Research questions about quantum states

Quantum State tomography Determine $\rho = (\rho_{ij})$.

- For a Hermitian matrix $A = (a_{ij})$, we can determine $\mathrm{tr}\,(a_{ij})X$ for $X \in \mathcal{B}$, where

$$\mathcal{B} = \{E_{rr} : 1 \le r \le n\} \cup \{E_{rs} + E_{sr} : 1 \le r < s \le n\}$$
$$\cup \{i(E_{rs} - E_{sr}) : 1 \le r < s \le n\}.$$

  Then $A = (a_{ij})$ is completely determined.

- If we know that $\mathrm{tr}\, A = 1$, we may skip the checking of $\mathrm{tr}\, AE_{nn}$.

- If $\rho = (\rho_{ij}) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} (\bar{x}_1, \dots, \bar{x}_n)$ is a pure state, we only need to get information for $\rho_{12}, \dots, \rho_{1n}$. If $x_1 > 0$, then we can solve x_1 in the equation

$$x_1^2 + \sum_{j=2}^{n} |\rho_{1j}/x_1|^2 = 1.$$

  Thus, one only need to check $\mathrm{tr}\,\rho X$ for

  $X = E_{1j} + E_{j1}$, $X = i(E_{1j} - E_{j1})$, $j = 2, \dots, n$.

- Can we write a computer program to do that?

- Can we set up physical experiments to to that?

- Consider the Pauli matrices: $\sigma_0 = I_2$,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

  For an n-qubit states in M_{2^n}, the test set can be

$$\{T_1 \otimes \cdots \otimes T_n : T_j \in \{\sigma_0, \sigma_x, \sigma_y, \sigma_z\}\}.$$

- Suppose $\mathcal{S} \subseteq M_n$ is a special set of quantum states. Can we find a small test set $\mathcal{S}$ of observable to determine whether $\rho \in \mathcal{S}$ or not?

- For example, determine all ρ with specific norm, eigenvalues, specific the Renyi entropy

  $H_\alpha = \frac{1}{1-\alpha} \log \mathrm{tr}\, \rho^\alpha$ for $\alpha \in (0,1) \cup (1, \infty)$.

  For $\alpha = 2$, we get $\qquad H_2(\rho) = -\mathrm{tr}\, \log \rho^2$

  When $\alpha \to 1$, we get the

  von Neumann entropy $\qquad H(\rho) = -\mathrm{tr}\,(\rho \log \rho)$.

I, $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\mathrm{tr}\, I \rho = 1$

$\mathrm{tr}\, \frac{X}{2} \rho = \mathrm{Re}\, \rho_{12} = \mathrm{Re}\, \rho_{21}$

$\mathrm{tr}\, \frac{Y}{2} \rho = \mathrm{Im}(\rho_{21}) = -\mathrm{Im}(\rho_{12})$

$\mathrm{tr}\, Z \rho = \rho_{11} - \rho_{22}$

$\rho \in M_4$ 6 measurements

if $\rho = xx^*$

or if $\rho = \gamma_1 \otimes I_1$

ρ_{12}, ρ_{23}, ρ_{32} | ρ_{11}, ρ_{22}, ρ_{33}, ρ_{44}

$\rho = [\rho_{ij}]$

$|0\rangle\langle 0|$

$|1\rangle$

$\sqrt{\mathrm{tr}\,\rho}$

$|\alpha|^2$ $|\beta|^2$

$\frac{1}{\sqrt{2}}$

$= \frac{1}{2}$

Multipartite states

Determine multipartite states with special properties.

- Let $\rho_1 \in M_{n_1}, \rho_2 \in M_{n_2}$. Determine the set

  $$\mathcal{S}(\rho_1, \rho_2) = \{\rho \in M_{n_1} \otimes M_{n_2} : \operatorname{tr}_1 \rho = \rho_2, \operatorname{tr}_1 \rho = \rho_2\}.$$

- One may consider the special case when $\rho_2 = I_{n_2}$.
- Determine all possible norms, eigenvalues, Renyi entropy of $\rho \in \mathcal{S}(\rho_1, \rho_2)$.

  Use projection methods to find the elements.
- Extend the problems to multipatite systems.
- For example, determine the set of states

  $$\rho \in M_{n_1} \otimes M_{n_2} \otimes M_{n_3}$$

  with specific $\operatorname{tr}_1(\rho) = \rho_{23} \in M_{n_2} \otimes M_{n_3}$,

  and $\operatorname{tr}_3(\rho) = \rho_{12} \in M_{n_1} \otimes M_{n_2}$.
- If the set is non-empty, determine all possible norms, eigenvalues, Renyi entropy of $\rho \in \mathcal{S}(\rho_1, \rho_2)$.

Quantum Operations/channels

- Quantum operations $\mathcal{E} : M_n \to M_n$ of a close system with density matrices in M_n is a unitary similarity transform

$$\mathcal{E}(A) = UAU^\dagger, \qquad A \in M_n,$$

where $U \in M_n$ is unitary.

- Here $U = U_t$ may be a function of t: time.

- A mixed unitary channel $\mathcal{E}$ has the form

$$\mathcal{E}(A) = \sum_{j=1}^{r} p_j U_j A U_j^\dagger, \qquad A \in M_n,$$

where $U_1, \ldots, U_r \in M_n$ are unitary, and $p_1, \ldots, p_r$ are positive numbers summing up to 1.

- For an open system (with may interact with the environment, $\mathcal{E} : M_n \otimes M_n$ has the form

$$\mathcal{E}(A) = \mathrm{tr}_2(U(A \otimes B)U^\dagger) = \sum_{j=1}^{r} F_j \rho F_j^\dagger, \quad A \in M_n,$$

where $F_1, \ldots, F_r \in M_n$ satisfy $\sum_{j=1}^{r} F_j^\dagger F_j = I_n$.

- More generally, a quantum operations $\mathcal{E} : M_n \to M_m$ has the operator sum representation

$$\mathcal{E}(A) = \mathrm{tr}_2(U(A \otimes B)U^\dagger) = \sum_{j=1}^{r} F_j \rho F_j^\dagger, \quad A \in M_n,$$

where $F_1, \ldots, F_r \in M_{m,n}$ satisfy $\sum_{j=1}^{r} F_j^\dagger F_j = I_n$.

- System tomography can be done by determining the states

$$\mathcal{E}(E_{kk}),\ \mathcal{E}(E_{ii} + E_{jj} + E_{ij} + E_{ji})/2,$$

$$\text{and}\ \ \mathcal{E}(E_{ii} + E_{jj} + iE_{ij} - iE_{ji})/2$$

for $k = 1, \ldots, n$, $1 \le i < j \le n$.