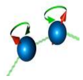
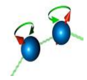
1. Quantum Information Science Research

Basic model

 $\longrightarrow$ *Quantum Computing Unit* / *Optical lattices, NMR, Anyons* $\longrightarrow$ 

Quantum bit
(*Qubit*)

- Design a *quantum process* to use *quantum properties* to get useful information for a given problem.
- Choose a *suitable* quantum system to build the hardware.
- Prepare the initial (*entangled*) quantum states.
- Create a suitable environment for the quantum system to *evolve* according to quantum mechanical rules.
- Apply a suitable *measurement* to extract useful information.

Remarks

- In physics labs, one would prepare quantum states, manipulate them with quantum operations and measure the output states, where measuring quantum states is also a quantum operation.
- Mathematical theory is needed to help model and design the process.
- Computer Science theory is needed to develop the computation and communication algorithms.
- Knowledge in engineering, material science, chemistry, etc. are needed to build the system.
- We are interested in the mathematical theory.

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$$

$$|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$|v\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{pmatrix}$$

$$\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} \qquad \left|\frac{i}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$|v\rangle = \begin{pmatrix}\frac{3}{5}\\ \frac{4i}{5}\end{pmatrix}$$

$$\left|\frac{3}{5}\right|^2 = \frac{9}{25} \qquad \left(\frac{4i}{5}\right)^2 = \frac{16}{25}$$

2×2

$$U\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}w_1\\w_2\end{bmatrix}$$

$$\|u_1\| = 1 = \|u_2\|$$

$$|u_1^* u_2| = 0$$

UV

$$[u_1, u_2]$$

$U^* U$

$$\boxed{U^\dagger U = I_2}$$

2. Mathematical framework and notation

- Quantum states with n measurable states are represented as complex unit vectors $v \in \mathbb{C}^n$.
- One does not distinguish v and $e^{it}v$ for any $t \in \mathbb{R}$.
- The conjugate transpose of $v \in \mathbb{C}^n$ and $A \in M_{m,n}$ are denoted by $v^\dagger$ and $A^\dagger$, where $M_{m,n}$ is the set of $m \times n$ complex matrices.
- In physics literature, one uses the bracket notation for v and $v^\dagger$, namely, $|v\rangle$ and $\langle v|$.
- For example, a photon has two measurable states so that it is represented by vectors in $\mathbb{C}^2$.
- Upon measurement, one only sees $|e_1\rangle$ or $|e_2\rangle$. Sometimes, written as $|\uparrow\rangle$, $|\rightarrow\rangle$.
- A general quantum state has the form $a_1|e_1\rangle + a_2|e_2\rangle$ with probability of $|a_1|^2$ in $|e_1\rangle$ and $|a_2|^2$ in $|e_2\rangle$.
- We say that the quantum state is a *superposition* of its measurable states.
- Schrödinger cat interpretation of superposition...
- One may have a different measuring "frame" or "basis", say, $|f_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, |f_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$, and the superposition has the form $b_1|f_1\rangle + b_2|f_2\rangle$.
- Quantum operations on a closed system with vector states in $\mathbb{C}^n$ are unitary matrices U, i.e., $U^\dagger U = I_n$.

Basic linear algebra [Nakahara and Ohmi, Chapter 1]

- Complex vectors.
- Inner product structure.
- Orthogonal and orthonormal vecotrs.
- Schur Triangularization Lemma.
- Special class of matrices: Hermitian, positive semidefinite, unitary, normal matrices.
- Spectral theorem of diagonalizable and normal matrices.

3. Quantum Mechanics

Here are the basic postulates of quantum mechanics in terms of vector states.

Copenhagen interpretation

A1 A vector state $|x\rangle$ is a unit vector in a Hilbert space $\mathcal{H}$ (usually $\mathbb{C}^n$). Linear combinations (superposition) of the physical states are allowed in the state space.

A2 Every physical quantity (observable) corresponds to a Hermitian operator (matrix) $A \in M_2$ such that A has orthonormal eigenvectors $|u_1\rangle$ and $|u_2\rangle$. Suppose a state $|x\rangle = c_1|u_1\rangle + c_2|u_2\rangle$. Then applying a measurement of $|x\rangle$ corresponding to A will cause the **wave function** (that describes the quantum state) to **collapse** to $|u_1\rangle$ or $|u_2\rangle$ with probability of $|c_1|^2$ and $|c_2|^2$ respectively. Here c_1, c_2 are called the probability amplitude of the state $|x\rangle$.

A3 The time dependence of a state is governed by the Schrödinger equation

$$i\hbar\frac{\partial|x\rangle}{\partial t} = H|x\rangle,$$

where $\hbar$ is the Planck constant with

$$\hbar = 6.62607004 \times 10^{-34} m^2 kg/s,$$

and H is a Hermitian operator (matrix) corresponding to the energy of the system known as the Hamiltonian.

$|0\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $|2\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$|v\rangle = c_1|0\rangle + c_2|1\rangle + c_3|2\rangle$

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$|\nearrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$, $|\searrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

$[a_{ij}]^\dagger = [\bar{a}_{ji}]$

$[u_1\, u_2]$

$U^\dagger H U = \begin{bmatrix}\lambda_1 & 0\\0 & \lambda_2\end{bmatrix}$

$\begin{bmatrix}v_1\\v_2\end{bmatrix} = |v\rangle = c_1|u_1\rangle + c_2|u_2\rangle$

$H = \begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix} \Leftrightarrow H^\dagger = \begin{bmatrix}\bar{a}_{11} & \bar{a}_{21}\\\bar{a}_{12} & \bar{a}_{22}\end{bmatrix}$

$a_{11} = \bar{a}_{11}$, $a_{22} = \bar{a}_{22}$, $\bar{a}_{12} = a_{21}$

Example $H = \begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}$, $u = \frac{1}{5}\begin{bmatrix}3i\\4\end{bmatrix} = \frac{3i}{5}\begin{bmatrix}1\\0\end{bmatrix} + \frac{4}{5}\begin{bmatrix}0\\1\end{bmatrix}$

Multipartite systems

- Suppose $|v_1\rangle \in \mathbb{C}^m, |v_2\rangle \in \mathbb{C}^n$ are quantum state. Then the $|v_1\rangle \otimes |v_2\rangle = |v_1 v_2\rangle$ is a **composite state** (**uncorrelated state**) in the bipartite system.
- For example, $|v_1\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, |v_2\rangle = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, then

$$|v_1\rangle \otimes |v_2\rangle = |v_1\rangle|v_2\rangle = |v_1 v_2\rangle = \begin{bmatrix} a_1|v_2\rangle \\ a_2|v_2\rangle \end{bmatrix} = \begin{bmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{bmatrix}.$$

- A state $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ is **entangled** if it is not a composite state.
- The orthonormal basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ for $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$ consists of decomposable states.
- The orthonormal basis

$$\{\tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$
$$\tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\}$$

consists of entangled states known as Bell states.

- Suppose an observable corresponds to the Hermitian matrix with eigenvectors $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, say, $H = \mathrm{diag}\,(3/2, 1/2, -1/2, -3/2)$.

  Then the measurement of $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ will yield $|00\rangle$ or $|11\rangle$ each with 50%.

  In particular, the first Schrödinger cat is alive (dead) if and only if the second one is alive (dead).

- We can construct **multipartite system** from k systems to get $\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k} = \mathbb{C}^{n_1 \cdots n_k}$.
- For example, $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ is a 3 qubit system.

$$H = \begin{pmatrix} \frac{3}{2} & & & 0 \\ & \frac{1}{2} & & \\ & & -\frac{1}{2} & \\ 0 & & & -\frac{3}{2} \end{pmatrix} \quad \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$A \otimes B = [a_{ij}] \otimes [b_{ij}] = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & & & \\ a_{m1}B & \cdots & & a_{mn}B \end{pmatrix}$$

$$(|v_1\rangle \otimes |v_2\rangle) \otimes |v_3\rangle = |v_1\rangle \otimes (|v_2\rangle \otimes |v_3\rangle)$$

$$\mathbb{C}^m \otimes \mathbb{C}^n \quad m\left\{\begin{pmatrix} n\,|v\rangle \\ \vdots \\ \cdot\, |v\rangle \end{pmatrix}\right\} mn$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|0\rangle \otimes |0\rangle = |0\rangle|0\rangle$$

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|01\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Quantum operations on multipartite systems

We focus on qubit systems.

- Local (unitary) operations. If $U_1, U_2 \in M_2$ are unitary, then $U_1 \otimes U_2$ is unitary $(U_1 \otimes U_2)|v_1v_2\rangle = |U_1v_1\rangle|U_2v_2\rangle$.

- General $U \in M_4$ is a product of local unitary gates $U_1 \otimes U_2$ and controlled unitary gates of the form $I_2 \oplus V$ and $V \oplus I_2$.

- Proof. Let U be unitary.

  Find $P_1 = U_1 \otimes V_1$ so that P_1U has zero $(4,1)$ entry.

  Find $P_2 = U_2 \oplus I_2$ so that P_2P_1U has zero $(4,1)$ and $(2,1)$ entry.

  Find $P_3 = U_3 \otimes I_2$ so that the first column of $P_3P_2P_1U$ is $(1,0,0,0)^t$. Then $P_2P_1 = [1] \oplus R$.

  Find $P_4 = I_2 \oplus V_4$ such that $P_4P_3P_2P_1U$ has zero $(3,2)$ entry.

  Find $P_5 = U_5 \otimes I_2$ such that $P_5 \cdots P_1U = I_2 \oplus V_6$.

  If $P_6^\dagger = I_2 \oplus V_6$, then $U = P_1^\dagger \cdots P_6^\dagger$.

- We can represent the operations on a circuit diagrams, and implement the operations using a quantum computers.

  See [Nakahara and Ohmi, Chapter 4].

- We only need to check the actions of the quantum operations on measurable states, say, $|000\rangle, |001\rangle, \ldots, |111\rangle$.

- The standard gates and basic gates might vary from different quantum computer.

- We are working on a research project requiring a decomposition of a unitary $U \in M_8$ into simple unitary gates.