Multipartite systems

- Suppose $|v_1\rangle \in \mathbb{C}^m, |v_2\rangle \in \mathbb{C}^n$ are quantum state. Then the $|v_1\rangle \otimes |v_2\rangle = |v_1 v_2\rangle$ is a **composite state** (**uncorrelated state**) in the bipartite system.

- For example, $|v_1\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, |v_2\rangle = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, then

$$|v_1\rangle \otimes |v_2\rangle = |v_1\rangle|v_2\rangle = |v_1 v_2\rangle = \begin{bmatrix} a_1|v_2\rangle \\ a_2|v_2\rangle \end{bmatrix} = \begin{bmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{bmatrix}.$$

- A state $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ is **entangled** if it is not a composite state.

- The orthonormal basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ for $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$ consists of decomposable states.

- The orthonormal basis

$$\Big\{ \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle),\ \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$
$$\tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle),\ \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \Big\}$$

  consists of entangled states known as Bell states.

- Suppose an observable corresponds to the Hermitian matrix with eigenvectors $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, say, $H = \operatorname{diag}(3/2, 1/2, -1/2, -3/2)$.

  Then the measurement of $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ will yield $|00\rangle$ or $|11\rangle$ each with 50%.

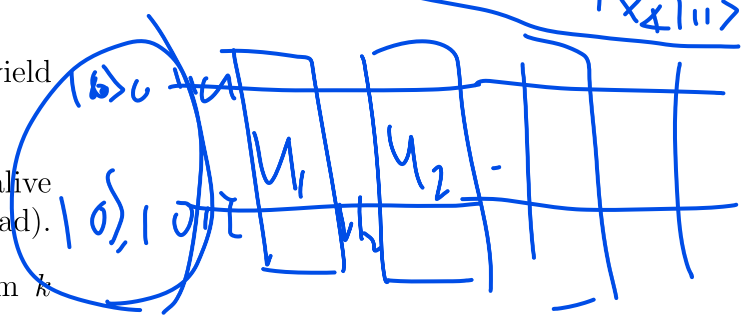
  In particular, the first Schrödinger cat is alive (dead) if and only if the second one is alive (dead).

- We can construct **multipartite system** from k systems to get $\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k} = \mathbb{C}^{n_1 \cdots n_k}$.

- For example, $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ is a 3 qubit system.

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad \mathbb{C}^2 \otimes \mathbb{C}^2$

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_1|00\rangle + x_2|01\rangle + x_3|10\rangle + x_4|11\rangle$



$U = U_1 \cdots U_m$

Quantum operations on multipartite systems

We focus on qubit systems.

- Local (unitary) operations. If $U_1, U_2 \in M_2$ are unitary, then $U_1 \otimes U_2$ is unitary
$$(U_1 \otimes U_2)|v_1 v_2\rangle = |U_1 v_1\rangle |U_2 v_2\rangle.$$
- General $U \in M_4$ is a product of local unitary gates $U_1 \otimes U_2$ and controlled unitary gates of the form $I_2 \oplus V$ and $V \oplus I_2$.
- Proof. Let U be unitary.

  Find $P_1 = U_1 \otimes V_1$ so that $P_1 U$ has zero $(4,1)$ entry.

  Find $P_2 = U_2 \oplus I_2$ so that $P_2 P_1 U$ has zero $(4,1)$ and $(2,1)$ entry.

  Find $P_3 = U_3 \otimes I_2$ so that the first column of $P_3 P_2 P_1 U$ is $(1,0,0,0)^t$. Then $P_2 P_1 = [1] \oplus B$.

  Find $P_4 = I_2 \oplus V_4$ such that $P_4 P_3 P_2 P_1 U$ has zero $(3,2)$ entry.

  Find $P_5 = U_5 \otimes I_2$ such that $P_5 \cdots P_1 U = I_2 \oplus V_6$.

  If $P_6 = I_2 \oplus V_6^\dagger$, then $U = P_1^\dagger \cdots P_6^\dagger$.
- We can represent the operations on a circuit diagrams, and implement the operations using a quantum computers.

  See [Nakahara and Ohmi, Chapter 4].
- We only need to check the actions of the quantum operations on measurable states, say, $|000\rangle, |001\rangle, \ldots, |111\rangle$.
- The standard gates and basic gates might vary from different quantum computer.
- We are working on a research project requiring a decomposition of a unitary $U \in M_8$ into simple unitary gates.

Mixed states and density matrices

A system is in a mixed state if there is a probability p_i that the system is in state $|x_i\rangle$ for $i = 1, \dots, N$.

If $N = 1$, then the system is in pure state.

Consider an observable corresponds to the Hermitian matrix A.

$A^\dagger = A$, $= (a_{ij})$, $a_{ii} \in \mathbb{R}$, $a_{ij} = \bar{a}_{ji}$

- The mean value of the quantum system with quantum state $|x\rangle$ is given by $\langle A\rangle = \langle x|A|x\rangle$.
- The mean value of the quantum system with a mixed state $\sum_{j=1}^{N} p_j |x_j\rangle$ is given by

$$\langle A\rangle = \sum_{j=1}^{N} p_j \langle x_j|A|x_j\rangle = \operatorname{tr}(A\rho) = \operatorname{tr}(\rho A),$$

where

$$\rho = \sum_{j=1}^{N} p_j |x_j\rangle\langle x_j|$$

is a density operator (matrix).

Description of quantum systems in mixed states.

A1' A physical state is specified by a density matrix $\rho : \mathcal{H} \to \mathcal{H}$, which is positive semidefinite with trace equal to one.

A2' The mean value of an observable associate with the Hermitian matrix A is $\langle A\rangle = \operatorname{tr}(\rho A)$.

A3' The temporal evolution of the density matrix is given by the Liouville-von Neumann equation

$$i\hbar \frac{d}{dt}\rho = [H, \rho] = H\rho - \rho H,$$

where H is the system Hamiltonian.

$|x\rangle = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ $|x_1|^2$ $|00\rangle$ $|x_2|^2$ $|01\rangle$ $|x_3|^2$ $|10\rangle$ $|x_4|^2$ $|11\rangle$

$U^\dagger A U = \begin{bmatrix} a_1 & 0 \\ 0 & a_n \end{bmatrix}$

$p_1|\psi_1\rangle + p_2|\psi_2\rangle + \dots + p_N|\psi_N\rangle$

Lemma $\operatorname{tr}(XY) = \operatorname{tr}(YX)$, $m\times n$, $n \times m$

$\operatorname{tr}(A I) = \operatorname{tr}(I A)$

$\rho = |x\rangle\langle x|$

Multipartitle systems

- Let $\rho_1 \in M_{n_1}, \rho_2 \in M_{n_2}$ be mixed states. Then

$$\rho_1 \otimes \rho_2 \in M_{n_1} \otimes M_{n_2} \equiv M_{n_1 n_2}$$

  is a composite (uncorrelated) state in the bipartite system.

- General states ρ in $M_{n_1} \otimes M_{n_2}$ are density matrices in $M_{n_1 n_2}$.

- Let ρ be a density matrix in the bipartite system $M_{n_1} \otimes M_{n_2}$. It is **separable** if it is a probabilistic (convex) combination of composite state, i.e.,

$$\rho = \sum_{j=1}^{N} p_j \sigma_j \otimes \tau_j$$

  with quantum states $\sigma_j \in M_{n_1}, \tau_j \in M_{n_2}$.

- Otherwise, it is **entangled**.

- Note that ρ is always a **linear combination** of composite states.

- Checking whether a state is separable is an NP-hard problem.

- A common test is to use the **partial transposes** defined by

$$(\rho_1 \otimes \rho_2)^{pt_1} = \rho_1^t \otimes \rho_2, \quad (\rho_1 \otimes \rho_2)^{pt_2} = \rho_1 \otimes \rho_2^t.$$

- If ρ is separable, then the partial transposes are also quantum states, i.e., positive semi-definite.

- If ρ is a state such that ρ^{pt_1} is positive semidefinite, we say that ρ is a ppt (positive partial trace) state.

- Question:

  Determine all possible norm, eigenvalues, etc., of ppt states,

- One can extend the concepts to multipartite systems.