Math 410-02 **Note on the Triangle Problem** **Chi-Kwong Li**

Motivated by the inverse eigenvalue problem, we study the following geometry problem.

Suppose H is a triangle is a convex hexagon with vertices $v_i = (a_i, b_i) \in \mathbb{R}^2$ in the counter-clockwise direction. Assume that the triangle $T(v_1, v_3, v_5)$ with vertices v_1, v_3, v_5 has the maximum area among all triangles formed by any 3 of the 6 vertices. Determine the smallest γ (conjectured value 1.2) such that

$$\text{Area}(T(v_1, v_2 v_3) + \text{Area}(T(v_3, v_4, v_5) + \text{Area}(T(v_4, v_5, v_6) \leq \text{Area}(T(v_1, v_2, v_3).$$

Some reduction by linear algebra. Construct the real matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix}.$$

Let $A[i, j, k]$ be the submatrix formed by column $i < j < k$. The problem reduced too:

Assume that $|\det([i, j, k])| \leq |\det(A[1, 3, 5])|$ for all $1 \leq i < j < k \leq 6$. Determine the smallest γ such that

$$|\det([1, 2, 3])| + |\det([3, 4, 5])| + |\det(A[4, 5, 6])| \leq \gamma |\det(A[1, 3, 5])|.$$

Since the hypothesis and conclusion will be change if we replace A by RA for any invertible $R \in M_3$, we may choose a suitable R to convert $A[1, 3, 5]$ to a nice form, say, with the vertices of the triangle corresponding to $(0, 0), (1, 0), (0, 1)$.