonalizable with spectrum contained in $\mathbb{T}$. Now suppose each group $\mathcal{G}_k$ is invariant under a continuous function $f\colon \mathbb{T} \to \mathbb{T}$. Fix an integer $m \geq 1$ and assume that in every group $\mathcal{G}_k$ we can find a matrix A_k with the following two properties:

(i') The centralizer $C_{\mathcal{G}_k}(A_k)$ is a finite subgroup, generated by A_k and a scalar matrix γI with $\gamma^m = 1$.

(ii') There exists an infinite subset $\mathfrak{F} \subseteq \mathbb{T}_T$ such that for every $\omega \in \mathfrak{F}$ and every $a \in \mathbb{Z}$, both ω and ω^a are eigenvalues of A_k for some index k.

Then, by letting $\mathcal{G} = \{A_k : k \in \Theta\}$ and using $f(A_k) \in C_{\mathcal{G}_k}(A_k)$, it is easy to see that the hypotheses, and hence the conclusion, of Theorem 3.6 hold.

Verification of Examples 3.1 and 3.2. One only needs to show that the collection of matrix groups $\big(\mathcal{G}_{\omega_1,\omega_2}\big)_{\omega_1,\omega_2}$, respectively $\big(\mathcal{G}_n\big)_n$, has the properties (i')–(ii'). To see this, note first that

$$A_{\omega_1,\omega_2} = U \operatorname{diag}(\omega_1^{-1}, \omega_1, \omega_2^{-1}, \omega_2, 1 \ldots, 1)U^*$$

where the unitary U is defined by $\hat{U} = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} i & -i \\ 1 & 1 \end{smallmatrix}\right) \in M_2(\mathbb{C})$ and $U = \hat{U} \oplus \hat{U} \oplus I_{n-4} \in M_n(\mathbb{C})$. Now, item (i') holds trivially because the whole group $\mathcal{G}_{\omega_1,\omega_2}$ is finite and generated by A_{ω_1,ω_2} and by $-I$ of order $m = 2$. For item (ii') one easily sees that the set $\mathfrak{F}$ consists of all roots of unity of odd order, and is clearly infinite.

As for Example 3.2, it is known that $\mathcal{G}_n$ consists of weighted permutational matrices (see, e.g., [9]), with weights ± 1. Let $A_n = \big(a_{ij}\big) \in \mathcal{G}_n$ be the long cycle permutation matrix given by $a_{ij} = 1$ if $j \equiv i + 1 \pmod{n}$ and $a_{ij} = 0$ otherwise, and let $\omega = e^{2\pi i/n}$ be the primitive n-th root of unity. Then A_n has n distinct eigenvalues ω^j, $j = 0, \ldots, n-1$. In particular, A_n is nonderogatory, hence its centralizer inside $\mathcal{G}_n$ coincides with the set $\{\pm A_n^j : 0 \leq j < n\}$. As such, $\mathcal{G}_n$ fulfills the assumptions (i')–(ii') with $\mathfrak{F} = \{1^{1/n} : f(\mathcal{G}_n) \subseteq \mathcal{G}_n\} \subseteq \mathbb{T}_T$ and with $(\gamma, m) = (-1, 2)$. □

4 Relaxation of the continuity assumption

In the following, we consider general (not necessarily continuous) functions f on $\mathbb{T}$ which preserve some matrix groups. Recall that a matrix $A \in M_n(\mathbb{C})$ is a generalized permutation if A has exactly one nonzero entry in each row and column, and each nonzero entry has modulus one. It is known that, for any symmetric norm on $\mathbb{C}^n$ that is not a multiple of the ℓ_2 norm, the isometry group is the group of generalized permutation matrices in $M_n(\mathbb{C})$; see [6]. We have the following result.

Theorem 4.1. *A function $f\colon \mathbb{T} \to \mathbb{T}$ preserves the set of complex generalized permutations (in all finite dimensions) if and only if, for each coset $\alpha\mathbb{T}_T \in \mathbb{T}/\mathbb{T}_T$ there exist $c_\alpha \in \mathbb{T}$ and a homomorphism $h_\alpha\colon \mathbb{T}_T \to \mathbb{T}_T$ such that $f(\alpha w) = c_\alpha h_\alpha(w)$ for all $w \in \mathbb{T}_T$.*

Proof. We prove necessity first. Let ω be a primitive n-th root of unity. Let $P \in M_n(\mathbb{R}) \subseteq M_n(\mathbb{C})$ be the cyclic permutation given by $P_{ij} = 1$ if $j \equiv i+1 \pmod{n}$ and $P_{ij} = 0$ otherwise. Then P has n distinct eigenvalues ω^k, $k = 0, \ldots, n-1$, with corresponding (orthonormal) eigenvectors

$$v_k = \frac{1}{\sqrt{n}}\left(1, \omega^k, \omega^{2k}, \ldots, \omega^{(n-1)k}\right)^T.$$

Then $P = \sum_{k=0}^{n-1} \omega^k v_k v_k^*$; by the functional calculus,

$$f(\alpha P) = \sum_{k=0}^{n-1} f(\alpha\omega^k) v_k v_k^*$$

for all $\alpha \in \mathbb{T}$. The $(1, m)$-entry of $f(\alpha P)$ is

$$\frac{1}{n}\sum_{k=0}^{n-1} f(\alpha\omega^k)\overline{\omega}^{(m-1)k};$$

for $f(\alpha P)$ to be a generalized permutation, exactly $n-1$ of these sums must be zero. Thus the vector

$$x_\alpha = \left(f(\alpha), f(\alpha\omega), \ldots, f(\alpha\omega^{n-1})\right)^T$$

is perpendicular to $n-1$ of the eigenvectors, and hence must be parallel to v_k for some k (which may depend on α and ω) in $\{0, \ldots, n-1\}$; that is,

$$x_\alpha = \sqrt{n} f(\alpha) v_k.$$

Thus, by comparing the $(j+1)$-th entries we get $f(\alpha\omega^j) = f(\alpha)\omega^{jk}$ for all $j = 0, \ldots, n-1$, so $f(\alpha v) = f(\alpha)v^k$ for all n-th roots of unity v, even nonprimitive ones. Let us fix α for the moment and write

$$h(v) = f(\alpha v)/f(\alpha) = v^k, \tag{5}$$

where the integer k depends only on the order of $v \in \mathbb{T}_T$. Choose two roots of unity $w, z \in \mathbb{T}_T$ and let $\rho \in \mathbb{T}_T$ be a primitive m-th root of unity such that $w = \rho^{j_w}$ and $z = \rho^{j_z}$ for some integers j_w, j_z. Then, if k_m is the corresponding exponent for $h(\rho)$ as guaranteed by (5),

$$h(wz) = h(\rho^{j_w + j_z}) = \rho^{(j_w + j_z)k_m} = \rho^{j_w k_m}\rho^{j_z k_m} = h(w)h(z),$$

and the result follows by letting $c_\alpha = f(\alpha)$.

Conversely, to show sufficiency, let A be a generalized permutation in $M_n(\mathbb{C})$. There exists a permutation P so that $P^{-1}AP = \oplus_j A_j$ is a decomposition into generalized cycles $A_j \in M_{n_j}$ (so the (k, l)-entry of A_j is zero unless $l \equiv k+1 \pmod{n_j}$) with spectral decomposition

$$A_j = \sum_{k=1}^{n_j} \alpha_j e^{2\pi i\, k/n_j} P_k \tag{6}$$

for some $\alpha_j \in \mathbb{T}$. Writing c_j for c_{α_j}, h_j for h_{α_j}, and letting $w_j = e^{2\pi i/n_j}$, we have

$$f(\alpha_j w_j^k) = c_j h_j(w_j)^k.$$

Note that $h_j(w_j)$ is an n_j-th root of unity, so $h_j(w_j) = w_j^{K_j}$ for some $K_j \in \mathbb{Z}$. Thus

$$f(A) = P^{-1}\left(\oplus_j f(A_j)\right) P = P^{-1}\left(\oplus_j c_j (A_j/\alpha_j)^{K_j}\right) P$$

is a generalized permutation. □

Note that the spectrum of every real generalized permutation lies in $\mathbb{T}_T$, so minor modifications of the preceding proof lead to the following.

Proposition 4.2. *A function $f\colon \mathbb{T} \to \mathbb{T}$ preserves the set of real generalized permutations (in all finite dimensions) if and only if f or $-f$ is a homomorphism of $\mathbb{T}_T$.*

Proof. Necessity follows by simply taking $\alpha = 1$ in the proof of Theorem 4.1 (and using $c_1 = f(1) = \pm 1$ which follows from $f(I) = f(1)I = \pm I$). We show sufficiency in the same manner as the proof of Theorem 4.1, except that A_j in (6) is a real generalized cycle; note that the eigenvalues of such cycles are either the n_j-th roots of 1, or the n_j-th roots of -1. Without loss of generality assume f is a homomorphism of $\mathbb{T}_\mathbb{T}$. In the first case (spectrum of A_j is the set of n_j-th roots of 1), the proof of Theorem 4.1 (take $\alpha_j = c_j = 1$) shows that $f(A_j) = A_j^{K_j}$ for some integer K_j. For the second case (spectrum of A_j is the set of n_j-th roots of -1), let $\nu_j = e^{\pi i/n_j}$ and note that $f(\nu_j) = \nu_j^{K_j}$ for some integer K_j. Then

$$f(A_j) = \sum_{k=1}^{n_j} f(\nu_j^{1+2k}) P_k = \sum_{k=1}^{n_j} \nu_j^{K_j(1+2k)} P_k = A_j^{K_j},$$

and the result follows as in the preceding proof. □

The character (i.e., the dual) group of a discrete group $\mathbb{T}_T$ is complicated; by [8, Theorem 1.2.5] and Pontryagin duality [8, Theorem 1.7.2] it is known to be compact but not discrete and as such contains the characters $z \mapsto z^k$, $k \in \mathbb{Z}$ as a proper subgroup. However, if we require that the functions be continuous in the standard topology, then they have a simple familiar form.

Corollary 4.3. *Suppose $f\colon \mathbb{T} \to \mathbb{T}$ is continuous. Then f preserves the set of all complex (respectively, real) generalized permutations in all finite dimensions if and only if $f(z) = cz^k$ for some integer k and $c \in \mathbb{T}$ (respectively, $c \in \{1, -1\}$).*

Proof. Without loss of generality we may suppose $f(1) = 1$. By Theorem 4.1 (respectively, Proposition 4.2), $f(wz) = f(w)f(z)$ for all $w, z \in \mathbb{T}_T$. By the continuity of f, $f(wz) = f(w)f(z)$ for all $w, z \in \mathbb{T}$. Then $f(z) = z^k$ for some integer k, as required. □

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