

Research questions about quantum states

Quantum State tomography Determine $\rho = (\rho_{ij})$.

- For a Hermitian matrix $A = (a_{ij})$, we can determine $\text{tr}(a_{ij})X$ for $X \in \mathcal{B}$, where

$$\mathcal{B} = \{E_{rr} : 1 \leq r \leq n\} \cup \{E_{rs} + E_{sr} : 1 \leq r < s \leq n\} \\ \cup \{i(E_{rs} - E_{sr}) : 1 \leq r < s \leq n\}.$$

Then $A = (a_{ij})$ is completely determined.

- If we know that $\text{tr} A = 1$, we may skip the checking of $\text{tr} AE_{nn}$.

- If $\rho = (\rho_{ij}) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} (\bar{x}_1, \dots, \bar{x}_n)$ is a pure state, we only need to get information for $\rho_{12}, \dots, \rho_{1n}$. If $x_1 > 0$, then we can solve x_1 in the equation

$$x_1^2 + \sum_{j=2}^n |\rho_{1j}/x_1|^2 = 1.$$

Thus, one only need to check $\text{tr} \rho X$ for

$$X = E_{1j} + E_{j1}, X = i(E_{1j} - E_{j1}), j = 2, \dots, n.$$

- Can we write a computer program to do that?
- Can we set up physical experiments to to that?
- Consider the Pauli matrices: $\sigma_0 = I_2$,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For an n -qubit states in M_{2^n} , the test set can be

$$\{T_1 \otimes \dots \otimes T_n : T_j \in \{\sigma_0, \sigma_x, \sigma_y, \sigma_z\}\}.$$

- Suppose $\mathcal{S} \subseteq M_n$ is a special set of quantum states. Can we find a small test set \mathcal{S} of observable to determine whether $\rho \in \mathcal{S}$ or not?

- For example, determine all ρ with specific norm, eigenvalues, specific the Renyi entropy

$$H_\alpha = \frac{1}{1-\alpha} \log \text{tr} \rho^\alpha \text{ for } \alpha \in (0, 1) \cup (1, \infty).$$

For $\alpha = 2$, we get

$$H_2(\rho) = -\text{tr} \log \rho^2$$

When $\alpha \rightarrow 1$, we get the

von Neumann entropy

$$H(\rho) = -\text{tr}(\rho \log \rho).$$

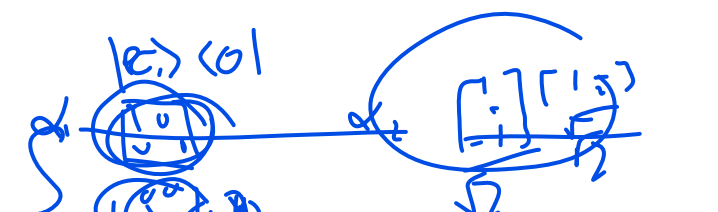
$I, X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
 $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\sqrt{2} \rho = 1$ | $\text{tr} \frac{Y}{2} \rho = \frac{\rho_{12} + \rho_{21}}{2}$
 $\text{tr} \frac{X}{2} \rho = \frac{\rho_{11} + \rho_{22}}{2}$ | $= \text{tr}(\rho)$
 $= \frac{\rho_{11} + \rho_{21}}{2}$ | $= \frac{\rho_{11} + \rho_{22}}{2}$

$\rho \in M_n$ 6 measurements
 $\text{tr} \rho = 1$
 $\rho = \sigma_i \otimes \tau_j$



$\rho = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}$
 $\rho = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}$



$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Multipartite states

Determine multipartite states with special properties.

- Let $\rho_1 \in M_{n_1}, \rho_2 \in M_{n_2}$. Determine the set $\mathcal{S}(\rho_1, \rho_2) = \{\rho \in M_{n_1} \otimes M_{n_2} : \text{tr}_1 \rho = \rho_1, \text{tr}_2 \rho = \rho_2\}$.

- One may consider the special case when $\rho_2 = I_{n_2}$.
- Determine all possible norms, eigenvalues, Renyi entropy of $\rho \in \mathcal{S}(\rho_1, \rho_2)$.

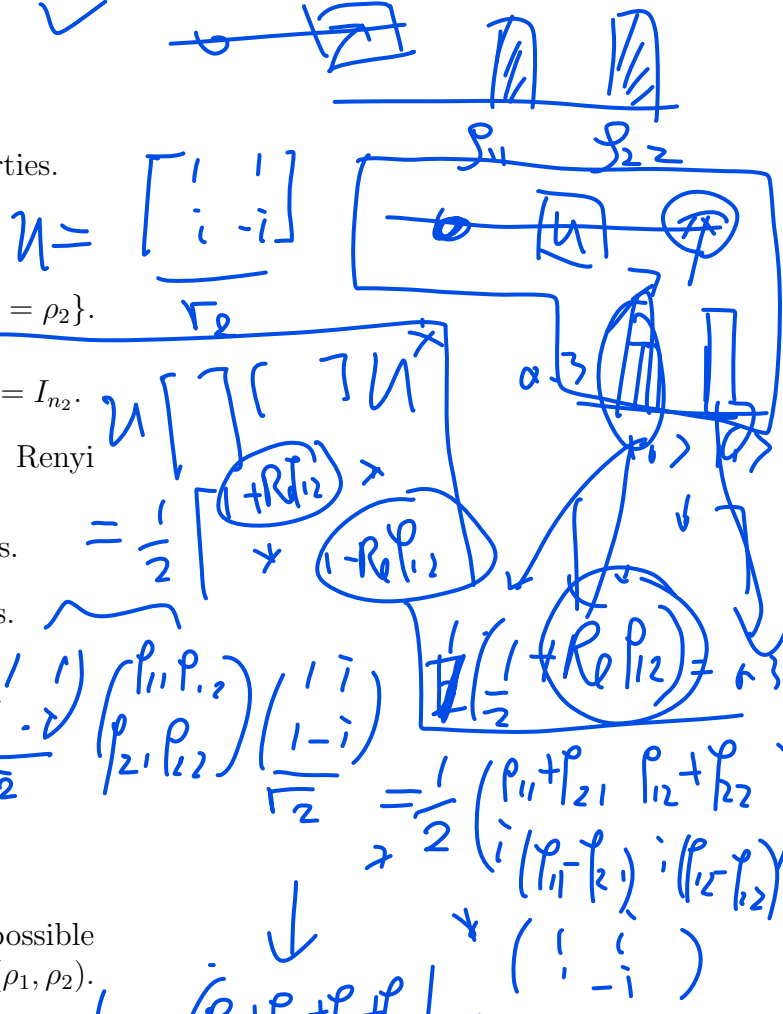
Use projection methods to find the elements.

- Extend the problems to multipartite systems.
- For example, determine the set of states

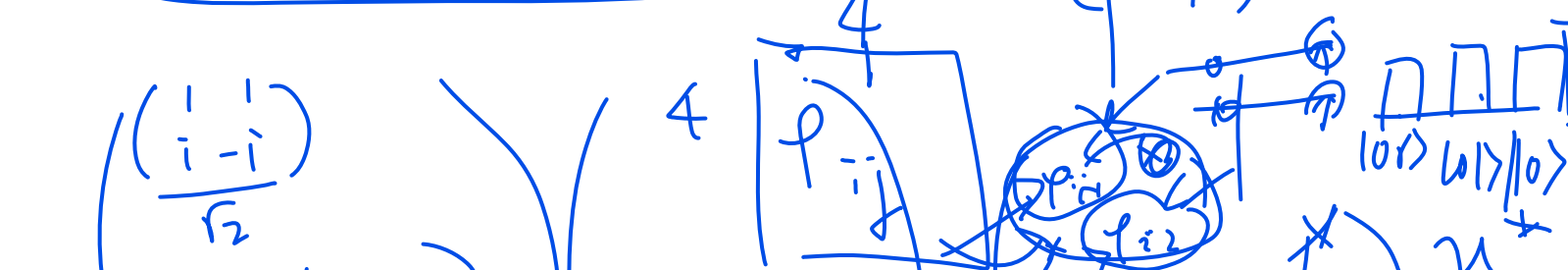
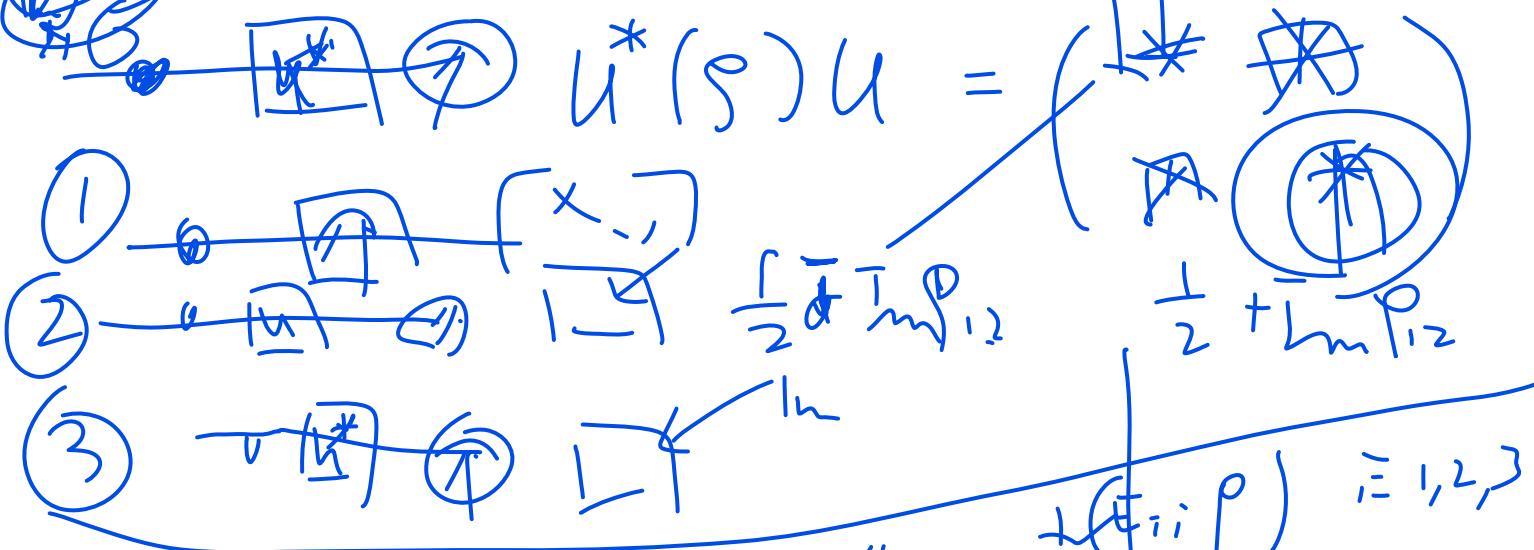
$$\rho \in M_{n_1} \otimes M_{n_2} \otimes M_{n_3}$$

with specific $\text{tr}_1(\rho) = \rho_{23} \in M_{n_2} \otimes M_{n_3}$,
and $\text{tr}_3(\rho) = \rho_{12} \in M_{n_1} \otimes M_{n_2}$.

- If the set is non-empty, determine all possible norms, eigenvalues, Renyi entropy of $\rho \in \mathcal{S}(\rho_1, \rho_2)$.



$$\frac{1}{2} (1 + 2 \text{Re} p_{12}) = \frac{1}{2} \begin{pmatrix} p_{11} + p_{21} & p_{12} + p_{22} \\ i(p_{11} - p_{21}) & i(p_{12} - p_{22}) \end{pmatrix}$$



Quantum Operations/channels

- Quantum operations $\mathcal{E} : M_n \rightarrow M_n$ of a close system with density matrices in M_n is a unitary similarity transform

$$\mathcal{E}(A) = UAU^\dagger, \quad A \in M_n,$$

where $U \in M_n$ is unitary.

- Here $U = U_t$ may be a function of t : time.
- A mixed unitary channel \mathcal{E} has the form

$$\mathcal{E}(A) = \sum_{j=1}^r p_j U_j A U_j^\dagger, \quad A \in M_n,$$

where $U_1, \dots, U_r \in M_n$ are unitary, and p_1, \dots, p_r are positive numbers summing up to 1.

- For an open system (with may interact with the environment, $\mathcal{E} : M_n \otimes M_n$ has the form

$$\mathcal{E}(A) = \text{tr}_2(U(A \otimes B)U^\dagger) = \sum_{j=1}^r F_j \rho F_j^\dagger, \quad A \in M_n,$$

where $F_1, \dots, F_r \in M_n$ satisfy $\sum_{j=1}^r F_j^\dagger F_j = I_n$.

- More generally, a quantum operations $\mathcal{E} : M_n \rightarrow M_m$ has the operator sum representation

$$\mathcal{E}(A) = \text{tr}_2(U(A \otimes B)U^\dagger) = \sum_{j=1}^r F_j \rho F_j^\dagger, \quad A \in M_n,$$

where $F_1, \dots, F_r \in M_{m,n}$ satisfy $\sum_{j=1}^r F_j^\dagger F_j = I_n$.

- System tomography can be done by determining the states

$$\mathcal{E}(E_{kk}), \quad \mathcal{E}(E_{ii} + E_{jj} + E_{ij} + E_{ji})/2,$$

$$\text{and } \mathcal{E}(E_{ii} + E_{jj} + iE_{ij} - iE_{ji})/2$$

for $k = 1, \dots, n, 1 \leq i < j \leq n$.

Handwritten notes and diagrams:

- Top right: $U \begin{pmatrix} \times & (p_{r,j}) \\ & \end{pmatrix} \begin{pmatrix} 1.40 \\ 0 \\ 0 \end{pmatrix}$
- Below that: $\begin{pmatrix} 1.6 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
- Center: $I \otimes U_1$ and U_1 circled.
- Right side: $A_{11}, A_{12}, A_{21}, A_{22}$ circled, with U_1^\dagger and U_1 arrows.
- Bottom right: $U_1 A_{11} U_1^\dagger$ and $U_1 A_{12} U_1^\dagger$ circled.
- Bottom right: 0 circled.