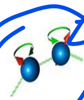
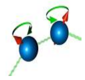
1. Quantum Information Science Research

Quantum states

Quantum Channels

Basic model

 $\longrightarrow$ | *Quantum Computing Unit*
 Optical lattices, NMR, Anyons | $\longrightarrow$ 

Quantum bit
(*Qubit*)

- Design a *quantum process* to use *quantum properties* to get useful information for a given problem.

- Choose a *suitable* quantum system to build the hardware.

- Prepare the initial (*entangled*) quantum states.

- Create a suitable environment for the quantum system to *evolve* according to quantum mechanical rules.

- Apply a suitable *measurement* to extract useful information.

complicated processes

Remarks

- In physics labs, one would prepare quantum states, manipulate them with quantum operations and measure the output states, where measuring quantum states is also a quantum operation.

- Mathematical theory is needed to help model and design the process.

- Computer Science theory is needed to develop the computation and communication algorithms.

- Knowledge in engineering, material science, chemistry, etc. are needed to build the system.

- We are interested in the mathematical theory.

Matrix Theory

2. Mathematical framework and notation

- Quantum states with n measurable states are represented as complex unit vectors $v \in \mathbb{C}^n$.
- One does not distinguish v and $e^{it}v$ for any $t \in \mathbb{R}$.
- The conjugate transpose of $v \in \mathbb{C}^n$ and $A \in M_{m,n}$ are denoted by $v^\dagger$ and $A^\dagger$, where $M_{m,n}$ is the set of $m \times n$ complex matrices. $\langle v| = v^\dagger = (-i, 2-i)$
- In physics literature, one uses the bracket notation for v and $v^\dagger$, namely, $|v\rangle$ and $\langle v|$.
- For example, a photon has two measurable states so that it is represented by vectors in $\mathbb{C}^2$.
- Upon measurement, one only sees $|e_1\rangle$ or $|e_2\rangle$. Sometimes, written as $|\uparrow\rangle, |\rightarrow\rangle$.
- A general quantum state has the form $a_1|e_1\rangle + a_2|e_2\rangle$ with probability of $|a_1|^2$ in $|e_1\rangle$ and $|a_2|^2$ in $|e_2\rangle$.
- We say that the quantum state is a *superposition* of its measurable states.
- Schrödinger cat interpretation of superposition...
- One may have a different measuring "frame" or "basis", say, $|f_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, |f_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$, and the superposition has the form $b_1|f_1\rangle + b_2|f_2\rangle$.
- Quantum operations on a closed system with vector states in $\mathbb{C}^n$ are unitary matrices U, i.e., $U^\dagger U = I_n$.

Basic linear algebra [Nakahara and Ohmi, Chapter 1]

- Complex vectors.
- Inner product structure.
- Orthogonal and orthonormal vecotrs.
- Schur Triangularization Lemma.
- Special class of matrices: Hermitian, positive semidefinite, unitary, normal matrices.
- Spectral theorem of diagonalizable and normal matrices.

$e_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\1\end{pmatrix} \in \mathbb{C}^2$

$|v\rangle = \begin{pmatrix} i \\ 2+i\end{pmatrix} \in \mathbb{C}^2$

$i e_1 + (2+i) e_2$

is unit vector

$\left\|\begin{pmatrix}z_1\\z_2\end{pmatrix}\right\| = (|z_1|^2+|z_2|^2)^{1/2}$

$\left\|\begin{pmatrix}i\\2+i\end{pmatrix}\right\| = \sqrt{|i|^2+|2+i|^2} = \sqrt{1^2+4+1^2} = \sqrt{6}$

$\frac{1}{\sqrt2}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt2}\begin{pmatrix}1\\-1\end{pmatrix}$

$\langle x, y\rangle = x^t y$

$\langle x|y\rangle = x^\dagger y$

$U = \frac{1}{\sqrt2}\begin{pmatrix}1&1\\1&-1\end{pmatrix} \quad U^\dagger U = I_2$

$U = \frac{1}{\sqrt2}\begin{pmatrix}1&i\\1&-i\end{pmatrix} \quad U U^\dagger = I_2$

2^4 16×16 8×8

$U = U_1 \cdots U_N$ 128