Minimum Positive Determinant of Integer Matrices with Constant Row and Column Sums

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Abstract

The least possible positive determinant of zero-one matrices that have constant row and column sums is determined, thus proving a conjecture of Newman. The result is extended to $n \times n$ integer matrices.

In 1978, Morris Newman ([N]) discussed determinants of matrices all of whose entries are -1,0, or 1. In particular, he studied the class of (0,1)-matrices with all row and column sums equal. For $n \geq 2$ and k an integer, let

$S(n,k)$ be the set of all $n \times n$ (0,1)-matrices with all row sums and column sums equal to k, for $1 \leq k \leq n$,

$Z_+(n,k)$ be the set of all $n \times n$ nonnegative integer matrices with all row and columns sums equal to k, for $k \geq 1$, and

$Z(n,k)$ be the set of all $n \times n$ integer matrices with all row and column sums equal to k, for $k \neq 0$.

Newman obtained a number of results that support his conjecture:

Conjecture [N]. *If $1 \leq k < n$ and $(n,k) \neq (4,2)$, then*

$$\min\{|\det(A)| : A \in S(n,k) \text{ and } \det(\mathrm{A}) \neq 0\} = \mathrm{k} \cdot \gcd(\mathrm{n},\mathrm{k}).$$

In 1993, Michael Grady ([G]) suggested an algorithm for constructing a matrix in $S(n,k)$ having determinant $\pm k \cdot \gcd(n,k)$. The existence of such a matrix, together with Newman's

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result appearing in Lemma 1 below, would establish the truth of the conjecture. However, the algorithm has been verified to construct such a matrix only for $n \le 90$. Our purpose is to prove an extension of Newman's conjecture. While we do not prove Grady's algorithm valid in all cases, we begin by verifying it for $(n, k) = (6, 2)$ and for $n = 2k$, (Lemmas 3 and 4, resp.). We then construct matrices in $S(n, k)$ for other choices of (n, k) whose determinants can be evaluated using Newman's result in Lemma 2 below and our Lemma 5. Our Theorem 1 proves the conjecture of Newman. We then extend the result to $Z(n, k)$ and $Z_+(n, k)$. The following properties are established in [N] or can be easily verified.

a. $S(n, 1)$ is the set of permutation matrices, and clearly for $A \in S(n, 1)$, $|\det(A)| = 1$.

b. The only element in $S(n, n)$ is denoted by J_n, or simply J, which is singular.

c. If $1 < k < n$ and $(n, k) \neq (4, 2)$, then $S(n, k)$ contains a nonsingular matrix.

d. If $A \in S(4, 2)$, then $\det(A) = 0$.

The following result is proved in [N] and in [G] for matrices in $S(n, k)$. We observe that the proofs given there hold for arbitrary integers $n > 0$ and k.

Lemma 1 ([N, Theorem 2]; [G]). *If* $A \in Z(n, k)$, *then* $\det(A)$ *is a multiple of* $k \cdot \gcd(n, k)$.

Lemma 2 ([N, Lemma 1]). *Let* $A \in Z(n, k)$.

(a) Then $\det(J + A) = (n + k)\det(A)/k$.

(b) In particular, if $A \in S(n, k)$, *then* $J - A \in S(n, n - k)$ *and*

$$\det(J - A) = (-1)^{n-1}(n - k)\det(A)/k.$$

The conjecture has been proved if $\gcd(n, k) = 1$ (see [N, Theorem 2]), so we now assume that $\gcd(n, k) > 1$, so that $n > k \ge 2$.

Let $R(n, k)$ be the collection of all $A = (a_{ij}) \in S(n, k)$ such that

(i) $a_{ij} = 1$ for all $i \ge n - k + 2, 1 \le j \le i - (n - k + 1)$, i.e., the left bottom corner of A is a triangle of ones of size $k - 1$. We say that A *has a* $k - 1$ *left bottom triangle of ones.*

(ii) $a_{ij} = 0$ for all $i \ge k + 2, k \le j \le i - 2$, i.e., if one removes the first $k - 1$ columns of A, the remaining matrix has an $n - k - 1$ left bottom triangle of zeros.

For a given n, let $\{E_{11}, E_{12}, \ldots, E_{nn}\}$ be the standard basis for $n \times n$ matrices, let $P_n = E_{12} + E_{23} + \cdots + E_{n-1,n} + E_{n1}$ be the basic circulant, let $N_n = E_{12} + E_{23} + \cdots + E_{n-1,n}$. We shall simply write P and N if the dimension is clear from the context.

The following special case can be verified directly.

Lemma 3. *Let* $(n, k) = (6, 2)$. *Then* $A = I + P - (E_{11} + E_{44} - E_{14} - E_{41}) \in R(6, 2)$ *with* $\det(A) = -2^2$.

The construction in the following lemma was proposed by Grady [G], and he conjectured that the resulting matrix A provides the minimum of Newman's conjecture.

Lemma 4. *Let $(n,k)=(2k,k)$ with $k\neq 2$. If*

$$Q=\begin{pmatrix} -I_{k-1} & 0_{k-1,2} & I_{k-1} \\ 0_{2,k-1} & 0_2 & 0_{2,k-1} \\ I_{k-1} & 0_{k-1,2} & -I_{k-1} \end{pmatrix},$$

then $A=P^{-1}+I+P+\cdots+P^{k-2}+Q\in R(n,k)$ with $\det(A)=-k^2$.

Proof. It is easily verified that $A\in R(n,k)$. To evaluate $\det(A)$, we evaluate $\det(J-A)$ and apply Lemma 2 to see that $\det(A)=-\det(J-A)$. If $J-A$ is partitioned as

$$J-A=\begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

with each of E,F,G,H a $k\times k$ matrix, it follows (employing the *Schur complement* , see [HJ] 0.8.5) that

$$\det(J-A)=(-1)^k\det\begin{pmatrix} F & E \\ H & G \end{pmatrix}=(-1)^k\det(G)\det(F-EG^{-1}H).$$

Letting $N=N_k$ and denoting its transpose by N^t, we have

$$E=I+N^{k-1}-E_{kk}+\textstyle\sum_{i=2}^{k-1}(N^t)^i,\ F=I+N^t+\sum_{i=2}^{k-2}N^i,$$

$$G=I+\textstyle\sum_{i=1}^{k-2}N^i,\ \text{ and }\ H=I+N^{k-1}-E_{11}+\sum_{i=2}^{k-1}(N^t)^i.$$

Thus

$$E=\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \ddots & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & \ddots & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \ddots & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 & 0 \end{pmatrix},\ F=\begin{pmatrix} 1 & 0 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & \ddots & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & \ddots & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & \ddots & 1 & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \ddots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix},$$

$$G=\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & \ddots & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & \ddots & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & \ddots & 1 & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix},\ H=\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \ddots & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \ddots & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & \ddots & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 & 1 \end{pmatrix}.$$

It is easily checked that

$$G^{-1} = I - N + N^{k-1} = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

Computing EG^{-1} column by column, we find

$$\begin{aligned} EG^{-1} &= I - N - N^t + \sum_{i=2}^{k} E_{i1} + 2E_{k1} + \sum_{i=3}^{k-1} E_{ik} \\ &= \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 2 \\ 0 & 1 & -1 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & \ddots & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 1 & \ddots & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & \ddots & \ddots & 0 & 0 & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \ddots & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & \ddots & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 1 \end{pmatrix}. \end{aligned}$$

Then, computing $(EG^{-1})H$ column by column, we obtain

$$EG^{-1}H = \begin{pmatrix} 2 & 1 & 2 & 2 & 2 & 2 & \cdots & 2 & 0 & 3 \\ -1 & 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 1 & \cdots & 1 & 0 & 2 \\ 0 & 1 & -1 & 2 & 0 & 1 & \ddots & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 2 & 0 & \ddots & 1 & 0 & 2 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & \ddots & \ddots & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -1 & 2 \end{pmatrix}.$$

Here the lower left $(k-2) \times (k-2)$ submatrix is $I - N + 2N^2 + \sum_{i=4}^{k-3} N^i$.
Now we obtain

$$
F - EG^{-1}H = \begin{pmatrix}
-1 & -1 & -1 & -1 & -1 & -1 & \cdots & -1 & 1 & -3 \\
2 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
-1 & 2 & -1 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\
0 & -1 & 2 & -1 & 0 & 0 & \ddots & 0 & 1 & -1 \\
0 & 0 & -1 & 2 & -1 & 0 & \ddots & 0 & 1 & -1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & \ddots & \ddots & -1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & \ddots & 2 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1
\end{pmatrix}.
$$

To evaluate the determinant of this matrix, replace the first row by the sum of all the rows, and then replace the last column by the sum of all the columns. The result is a matrix whose first row is $(0, 0, \cdots, 0, k, 0)$, whose last column is $(0, k, 0, \cdots, 0)^t$, and whose lower left $(k-2) \times (k-2)$ submatrix is triangular with -1's on the diagonal. Now expand by the first row, and then by the last column, to obtain $\det(F - EG^{-1}H) = (-1)^k k^2$. Thus

$$
\det(A) = -\det(J - A) = -(-1)^k \det \begin{pmatrix} F & E \\ H & G \end{pmatrix} = (-1)^{k+1} \det(G) \det(F - EG^{-1}H) = -k^2.
$$

$\square$

Lemma 5. *Let $A \in R(n, k)$.*

(a) If $\hat{A}$ is obtained from $J - A$ by moving columns k through $(n-2)$ over the preceding $k-1$ columns so that they become columns 1 through $(n-k-1)$, then $\hat{A} \in R(n, n-k)$ with $|\det(\hat{A})| = (n-k)|\det(A)|/k$.

(b) Suppose $\hat{A} = \begin{pmatrix} \tilde{A} & B \\ C & D \end{pmatrix}$, where $\tilde{A}$ is $n \times n$ and is obtained from A by setting its $k-1$ left bottom triangle of ones to zeros, B is $n \times k$ with a $k-1$ left bottom triangle of ones and zeros elsewhere, C is $k \times n$ with a $k-1$ left bottom triangle of ones and zeros elsewhere, and D is $k \times k$ with a k right top triangle of ones and zeros elsewhere. Then $\hat{A} \in R(n+k, k)$ with $\det(\hat{A}) = \det(A)$.

Proof. (a) Direct verification.

(b) By the facts that $A = \tilde{A} - BD^{-1}C$ and $\det(\hat{A}) = \det(D)\det(\tilde{A} - BD^{-1}C)$. $\square$

We shall make use of the basic matrices constructed in Lemmas 3 and 4 and apply the procedures in Lemma 5 (a) and (b) repeatedly to produce a matrix A in $R(n, k)$ that provides a minimum positive $|\det(A)|$ over $A \in S(n, k)$. As a result, we have a formally stronger result, namely, we can find a matrix in $R(n, k)$, a subset of $S(n, k)$, to achieve our goal. As can be seen in the proof of the following theorem, it is very important to establish that the procedures in Lemma 5 produce a matrix belonging to $R(n, k)$. In particular, it is worth noting that the triangle of zeros in the definition of $R(n, k)$ will grow to the appropriate size when one applies Lemma 5(b), and the sizes and locations of the triangles of zeros and ones will adjust themselves properly when one applies Lemma 5(a).

Theorem 1. *Suppose* $1 \le k < n$ *and* $(n,k) \neq (4,2)$. *There exists* $A \in R(n,k)$ *with*

$$\min\{|\det(B)| : B \in S(n,k) \text{ and } \det(\mathrm{B}) \neq 0\} = |\det(\mathrm{A})| = \mathrm{k} \cdot \gcd(\mathrm{n}, \mathrm{k}).$$

Proof. For $k = 1$ we may take $A = I$, so we assume $k \ge 2$. Given $(n,k) \neq (4,2)$, define $(n_1,k_1),(n_2,k_2),\ldots$ by the following algorithm:

Step 1. Set $(n_1,k_1) = (n,k)$, and set $r = 1$. Go to Step 2.

Step 2. If $n_r = 2k_r$, stop. Otherwise go to Step 3.

Step 3. Set $i = r$. Define

$$(n_{i+1},k_{i+1}) = \begin{cases} (n_i - k_i, k_i) & \text{if } n_i > 2k_i, \\ (n_i, n_i - k_i) & \text{if } n_i < 2k_i. \end{cases}$$

Step 4. Set $r = i + 1$. Go to Step 2.

By the Euclidean algorithm, after finitely many iterations, we get $(n_s,k_s) = (2t,t)$, where $t = \gcd(n,k)$.

If $t > 2$, one can construct $A_s \in R(n_s,k_s)$ with $|\det(A_s)| = t^2$ by Lemma 4. Then for $i = s-1,\ldots,1$, one can apply Lemma 5(a) (if $n_i < 2k_i$) or 5 (b) (if $n_i > 2k_i$) to construct $A_i \in R(n_i,k_i)$ with $|\det(A_i)| = tk_i$. One readily checks that $A_1 \in R(n,k)$ satisfies $|\det(A_1)| = kt$.

If $t = 2$, then $(n_s,k_s) = (4,2)$ and $(n_{s-1},k_{s-1}) = (6,2)$. One can construct $A_{s-1} \in R(n_{s-1},k_{s-1})$ with $|\det(A_{s-1})| = 2^2$ by Lemma 3. Then for $i = s-2,\ldots,1$, one can apply Lemma 5 (a) or (b) as before to construct $A_i \in R(n_i,k_i)$ with $|\det(A_i)| = tk_i$. One readily checks that $A_1 \in R(n,k)$ satisfies $|\det(A_1)| = kt$.

Lemma 1 completes the proof. $\square$

To illustrate the process described in the proof, we construct a matrix A_1 in $R(15,6)$ with $|\det(A_1)| = 6 \cdot \gcd(15,6) = 18$.

First we construct the sequence: $(n_1,k_1) = (15,6)$, $(n_2,k_2) = (9,6)$, $(n_3,k_3) = (9,3)$, $(n_4,k_4) = (6,3)$.

Then we construct the matrices:

$$A_4 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By our results (Lemmas 4 and 5) or by direct verification, one sees that $|\det(A_4)| = |\det(A_3)| = 9$ and $|\det(A_2)| = |\det(A_1)| = 18$.

Because of the more general hypothesis in Lemmas 1 and 2, we now relax the conditions on k and obtain the following theorem concerning integer matrices.

Theorem 2. *Let $n \geq 2$ and k be nonzero integers.*

(a) Then $\min\{|\det(A)| : A \in Z(n,k) \text{ and } \det(A) \neq 0\} = |k| \cdot \gcd(n,k)$.

(b) If $k \geq 1$, then

$$\min\{|\det(A)| : A \in Z_+(n,k) \text{ and } \det(A) \neq 0\} = k \cdot \gcd(n,k).$$

Proof. It suffices to prove (b). In case we are dealing with (a) and a negative integer k, we only need to construct $A \in Z_+(n,-k)$ satisfying (b). Then $-A \in Z(n,k)$ will satisfy the requirement.

First suppose $1 \leq k \leq n$. Then all these cases are covered by Theorem 1 except the cases $(n,k) = (4,2)$ and $k = n$. For $(n,k) = (4,2)$, the lower bound of 4 provided by Lemma 1 is attained by the direct sum $(J_3 - I_3) \oplus [2]$.

For $k = n$, the lower bound of n^2 provided by Lemma 1 is attained by the direct sum $(I_{n-1} + J_{n-1}) \oplus [n]$, which is clearly in $Z_+(n,n)$. The eigenvalues of $I_{n-1} + J_{n-1}$ are $n, 1, 1, \cdots, 1$, and so the determinant of the direct sum is n^2.

Suppose $k = qn + r$ for some $q \geq 1$ and $1 \leq r \leq n$. Then $\gcd(n,r) = \gcd(n,k)$ and we can construct $A \in Z_+(n,r)$ such that $|\det(A)| = r \cdot \gcd(n,k)$. One can then apply Lemma 2(a) repeatedly to conclude that $qJ + A \in Z_+(n,k)$ satisfies $|\det(qJ + A)| = k \cdot \gcd(n,k)$. $\square$

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