Notes on Unitary Matrices

$x_i = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$, $x_i \in \mathbb{R}^n$

$x_j = \begin{bmatrix} 1+i \\ 0 \\ 2 \\ 1 \end{bmatrix}$, $x_j \in \mathbb{C}^n$

1. Orthonormal

- Let $x_1, x_2, \cdots, x_k \in \mathbb{C}^n$ be an orthogonal set (so $x_i^* x_j = 0$ for $1 \le i < j \le k$).
- If we normalize it: $x_i^* x_i = 1$, then the set is orthonormal.

$x_i^* x_j \Rightarrow x_i^T x_j$ ex: $x_j = \begin{bmatrix} 1+i \\ 0 \\ 2 \\ 1 \end{bmatrix}$, $x_j^* = [1-i \quad 0 \quad 2 \quad 1]$

$*$: Conjagate transpose

similar to "transpose" in real case

2. Thm:

- An orthonormal set of vectors is linearly independent.

Pf: Suppose $0 = a_1 x_1 + \cdots + a_k x_k$

then, $0 = 0 \cdot 0 = \sum_{i,j} \bar{a}_i a_j x_i^* x_y$ x_i, x_j are orthogonal.

$= \sum_{i=1}^k |a_i|^2 x_i^* x_i$

Also, $\sum_{i=1}^k |a_i|^2 \underbrace{x_i^* x_i}_{=1} = \sum_{i=1}^k |a_i|^2 = 0$ → x_i's are normalized

$\Rightarrow$ so all $a_i = 0$, and the set $\{x_1, x_2, \cdots, x_k\}$ is linearly independent.

3. Unitary

- A matrix $U \in M_n$ is said to be unitary if $U^* U = I$. (example?)
- If $U \in M_n(\mathbb{R})$, then U is real orthogonal.

$*$: conjagate transpose. (similar to "transpose" in real)

$$M = \begin{bmatrix} 1 & 0 & 1+i \\ 2 & 2-i & 3i \end{bmatrix} \qquad M^* = \begin{bmatrix} 1 & 2 \\ 0 & 2+i \\ 1-i & -3i \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4. Properties of unitary matrices:

TFAE: 1). U is unitary. $\quad U^* U = I = U^{-1} U = I$

2). U is nonsingular and $U^* = U^{-1}$.

3). $UU^* = I$. $\quad U^* U = I \quad (U^*)^*(U^*) = I$

4). U^* is unitary.

$U^* U = I$ means $u^{(i)*} u^{(j)} = \begin{cases} 0 & \text{if } j \neq i \\ 1 & \text{if } j = i \end{cases}$

5). the columns of U form an orthonormal set. $\quad \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

6). the rows of U form an orthonormal set.

7). For all $x \in \mathbb{C}^n$, the Euclidean length of $y \equiv Ux$ is the same as x. $(y^* y = x^* x)$

→ Unitary matrices preserve length.

Examples :

5. Unitary & Vectors (Preserves length).

Pauli gates :

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

symmetric w/ complex number.
(⇒ Hermitian as well.)

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

① $X|0\rangle = |1\rangle$

② $X|1\rangle = |0\rangle$

$|0\rangle : \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$|1\rangle : \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

length 1   length 1

① $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$

② $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$

③ $Z|0\rangle = |0\rangle$

④ $Z|1\rangle = -|1\rangle$

③ $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

④ $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Hadamard gate : $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

① $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ ② $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

①

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \qquad \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad \left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\right)^*\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}^*\begin{bmatrix}1\\1\end{bmatrix} = \frac{2}{2} = 1.$$

b. What else ?

(offered by IBM)

→ Rotations & "famous" unitary matrices

✓. Unitary matrices preserves orthogonality.

$x^*y = 0$: $(Ux)^*(Uy) = x^*\underbrace{U^*U}_{=I}y = x^*y = 0.$

↳ only rotate vector rigidly.

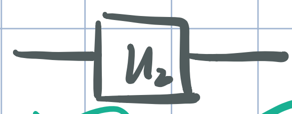
Plane rotation :

$$\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & 0 & & \\ & & 1 & & & & \\ & & & \cos\theta & 0 & \sin\theta & \\ & 0 & & 0 & 1 \; \ddots & 0 & \\ & & & & 1 & & \\ & & & -\sin\theta & 0 & \cos\theta & \end{bmatrix}$$

"famous" unitary matrices in QC :

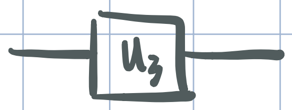
$$U_1 = \begin{bmatrix}1 & 0\\ 0 & e^{i\lambda}\end{bmatrix} \qquad U_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & e^{-i\lambda}\\ e^{i\phi} & e^{i\lambda+i\phi}\end{bmatrix}$$

Circuit representations



$$U_3 = \begin{bmatrix} \cos(\theta/2) & -e^{i\lambda}\sin(\theta/2) \\ e^{i\phi}\sin(\theta/2) & e^{i\lambda+i\phi}\cos(\theta/2) \end{bmatrix}$$

These are rotations.



(Next step : "pure" linear algebra, but may be useful).

7. Unitary Similarity

- In general, similarity : $B = S^{-1}AS$

  $\hookrightarrow$ B is similar to A.

- In unitary similarity, $B = U^* A U$.

  $BU^* = U^*AUU^* = U^*A$.

In this case : $U^*A = BU^*$.

$$U^*A = U^*\begin{bmatrix} | \\ a_1 \\ | \end{bmatrix} \; U^*\begin{bmatrix} | \\ a_2 \\ | \end{bmatrix} \cdots U^*\begin{bmatrix} | \\ a_n \\ | \end{bmatrix}$$

so $\| U^*A \|^2 = \sum_{i,j} |a_{ij}|^2$

$\Rightarrow$ so $\sum_{i,j} |a_{ij}|^2 = \sum_{i,j} |b_{ij}|^2$