Mappings preserving spectra of products of matrices [1]

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Dedicated to Professor Ahmed Sourour on the occasion of his sixtieth birthday.

Abstract

Let M_n be the set of $n \times n$ complex matrices, and for every $A \in M_n$, let $\mathrm{Sp}\,(A)$ denote the spectrum of A. For various types of products $A_1 * \cdots * A_k$ on M_n, it is shown that a mapping $\phi : M_n \to M_n$ satisfying $\mathrm{Sp}\,(A_1 * \cdots * A_k) = \mathrm{Sp}\,(\phi(A_1) * \cdots * \phi(A_k))$ for all $A_1, \ldots, A_k \in M_n$ has the form

$$X \mapsto \xi S^{-1} X S \quad \text{or} \quad A \mapsto \xi S^{-1} X^t S$$

for some invertible $S \in M_n$ and scalar ξ. The result covers the special cases of the usual product $A_1 * \cdots * A_k = A_1 \cdots A_k$, the Jordan triple product $A_1 * A_2 = A_1 * A_2 * A_1$, and the Jordan product $A_1 * A_2 = (A_1A_2 + A_2A_1)/2$. Similar results are obtained for Hermitian matrices.

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1 Introduction

Let M_n be the set of all $n \times n$ complex matrices. In [5], Marcus and Moyls proved that if a linear mapping $\phi : M_n \to M_n$ preserves the eigenvalues (counting multiplicities) of each matrix in M_n, then there exists an invertible matrix S such that ϕ has the form

$$A \mapsto S^{-1}AS \qquad \text{or} \qquad A \mapsto S^{-1}A^tS,$$

where A^t denotes the transpose of A. The assumption on multiplicity is not really necessary. Let $\mathrm{Sp}\,(A)$ denote the spectrum of A, i.e., the set of all eigenvalues of A without counting multiplicities. Then by a result of Jafarian and Sourour [3], the above conclusion holds if $\mathrm{Sp}\,(\phi(A)) = \mathrm{Sp}\,(A)$.

The result has been generalized in different directions. For example, in [8], Omladič and P. Šemrl considered spectrum preserving mappings that are just additive. In [6] Molnár studied surjective maps ϕ on bounded linear operators such that

$$\mathrm{Sp}\,(\phi(A)\phi(B)) = \mathrm{Sp}\,(AB) \qquad \text{for all linear operators } A, B. \tag{1.1}$$

In particular, such a map on M_n has the form

$$A \mapsto \xi S^{-1}AS \qquad \text{or} \qquad A \mapsto \xi S^{-1}A^tS \tag{1.2}$$

for some invertible matrix S and $\xi \in \{1, -1\}$. Continuous differentiable maps on M_n preserving spectrum was characterized in [1].

In this paper, we consider different types of products $A * B$ on M_n including the usual product $A*B = AB$, the Jordan triple product $A*B = ABA$, and the Jordan product $A*B = (AB+BA)/2$. We obtain a general result, which implies that a mapping $\phi : M_n \to M_n$ satisfying

$$\mathrm{Sp}\,(A * B) = \mathrm{Sp}\,(\phi(A) * \phi(B)) \quad \text{for all } A, B \in M_n \tag{1.3}$$

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has the form (1.2) for some invertible $S \in M_n$ and scalar ξ. As we do not require the surjective assumption on ϕ, our result refines that of Molnár in the finite dimensional case.

Note that a characterization of those $\phi : M_n \to M_n$ such that AB and $\phi(A)\phi(B)$ have the same eigenvalues counting multiplicities is given in [7]. A crucial observation is the following proposition. We include the proof for the sake of completeness.

Proposition 1.1 *Suppose $\phi : M_n \to M_n$ satisfies*

$$\mathrm{tr}\,(AB) = \mathrm{tr}\,(\phi(A)\phi(B)) \qquad \textit{for all } A, B \in \mathcal{M}.$$

Then ϕ is an invertible linear map.

Proof. For every $X = (x_{ij}) \in M_n$, let R_X be the n^2 row vector

$$R_X = (x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{n1}, \ldots, x_{nn}),$$

and C_X the n^2 column vector

$$C_X = (x_{11}, x_{21} \ldots, x_{n1}, x_{12}, \ldots, x_{n2}, \ldots, x_{1n}, \ldots, x_{nn})^t,$$

Then for any $X, Y \in M_n$,

$$R_{\phi(X)}C_{\phi(Y)} = \mathrm{tr}\,\phi(X)\phi(Y) = \mathrm{tr}\,XY = R_X C_Y. \tag{1.4}$$

Let $\{Y_1, \ldots, Y_{n^2}\}$ be a basis for M_n. Let $\mathcal{Y}$ have columns $C_{Y_1}, \ldots C_{Y_{n^2}}$, and $\mathcal{Z} \in M_{n^2}$ have columns $C_{\phi(Y_1)}, \ldots, C_{\phi(Y_{n^2})}$. Then by (1.4), for any $X \in M_n$ we have

$$R_{\phi(X)}\mathcal{Z} = R_X\mathcal{Y}.$$

Next, we show that $\mathcal{Z}$ is invertible. To this end, let $\{X_1, \ldots, X_{n^2}\}$ be a basis for M_n, $\mathcal{X} \in M_{n^2}$ with rows $R_{X_1}, \ldots R_{X_{n^2}}$, and $\mathcal{W} \in M_{n^2}$ with rows $R_{\phi(X_1)}, \ldots, R_{\phi(X_{n^2})}$. Then $\mathcal{W}\mathcal{Z} = \mathcal{X}\mathcal{Y}$ for the invertible matrices $\mathcal{X}$ and $\mathcal{Y}$. So, $\mathcal{Z}$ is invertible, and for any $X \in M_n$

$$R_{\phi(X)} = R_X\mathcal{Y}\mathcal{Z}^{-1}.$$

Hence, ϕ is an invertible linear map. ∎

The problem of characterizing mappings that preserve the spectra of the product of matrices is more challenging. Our results will give characterization of mappings preserving the spectrum of various products of k matrices $X_1 * \cdots * X_k$ defined as follows.

Let $k \geq 2$, and a sequence $(j_1, \ldots, j_m)$ be given so that $\{j_1, \ldots, j_m\} = \{1, \ldots, k\}$. We consider products of the form

$$X_1 * \cdots * X_k = X_{j_1} \cdots X_{j_m},$$

which cover the usual product $A * B = AB$ and the Jordan triple product $A * B = ABA$. We also consider products of the form

$$X_1 * \cdots * X_k = (X_{j_1} \cdots X_{j_m} + X_{j_m} \cdots X_{j_1})/2,$$

which cover the Jordan product $A * B = (AB + BA)/2$.

In Section 2, we obtain the results on the set M_n of $n \times n$ complex matrices. Using a transfer principle in model theoretic algebra (see [2]), one sees that the results also hold for square matrices over an algebraically closed field. In Section 3, similar results are proved for the set H_n of $n \times n$ complex Hermitian matrices. The same results and proofs are valid for $n \times n$ real symmetric matrices as well.

2 Results on complex matrices

Theorem 2.1 *Suppose $k \geq 2$, and a sequence $(j_1, \ldots, j_m)$ is given so that $\{j_1, \ldots, j_m\} = \{1, \ldots, k\}$ and there is j_r not equal to j_s for all $s \neq r$. Consider*

$$X_1 * \cdots * X_k = X_{j_1} \cdots X_{j_m}.$$

Then a mapping $\phi : M_n \to M_n$ satisfies

$$\mathrm{Sp}\,(\phi(X_1) * \cdots * \phi(X_k)) = \mathrm{Sp}\,(X_1 * \cdots * X_k) \qquad \text{for all } X_1, \ldots, X_k \in M_n \tag{2.1}$$

if and only if there exist an invertible matrix $S \in M_n$ and a scalar ξ satisfying $\xi^m = 1$ such that

(a) *ϕ has the form $A \mapsto \xi S^{-1}AS$, or*

(b) *$(j_{r+1}, \ldots, j_m, j_1, \ldots, j_{r-1}) = (j_{r-1}, \ldots, j_1, j_m, \ldots, j_{r+1})$ and ϕ has the form $A \mapsto \xi S^{-1}A^tS$.*

Note that the assumption that there is $j_r \notin \{j_1, \ldots, j_{r-1}, j_{r+1}, \ldots, j_m\}$ is necessary. For example, if $A * B = ABBA$, then mappings ϕ satisfying $\mathrm{Sp}\,(\phi(A) * \phi(B)) = \mathrm{Sp}\,(A * B)$ may not have nice structure. For instance, ϕ can send all involutions, i.e., those matrices $X \in M_n$ such that $X^2 = I_n$, to a fixed involution, and $\phi(X) = X$ for other X.

Proof of Theorem 2.1. It is clear that if (a) or (b) holds, then ϕ satisfies (2.1). We need only prove the necessity part. We divide the proof of it into several assertions. Since $\mathrm{Sp}\,(X_{j_1} \cdots X_{j_m}) = \mathrm{Sp}\,(X_{j_r} \cdots X_{j_m} X_{j_1} \cdots X_{j_{r-1}})$, we may assume that $j_1 \notin \{j_2, \ldots, j_m\}$. Define

$$\mathcal{S} = \{X \in M_n : X \text{ has } n \text{ distinct eigenvalues}\}.$$

Assertion 1 *For every $A \in \mathcal{S}$, there is a neighborhood of $\mathcal{N}_A$ such that the restriction of ϕ on $\mathcal{N}_A$ equals an invertible linear map L_A.*

Proof. For every $A \in \mathcal{S}$, $\mathrm{Sp}\,(AI_n^{m-1})$ has n distinct elements. By the continuity of the eigenvalues, there are neighborhoods $\mathcal{N}_{I_n}$ of I_n and $\mathcal{N}_A$ of A such that XY^{m-1} has n distinct eigenvalues for every $X \in \mathcal{N}_A$ and $Y \in \mathcal{N}_{I_n}$. By (2.1), $\phi(X)\phi(Y)^{m-1}$ has n distinct eigenvalues equal to those of XY^{m-1}. Hence

$$\mathrm{tr}\,\phi(X)\phi(Y)^{m-1} = \mathrm{tr}\,XY^{m-1} \quad \text{for every } X \in \mathcal{N}_A \text{ and } Y \in \mathcal{N}_{I_n}. \tag{2.2}$$

As in the proof of Proposition 1.1, for every $X = (x_{ij}) \in M_n$, let R_X be the n^2 row vector

$$R_X = (x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{n1}, \ldots, x_{nn}),$$

and C_X the n^2 column vector

$$C_X = (x_{11}, x_{21} \ldots, x_{n1}, x_{12}, \ldots, x_{n2}, \ldots, x_{1n}, \ldots, x_{nn})^t,$$

Then

$$R_{\phi(X)}C_{\phi(Y)^{m-1}} = \mathrm{tr}\,\phi(X)\phi(Y)^{m-1} = \mathrm{tr}\,XY^{m-1} = R_X C_{Y^{m-1}} \tag{2.3}$$

for every $X \in \mathcal{N}_A$ and $Y \in \mathcal{N}_{I_n}$.

Now, suppose $\operatorname{tr} XZ = 0$ for each $Z \in \{Y^{m-1} : Y \in \mathcal{N}_{I_n}\}$. Then for any $R \in M_n$,

$$\operatorname{tr} X(I+tR)^{m-1} = \sum_{j=0}^{m} t^j \binom{m-1}{j} \operatorname{tr} XR^j = 0$$

for sufficiently small $t > 0$. We have $\operatorname{tr} XR = 0$. It follows that $X = 0$. So, $\{Y^{m-1} : Y \in \mathcal{N}_{I_n}\}$ is a spanning set of M_n, and contains a basis $\{Y_j^{m-1} : 1 \leq j \leq n^2\}$ for M_n with $Y_j \in \mathcal{N}_{I_n}$ for each $j = 1, \ldots, n^2$. Let $\mathcal{Y}$ and $\mathcal{Z}$ be the $n^2 \times n^2$ matrices with columns $C_{Y_1^{m-1}}, \ldots, C_{Y_{n^2}^{m-1}}$ and $C_{\phi(Y_1)^{m-1}}, \ldots, C_{\phi(Y_{n^2})^{m-1}}$ respectively. By (2.2) and (2.3),

$$R_{\phi(X)}\mathcal{Z} = R_X \mathcal{Y} \quad \text{for every } X \in \mathcal{N}_A.$$

We claim that the matrix $\mathcal{Z}$ is invertible. To this end take a basis $\{X_1, \ldots, X_{n^2}\}$ of M_n in $\mathcal{N}_A$ and let $\mathcal{X}$ and $\mathcal{W}$ be the $n^2 \times n^2$ matrices with rows $R_{X_1}, \ldots, R_{X_{n^2}}$ and $R_{\phi(X_1)}, \ldots, R_{\phi(X_{n^2})}$ respectively. Then $\mathcal{W}\mathcal{Z} = \mathcal{X}\mathcal{Y}$ for invertible matrices $\mathcal{X}$ and $\mathcal{Y}$. It follows that $\mathcal{Z}$ is invertible, and

$$R_{\phi(X)} = R_X \mathcal{X}\mathcal{Z}^{-1} \quad \text{for every } X \in \mathcal{N}_A.$$

Hence the restriction of ϕ to $\mathcal{N}_A$ is some invertible linear mapping L_A. The proof of Assertion 1 is complete.

Assertion 2 *All the linear maps L_A in Assertion 1 are the same, i.e., ϕ is equal to an invertible linear mapping L on the dense subset $\mathcal{S}$.*

Proof. Note that for any $A, B \in \mathcal{S}$, there is a continuous curve $f : [0,1] \to \mathcal{S}$ such that $f(0) = A$ and $f(1) = B$. Consider the set

$$\mathcal{C} = \{t \in [0,1] : \phi = L_A \text{ on an open neighborhood of } f(t)\}.$$

Then clearly $\mathcal{C}$ is an open subset of $[0,1]$. But $\mathcal{C}$ is also closed in $[0,1]$. Let $t_0 \in \mathcal{C}^-$. There is an open neighborhood $\mathcal{N}_{f(t_0)}$ of $f(t_0)$ on which ϕ is equal to the linear mapping $L_{f(t_0)}$. Take $t \in f^{-1}(\mathcal{N}_{f(t_0)}) \cap \mathcal{C}$. Then on some open neighborhood $\mathcal{N}_{f(t)}$ of $f(t)$, $\phi = L_A$. On the non-empty open set $\mathcal{N}_{f(t_0)} \cap \mathcal{N}_{f(t)}$, $L_{f(t_0)} = \phi = L_A$. Hence $L_{f(t_0)} = L_A$, and $t_0 \in \mathcal{C}$. We conclude that $\mathcal{C} = [0,1]$, and $L_A = L_B$. The proof of the Assertion 2 is complete.

Assertion 3 *The mapping L in Assertion 2 has the form $A \mapsto \xi S^{-1}AS$ or $A \mapsto \xi S^{-1}A^tS$ for some invertible $S \in M_n$ and $\xi \in \mathbb{C}$ with $\xi^m = 1$. Moreover, if the latter case holds, then $(j_2, \ldots, j_m) = (j_m, \ldots, j_2)$.*

Proof. By the continuity of L and the spectrum, we have that

$$\operatorname{Sp}(L(X_1) * \cdots * L(X_k)) = \operatorname{Sp}(X_1 * \cdots * X_k)$$

for all $X_1, \ldots, X_k \in M_n$. If A is invertible, then

$$0 \notin \operatorname{Sp}(A * \cdots * A) = \operatorname{Sp}(L(A) * \cdots * L(A)),$$

and hence $L(A)$ is also invertible. It follows that L is nonsingular and preserves invertible matrices. By [5], there are invertible matrices M, N such that L has the form

$$A \mapsto MAN \qquad \text{or} \qquad A \mapsto MA^tN. \tag{2.4}$$

We claim that NM is a scalar matrix. Otherwise, there exists an invertible $R \in M_n$ such that $RNMR^{-1}$ is a direct sum of companion matrices so that its second row has the form $(1, 0, \dots, *)$. Let $A = R^{-1}E_{12}R$ or $A^t = R^{-1}E_{12}R$ depending on L has the first or the second form in (2.4), where E_{12} is the $n \times n$ matrix with 1 at $(1, 2)$ position and 0 everywhere else. Then $\mathrm{Sp}\,(A^m) = \mathrm{Sp}\,(A) = \{0\}$. Now

$$\mathrm{Sp}\,(L(A)) = \mathrm{Sp}\,(M(R^{-1}E_{12}R)N) = \mathrm{Sp}\,(E_{12}RNMR^{-1}).$$

It follows that $1 \in \mathrm{Sp}\,(L(A))$ and hence $1 \in \mathrm{Sp}\,(L(A)^m)$ where as $\mathrm{Sp}\,(A^m) = \{0\}$, which contradicts (2.1).

We have proved that L has the form $A \mapsto \xi S^{-1}AS$ or $A \mapsto \xi S^{-1}A^tS$ for some ξ. Since $\{\xi^m\} = \mathrm{Sp}\,(L(I_n)^m) = \mathrm{Sp}\,(I_n^m) = \{1\}$, $\xi^m = 1$.

Now, suppose L has the form $A \mapsto \xi S^{-1}A^tS$. Replacing L by the mapping $A \mapsto \bar{\xi}SL(A)S^{-1}$, we may assume that $L(A) = A^t$ for all $A \in M_n$. Then

$$\begin{aligned} & \mathrm{Sp}\,(X_{j_1} \cdots X_{j_m}) = \mathrm{Sp}\,(X_1 * \cdots * X_k) = \mathrm{Sp}\,(L(X_1) * \cdots * L(X_k)) \\ = \ & \mathrm{Sp}\,(X_{j_1}^t \cdots X_{j_m}^t) = \mathrm{Sp}\,(X_{j_m} \cdots X_{j_1}) = \mathrm{Sp}\,(X_{j_1}X_{j_m} \cdots X_{j_2}) \end{aligned}$$

for any $X_1, \dots, X_k \in M_n$. We have to show that $(j_2, \dots, j_m) = (j_m, \dots, j_2)$. Suppose it is not true. Let $l \geq 2$ be the smallest integer such that $j_l \neq j_{m+2-l}$. Then $l \leq (m+1)/2$. Let $A_{j_l} = A = \mathrm{diag}(\lambda, 1, \dots, 1)$, and for every $k \notin \{1, j_l\}$, let $A_k = B = B_1 \oplus I_{n-2}$, where $B_1 \in M_2$ is a symmetric invertible matrix with positive entries. Then

$$A_{j_2} \cdots A_{j_m} = RA^{r_1}B^{s_1}A^{r_2}B^{s_2} \cdots A^{r_t}B^{s_t}R^t$$

for positive integers $r_i,\ s_i$, where $R = A_{j_2} \cdots A_{j_{l-1}}$. Note that

$$A^{r_i}B^{s_i} = \begin{pmatrix} \lambda^{r_i}b_{11}^{(s_i)} & \lambda^{r_i}b_{12}^{(s_i)} \\ b_{21}^{(s_i)} & b_{22}^{(s_i)} \end{pmatrix} \oplus I_{n-2},$$

for positive numbers $b_{11}^{(s_i)}, b_{12}^{(s_i)}, b_{21}^{(s_i)}$ and $b_{22}^{(s_1)}$. An induction argument shows that the $(1, 2)$ entry of $A_{j_l} \cdots A_{j_{m-l+2}}$ is a polynomial of degree $r_1 + \cdots + r_t$ in λ. Similarly, the $(1, 2)$ entry of $A_{j_{m-l+2}} \cdots A_{j_l}$, is a polynomial of degree $r_2 + \cdots + r_t$. So, there is $\lambda > 0$ such that

$$A_{j_l} \cdots A_{j_{m-l+2}} \neq A_{j_{m-l+2}} \cdots A_{j_l}.$$

It follows that

$$A_{j_2} \cdots A_{j_m} = RA_{j_l} \cdots A_{j_{m-l+2}}R^t \neq RA_{j_{m-l+2}} \cdots A_{j_l}R^t = A_{j_m} \cdots A_{j_2}.$$

Note that if $X \in M_n$ is a rank one idempotent matrix, and $\mathrm{Sp}\,(A) = \mathrm{Sp}\,(BX)$, then $\mathrm{tr}\,(AX) = \mathrm{tr}\,(BX)$. Moreover, if $\mathrm{tr}\,(AX) = \mathrm{tr}\,(BX)$ for all rank one idempotent $X \in M_n$, then $A = B$. By these facts, we see that there exists a rank one idempotent A_1 such that

$$\mathrm{Sp}\,(A_{j_m} \cdots A_{j_2}A_{j_1}) \neq \mathrm{Sp}\,(A_{j_2} \dots, A_{j_m}A_{j_1}) = \mathrm{Sp}\,(A_{j_1}A_{j_2} \cdots A_{j_m}),$$

which is a contradiction. Hence, $(j_2, \dots, j_m) = (j_m, \dots, j_2)$ as asserted.

Assertion 4 *The mapping ϕ equals the invertible linear mapping L in Assertion 3.*

Proof. From (2.1), and the continuity of L and the spectrum, we have

$$\mathrm{Sp}\,(L(A)L(B)^{m-1}) = \mathrm{Sp}\,(AB^{m-1}) = \mathrm{Sp}\,(\phi(A)L(B)^{m-1}) \quad \text{for every } A, B \in M_n.$$

Since L is surjective,

$$\mathrm{Sp}\,(\phi(A)C^{m-1}) = \mathrm{Sp}\,(L(A)C^{m-1}) \qquad \text{for every } A, C \in M_n. \tag{2.5}$$

Let $A \in M_n$. If $C \in M_n$ is a rank one idempotent, $\phi(A)C^{m-1}$ has at most one nonzero eigenvalue, which is given by $\mathrm{tr}\,\phi(A)C$. The same is true for $L(A)C$. By (2.5),

$$\mathrm{tr}\,\phi(A)C = \mathrm{tr}\,L(A)C \qquad \text{for every rank one idempotent matrix } C \in M_n.$$

It follows that $\phi(A) = L(A)$ for all $A \in M_n$. The proof of Assertion 4 is complete.

By Assertions 1–4, the theorem follows. ∎

Theorem 2.2 *Suppose $k \geq 2$, and $X_1 * \cdots * X_k = X_{j_1} \cdots X_{j_m} + X_{j_m} \cdots X_{j_1}$ for a given sequence $(j_1, \ldots, j_m)$ so that $\{j_1, \ldots, j_m\} = \{1, \ldots, k\}$ and there exists j_r not equal to j_s for all $s \neq r$. Then a mapping $\phi : M_n \to M_n$ satisfies*

$$\mathrm{Sp}\,(\phi(X_1) * \cdots * \phi(X_k)) = \mathrm{Sp}\,(X_1 * \cdots * X_k) \qquad \textit{for all } X_1, \ldots, X_k \in M_n \tag{2.6}$$

if and only if there exist an invertible matrix $S \in M_n$ and a scalar ξ satisfying $\xi^m = 1$ such that ϕ has the form

$$A \mapsto \xi S^{-1}AS \qquad \textit{or} \qquad A \mapsto \xi S^{-1}A^tS.$$

Proof. The necessity of the result is clear. We consider the sufficiency part. Using similar arguments as in the proof of Theorem 2.1 (cf. Assertions 1 and 2), we can prove that ϕ is equal to a bijective linear mapping L on the dense subset

$$\mathcal{S} = \{X \in M_n : X \text{ has } n \text{ distinct eigenvalues }\}.$$

By continuity of L and the spectrum, we see that

$$\mathrm{Sp}\,(L(X_1) * \cdots * L(X_k)) = \mathrm{Sp}\,(X_1 * \cdots * X_k)$$

for all $X_1, \ldots, X_k \in M_n$. Thus, $\mathrm{Sp}\,(L(A)^m) = \mathrm{Sp}\,(A^m)$ for all $A \in M_n$. Using the argument in Assertion 3 in the proof of Theorem 2.1, we see that L has the form $A \mapsto \xi S^{-1}AS$ or $A \mapsto \xi S^{-1}A^tS$. Now, replace ϕ and L by the mappings $A \mapsto \bar{\xi}S\phi(A)S^{-1}$ and $A \mapsto \bar{\xi}SL(A)S^{-1}$, respectively, we may assume that $L(A) = A$ for all $A \in M_n$.

We will show that $\phi = L$ on M_n. From (2.2), we have

$$\begin{aligned}
& \mathrm{Sp}\,(X^{r-1}L(Y)X^{m-r} + X^{m-r}L(Y)X^{r-1}) \\
= \; & \mathrm{Sp}\,(L(X)^{r-1}L(Y)L(X)^{m-r} + L(X)^{m-r}L(Y)L(X)^{r-1}) \\
= \; & \mathrm{Sp}\,(X^{r-1}YX^{m-r} + X^{m-r}YX^{r-1}) \\
= \; & \mathrm{Sp}\,(\phi(X)^{r-1}\phi(Y)\phi(X)^{m-r} + \phi(X)^{m-r}\phi(Y)\phi(X)^{r-1}) \\
= \; & \mathrm{Sp}\,(L(X)^{r-1}\phi(Y)L(X)^{m-r} + L(X)^{m-r}\phi(Y)L(X)^{r-1}) \\
= \; & \mathrm{Sp}\,(X^{r-1}\phi(Y)X^{m-r} + X^{m-r}\phi(Y)X^{r-1})
\end{aligned}$$

for every $X \in \mathcal{S}$ and $Y \in M_n$. Since the set of such matrices X is dense in M_n, by continuity of the spectrum, we see that

$$\mathrm{Sp}\,(X^{r-1}\phi(Y)X^{m-r} + X^{m-r}\phi(Y)X^{r-1}) = \mathrm{Sp}\,(X^{r-1}L(Y)X^{m-r} + X^{m-r}L(Y)X^{r-1}) \tag{2.7}$$

for any $X, Y \in M_n$. It remains to prove the following.

Assertion *Let $A, B \in M_n$. Then $A = B$ if*

$$\mathrm{Sp}\,(X^{r-1}AX^{m-r} + X^{m-r}AX^{r-1}) = \mathrm{Sp}\,(X^{r-1}BX^{m-r} + X^{m-r}BX^{r-1}) \quad \textit{for every} \quad X \in M_n.$$

Proof. If both $r-1$ and $m-r$ are positive, then for any rank one idempotent $X \in M_n$ we have

$$\mathrm{Sp}\,(2AX) = \mathrm{Sp}\,(XAX + XAX) = \mathrm{Sp}\,(X^{r-1}AX^{m-r} + X^{m-r}AX^{r-1})$$

$$= \mathrm{Sp}\,(X^{r-1}BX^{m-r} + X^{m-r}BX^{r-1}) = \mathrm{Sp}\,(XBX + XBX) = \mathrm{Sp}\,(2BX).$$

Since AX and BX have the same spectrum and have rank at most one, we see that

$$\mathrm{tr}\,(AX) = \mathrm{tr}\,(BX).$$

It follows that $A = B$.

Suppose $r-1$ or $m-r$ is zero. Then

$$\mathrm{Sp}\,(AX + XA) = \mathrm{Sp}\,(BX + XB)$$

for all $X \in \{Z^{m-1} : Z \in M_n\}$, which is a dense set in M_n. By continuity of the spectrum, we may assume that the above equality is true for all $X \in M_n$. We shall assume without loss of generality that A is upper triangular. We claim that B is also upper triangular. Suppose $A = (a_{ij})$, $B = (b_{ij})$, and for every $t \in \mathbb{C}$, let $X_t = E_{11} + tE_{12} + \cdots + t^{n-1}E_{1n}$. Then only the first row of $AX_t + X_tA$ is nonzero and equals $(2a_{11} \;\; * \;\; \cdots \;\; *)$.

Hence $\mathrm{Sp}\,(AX_t + X_tA) = \{2a_{11}, 0\}$. As $\mathrm{Sp}\,(BX_t + X_tB) = \mathrm{Sp}\,(AX_t + X_tA) = \{2a_{11}, 0\}$, $BX_t + X_tB$ has eigenvalues $2a_{11}$ and 0 with certain multiplicities. So,

$$\mathrm{tr}\,(BX_t + X_tB) \in \{2a_{11}, \ldots, 2(n-1)a_{11}\}.$$

Now

$$BX_t + X_tB = \begin{pmatrix} b_{11} & b_{11}t & \cdots & b_{11}t^{n-1} \\ b_{21} & b_{21}t & \cdots & b_{21}t^{n-1} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n1}t & \cdots & b_{n1}t^{n-1} \end{pmatrix} + \begin{pmatrix} b_n + b_{21}t + \cdots + b_{n1}t^{n-1} & * & \cdots & * \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

has diagonal entries

$$2b_{11} + b_{21}t + \cdots + b_{n1}t^{n-1}, b_{21}t, \ldots, b_{n1}t^{n-1},$$

and hence

$$\mathrm{tr}\,(BX_t + X_tB) = 2b_{11} + 2b_{21}t + \cdots + 2b_{n1}t^{n-1}$$

is a polynomial in t. It cannot take on a finite number of values only unless it is a constant. The coefficients, except the constant term, are all zero. Hence $b_{21} = \cdots = b_{n1} = 0$. Similarly, by considering $X_t = E_{ii} + tE_{i,i+1} + \cdots + t^{n-i}E_{in}$, we get $b_{i+1,i} = \cdots = b_{ni} = 0$ for $i = 2, \ldots, n-1$. The matrix B is upper triangular.

To show that $A = B$, we first obtain, by putting $X = E_{ii}$, $a_{ii} = b_{ii}$ for every i. For $i < j$, we have

$$AE_{ji} + E_{ji}A = \left(\begin{array}{ccccccc} & & a_{1j} & & & & \\ & & \vdots & & & & \\ & & a_{ij} & & & & \\ & & \vdots & & & & \\ 0 & \cdots & a_{ii} + a_{jj} & \cdots & a_{ij} & \cdots & a_{in} \\ & & 0 & & & & \\ & & \vdots & & & & \\ & & 0 & & & & \end{array}\right).$$

Expanding along the jth column, say, we get

$$\det(AE_{ji} + E_{ji}A - \lambda I_n) = (-\lambda)^{n-2}(a_{ij} - \lambda)^2.$$

Hence $\mathrm{Sp}\,(AE_{ji} + E_{ji}A) = \{a_{ij}, 0\}$. Note that $\mathrm{Sp}\,(BE_{ji} + E_{ji}B) = \{b_{ij}, 0\}$. So, $a_{ij} = b_{ij}$. ∎

3 Results on Hermitian matrices

In this section, we study mappings on H_n that have similar preserving properties as in Section 2. First, we consider products of the form $X_1 * \cdots * X_k = X_{j_1} \cdots X_{j_m}$ such that one of the j_r appears only once in $(j_1, \ldots, j_m)$. Even though H_n may not be closed under this product, mappings that preserve the spectrum of the product are in nice form. If we insist that $X_1 * \cdots * X_m \in H_n$, then m is odd, and $r = (m+1)/2$ is the only possible value for j_r to appear once; in particular, $A * B = A^k B A^k$ is the only product we can define on two matrices.

Theorem 3.1 *Suppose $k \geq 2$, and $X_1 * \cdots * X_k = X_{j_1} \cdots X_{j_m}$ for a given sequence $(j_1, \ldots, j_m)$ so that $\{j_1, \ldots, j_m\} = \{1, \ldots, k\}$ and there exists j_r not equal to j_s for all $s \neq r$. Then a mapping $\phi : H_n \to H_n$ satisfies*

$$\mathrm{Sp}\,(\phi(X_1) * \cdots * \phi(X_k)) = \mathrm{Sp}\,(X_1 * \cdots * X_k) \qquad \text{for all } X_1, \ldots, X_k \in H_n \tag{3.1}$$

if and only if there exist a unitary matrix $S \in M_n$ and a scalar ξ satisfying $\xi^m = 1$ such that

(a) *ϕ has the form $A \mapsto \xi S^* A S$, or*

(b) *$(j_1, \ldots, j_{r-1}, j_{r+1}, \ldots, j_m) = (j_{r+1}, \ldots, j_m, j_1, \ldots, j_{r-1})$ and ϕ has the form $A \mapsto \xi S^* A^t S$.*

Proof. Again, we only need to consider the sufficiency part. Using similar arguments as in the proof of Theorem 2.1 (cf. Assertions 1 and 2), we can prove that ϕ is equal to a bijective (real) linear mapping L on the dense subset

$$\mathcal{S} = \{X \in H_n : X \text{ has } n \text{ distinct eigenvalues }\},$$

and that L preserves the invertible matrices in H_n. By [4, Theorem 6], there is an invertible matrix $S \in M_n$ such that L is of the form

$$A \mapsto \pm S^* A S \qquad \text{or} \qquad A \mapsto \pm S^* A^t S.$$

From the observations

$$\{1\} = \mathrm{Sp}\,(I_n^m) = \mathrm{Sp}\,(L(I_n)^m) = \mathrm{Sp}\,((\pm S^* S)^m)$$

and that S^*S is positive definite, we conclude that $S^*S = I_n$, i.e., S is unitary. Hence L has the asserted forms.

Also, we can show that $(j_{r+1}, \ldots, j_m, j_1, \ldots, j_{r-1}) = (j_m, \ldots, j_{r+1}, j_{r-1}, \ldots, j_1)$ if L has the form $A \mapsto \xi S^* A^t S$ with the help of the following fact.

Two matrices $A, B \in M_n$ are equal if $\mathrm{Sp}\,(XA) = \mathrm{Sp}\,(XB)$ *for every rank one* $X \in H_n$.

[Note that we use real symmetric matrices in the proof of Assertion 3 in the proof of Theorem 2.1.] Using the above fact again, we can adapt the proof of Assertion 4 in the proof of Theorem 2.1. ■

Theorem 3.2 *Suppose $k \geq 2$, and $X_1 * \cdots * X_k = X_{j_1} \cdots X_{j_m} + X_{j_m} \cdots X_{j_1}$ for a given sequence $(j_1, \ldots, j_m)$ so that $\{j_1, \ldots, j_m\} = \{1, \ldots, k\}$ and there exists j_r not equal to j_s for all $s \neq r$. Then a mapping $\phi : H_n \to H_n$ satisfies*

$$\mathrm{Sp}\,(\phi(X_1) * \cdots * \phi(X_k)) = \mathrm{Sp}\,(X_1 * \cdots * X_k) \qquad \text{for all } X_1, \ldots, X_k \in H_n \tag{3.2}$$

if and only if there exist a unitary matrix $S \in M_n$ and a scalar ξ satisfying $\xi^m = 1$ such that ϕ has the form

$$A \mapsto \xi S^* A S \qquad \text{or} \qquad A \mapsto \xi S^* A^t S.$$

Proof. We use arguments similar to those in the proof of Theorem 2.2. We need only replace the Assertion in the proof by the following.

Assertion *Let $A, B \in H_n$. Then $A = B$, if*

$$\mathrm{Sp}\,(X^{r-1} A X^{m-r} + X^{m-r} A X^{r-1}) = \mathrm{Sp}\,(X^{r-1} B X^{m-r} + X^{m-r} B X^{r-1})$$

for every rank one idempotent $X \in H_n$.

Proof. If both $r-1$ and $m-r$ are positive, we can prove the result using a similar argument as in the proof of Theorem 2.2.

If $r-1$ or $m-r$ is zero, then we have

$$\mathrm{Sp}\,(XA + AX) = \mathrm{Sp}\,(XB + BX) \quad \text{for every rank one idempotent } X \in H_n.$$

We shall assume without loss of generality that A is the diagonal matrix $\mathrm{diag}\,(a_1, \ldots, a_n)$. Put $X = E_{11}$, we have $AE_{11} + E_{11}A = \mathrm{diag}\,(2a_1, 0, \ldots, 0)$, and hence $\mathrm{Sp}\,(AE_{11} + E_{11}A) = \{2a_1, 0\}$. Let $B = (b_{ij})$. Then

$$BE_{11} + E_{11}B = \begin{pmatrix} 2b_{11} & b_{12} & \ldots & a_{1n} \\ \overline{b_{12}} & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ \overline{b_{1n}} & 0 & \ldots & 0 \end{pmatrix}.$$

The characteristic polynomial of $BE_{11} + E_{11}B$ is

$$(-\lambda)^{n-2}(\lambda^2 - 2b_{11}\lambda - (|b_{12}|^2 + \cdots + |b_{1n}|^2)).$$

The zeros of the polynomial are $2a_1$ and 0. Now it is easy to see that the polynomial cannot have a nonzero double zero. Hence if $a_1 \neq 0$, $2a_1$ is a simple zero. We have $b_{11} = a_1$ and $b_{12} = \cdots b_{1n} = 0$. It is obvious that if $a_1 = 0$, then $b_{11} = b_{12} = \cdots b_{1n} = 0$. Similarly, by putting $X = E_{jj}$, we conclude that $A = B$. ■

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