Multipartite systems

- Suppose $|v_1\rangle \in \mathbb{C}^m, |v_2\rangle \in \mathbb{C}^n$ are quantum state. Then the $|v_1\rangle \otimes |v_2\rangle = |v_1 v_2\rangle$ is a **composite state** (**uncorrelated state**) in the bipartite system.

- For example, $|v_1\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, |v_2\rangle = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, then

$$|v_1\rangle \otimes |v_2\rangle = |v_1\rangle|v_2\rangle = |v_1 v_2\rangle = \begin{bmatrix} a_1|v_2\rangle \\ a_2|v_2\rangle \end{bmatrix} = \begin{bmatrix} a_1b_1 \\ a_1b_2 \\ a_2b_1 \\ a_2b_2 \end{bmatrix}.$$

$$|v\rangle = \begin{bmatrix} v_1 \\ v_2 \\ v_2 \\ v_3 \end{bmatrix} = \begin{pmatrix} a \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ b \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \end{pmatrix}$$

- A state $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ is **entangled** if it is not a composite state.

- The orthonormal basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ for $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$ consists of decomposable states.

- The orthonormal basis

$$\Big\{\tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle),\ \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$
$$\tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle),\ \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\Big\}$$

  consists of entangled states known as Bell states.

- Suppose an observable corresponds to the Hermitian matrix with eigenvectors $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, say, $H = \mathrm{diag}\,(3/2, 1/2, -1/2, -3/2)$.

  Then the measurement of $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ will yield $|00\rangle$ or $|11\rangle$ each with 50%.

  In particular, the first Schrödinger cat is alive (dead) if and only if the second one is alive (dead).

- We can construct **multipartite system** from k systems to get $\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k} = \mathbb{C}^{n_1 \cdots n_k}$.

- For example, $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ is a 3 qubit system.

Quantum operations on multipartite systems

We focus on qubit systems.

- Local (unitary) operations. If $U_1, U_2 \in M_2$ are unitary, then $U_1 \otimes U_2$ is unitary

$$(U_1 \otimes U_2)|v_1 v_2\rangle = |U_1 v_1\rangle|U_2 v_2\rangle.$$

- General $U \in M_4$ is a product of local unitary gates $U_1 \otimes U_2$ and controlled unitary gates of the form $I_2 \oplus V$ and $V \oplus I_2$.

- Proof. Let U be unitary.

  Find $P_1 = U_1 \otimes V_1$ so that $P_1 U$ has zero $(4,1)$ entry.

  Find $P_2 = U_2 \oplus I_2$ so that $P_2 P_1 U$ has zero $(4,1)$ and $(2,1)$ entry.

  Find $P_3 = U_3 \otimes I_2$ so that the first column of $P_3 P_2 P_1 U$ is $(1,0,0,0)^t$. Then $P_2 P_1 = [1] \oplus B$.

  Find $P_4 = I_2 \oplus V_4$ such that $P_4 P_3 P_2 P_1 U$ has zero $(3,2)$ entry.

  Find $P_5 = U_5 \otimes I_2$ such that $P_5 \cdots P_1 U = I_2 \oplus V_6$.

  If $P_6 = I_2 \oplus V_6^\dagger$, then $U = P_1^\dagger \cdots P_6^\dagger$.

- We can represent the operations on a circuit diagrams, and implement the operations using a quantum computers.

  See [Nakahara and Ohmi, Chapter 4].

- We only need to check the actions of the quantum operations on measurable states, say, $|000\rangle, |001\rangle, \ldots, |111\rangle$.

- The standard gates and basic gates might vary from different quantum computer.

- We are working on a research project requiring a decomposition of a unitary $U \in M_8$ into simple unitary gates.

$U = u_1 \cdots u_m$

$(u_1 \otimes u_2)$

$(u_1 \otimes u_2)|v\rangle = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \sum c_i \ldots$

$u_1|0\rangle \qquad u_2|0\rangle$

earth

mars

Mixed states and density matrices

A system is in a mixed state if there is a probability p_i that the system is in state $|x_i\rangle$ for $i = 1, \ldots, N$.

If $N = 1$, then the system is in pure state.

Consider an observable corresponds to the Hermitian matrix A.

- The mean value of the quantum system with quantum state $|x\rangle$ is given by $\langle A\rangle = \langle x|A|x\rangle$.
- The mean value of the quantum system with a mixed state $\sum_{j=1}^{N} p_j|x_j\rangle$ is given by

  $$\boxed{\langle A\rangle = \sum_{j=1}^{N} p_j\langle x_j|A|x_j\rangle = \mathrm{tr}\,(A\rho) = \mathrm{tr}\,(\rho A),}$$

  where density matrices

  $$\rho = \sum_{j=1}^{N} p_j|x_j\rangle\langle x_j|$$

  is a density operator (matrix).

Description of quantum systems in mixed states.

A1' A physical state is specified by a density matrix $\rho : \mathcal{H} \to \mathcal{H}$, which is positive semidefinite with trace equal to one.

A2' The mean value of an observable associate with the Hermitian matrix A is $\langle A\rangle = \mathrm{tr}\,(\rho A)$.

A3' The temporal evolution of the density matrix is given by the Liouville-von Neumann equation

$$i\hbar\frac{d}{dt}\rho = [H, \rho] = H\rho - \rho H,$$

where H is the system Hamiltonian.

Multipartitle systems

- Let $\rho_1 \in M_{n_1}, \rho_2 \in M_{n_2}$ be mixed states. Then

$$\rho_1 \otimes \rho_2 \in M_{n_1} \otimes M_{n_2} \equiv M_{n_1 n_2}$$

  is a composite (uncorrelated) state in the bipartite system.

- General states ρ in $M_{n_1} \otimes M_{n_2}$ are density matrices in $M_{n_1 n_2}$.

- Let ρ be a density matrix in the bipartite system $M_{n_1} \otimes M_{n_2}$. It is **separable** if it is a probabilistic (convex) combination of composite state, i.e.,

$$\rho = \sum_{j=1}^{N} p_j \sigma_j \otimes \tau_j$$

  with quantum states $\sigma_j \in M_{n_1}, \tau_j \in M_{n_2}$.

- Otherwise, it is **entangled**.

- Note that ρ is always a **linear combination** of composite states.

- Checking whether a state is separable is an NP-hard problem.

- A common test is to use the **partial transposes** defined by

$$(\rho_1 \otimes \rho_2)^{pt_1} = \rho_1^t \otimes \rho_2, \ \ (\rho_1 \otimes \rho_2)^{pt_2} = \rho_1 \otimes \rho_2^t.$$

- If ρ is separable, then the partial transposes are also quantum states, i.e., positive semi-definite.

- If ρ is a state such that ρ^{pt_1} is positive semidefinite, we say that ρ is a ppt (positive partial trace) state.

- One can extend the concepts to multipartite systems.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} B \end{bmatrix} = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix}$$

$$\begin{bmatrix} A_{11}^T & A_{12}^T \\ A_{21}^T & A_{22}^T \end{bmatrix}$$

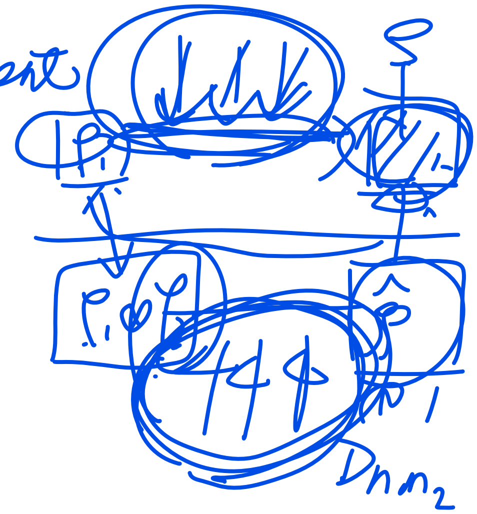
Partial traces

- If ρ is quantum state in the bipartite system $M_{n_1} \otimes M_{n_2}$, the partial traces of ρ are defined by

$$\mathrm{tr}_1(\rho_1 \otimes \rho_2) = \mathrm{tr}\,(\rho_1)\rho_2 = \rho_2 \in M_{n_2},$$

Environment

$$\mathrm{tr}_2(\rho_1 \otimes \rho_2) = \rho_1 \mathrm{tr}\,(\rho_2) = \rho_1 \in M_{n_1}.$$



- One may regard ρ_1 lies in the principal system, and ρ_2 is the environment.

- There are many problems concerning partial traces.

  * Given quantum states $\rho_1 \in M_m, \rho_2 \in M_n$, determine

$$\mathcal{S}(\rho_1, \rho_2) = \{\rho \in D_{n_1 n_2} : \mathrm{tr}_1(\rho) = \rho_2, \mathrm{tr}_2(\rho) = \rho_1\}.$$

  * Find a quantum state in $\mathcal{S}(\rho_1, \rho_2)$ with lowest rank.

  * Find a quantum state in $\mathcal{S}(\rho_1, \rho_2)$ with the lowest von Neumann entropy

$$S(\rho) = -\mathrm{tr}\,\rho \log \rho.$$

  * Find all possible eigenvalues of $\rho \in \mathcal{S}(\rho_1, \rho_2)$.

- One can extend the concepts to multipartite systems.

Research questions about quantum states

Quantum State tomography

Determine $\rho = (\rho_{ij})$.

- For a Hermitian matrix $A = (a_{ij})$, we can determine $\operatorname{tr}(a_{ij})X$ for $X \in \mathcal{B}$, where

$$\mathcal{B} = \{E_{rr} : 1 \le r \le n\} \cup \{E_{rs} + E_{sr} : 1 \le r < s \le n\}$$
$$\cup \{i(E_{rs} - E_{sr}) : 1 \le r < s \le n\}.$$

  Then $A = (a_{ij})$ is completely determined.

- If we know that $\operatorname{tr} A = 1$, we may skip the checking of $\operatorname{tr} AE_{nn}$.

- If $\rho = (\rho_{ij}) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} (\bar{x}_1, \dots, \bar{x}_n)$ is a pure state, we only need to get information for $\rho_{12}, \dots, \rho_{1n}$. If $x_1 > 0$, then we can solve x_1 in the equation

$$x_1^2 + \sum_{j=2}^{n} |\rho_{1j}/x_1|^2 = 1.$$

  Thus, one only need to check $\operatorname{tr} \rho X$ for

  $X = E_{1j} + E_{j1}$, $X = i(E_{1j} - E_{j1})$, $j = 2, \dots, n$.

- Can we write a computer program to do that?

- Can we set up physical experiments to to that?

- Consider the Pauli matrices: $\sigma_0 = I_2$,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

  For an n-qubit states in M_{2^n}, the test set can be

$$\{T_1 \otimes \cdots \otimes T_n : T_j \in \{\sigma_0, \sigma_x, \sigma_y, \sigma_z\}\}.$$

- Suppose $\mathcal{S} \subseteq M_n$ is a special set of quantum states. Can we find a small test set $\mathcal{S}$ of observable to determine whether $\rho \in \mathcal{S}$ or not?

- For example, determine all ρ with specific norm, eigenvalues, specific the Renyi entropy

  $H_\alpha = \frac{1}{1-\alpha} \log \operatorname{tr} \rho^\alpha$ for $\alpha \in (0,1) \cup (1, \infty)$.

  For $\alpha = 2$, we get $\qquad H_2(\rho) = -\operatorname{tr} \log \rho^2$.

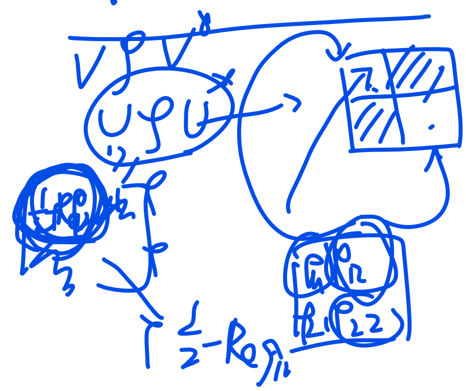
  When $\alpha \to 1$, we get the

  von Neumann entropy $\qquad H(\rho) = -\operatorname{tr}(\rho \log \rho)$.

$\rho \in D_\ell$

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = |0\rangle\langle 0|$ ✓

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = |1\rangle\langle 1|$ ✓



Multipartite states

Determine multipartite states with special properties.

- Let $\rho_1 \in M_{n_1}, \rho_2 \in M_{n_2}$. Determine the set

$$\mathcal{S}(\rho_1, \rho_2) = \{\rho \in M_{n_1} \otimes M_{n_2} : \operatorname{tr}_1 \rho = \rho_2, \operatorname{tr}_2 \rho = \rho_1\}.$$

- One may consider the special case when $\rho_2 = I_{n_2}/n_2$.
- Determine all possible norms, eigenvalues, Renyi entropy of $\rho \in \mathcal{S}(\rho_1, \rho_2)$.

  Use projection methods to find the elements.

- Extend the problems to multipatite systems.
- For example, determine the set of states

$$\rho \in M_{n_1} \otimes M_{n_2} \otimes M_{n_3}$$

  with specific $\operatorname{tr}_1(\rho) = \rho_{23} \in M_{n_2} \otimes M_{n_3}$,

  and $\operatorname{tr}_3(\rho) = \rho_{12} \in M_{n_1} \otimes M_{n_2}$.

- If the set is non-empty, determine all possible norms, eigenvalues, Renyi entropy of $\rho \in \mathcal{S}(\rho_1, \rho_2)$.

$\rho = \rho_1 \otimes \rho_2$

$H(\rho) = \operatorname{tr} \rho \log(\rho)$

$D_2 \otimes D_2 \otimes D_2 \cong D_8$

Projection methods and gradient methods

- It is difficult to construct mutlipartite states with prescribed reduced states with overlapped subsystems. For example, construct $\rho \in M_{n_1} \otimes M_{n_2} \otimes M_{n_3}$ with prescribed ρ_{12} and ρ_{23}.

- One may construct a tripartite state ρ with prescribed $\rho \in M_{n_1} \otimes M_{n_2} \otimes M_{n_2}$ using alternating projection methods between the two convex sets:

$$S_1 = \{\rho \in D_{n_1 n_2 n_3} : \operatorname{tr}_3(\rho) = \rho_{12}\},$$

$$S_2 = \{\rho \in D_{n_1 n_2 n_3} : \operatorname{tr}_1(\rho) = \rho_{23}\}.$$

- We know that set

$$\mathcal{S}(\rho_1, \rho_2) = \{\rho \in D_{n_1 n_2} : \operatorname{tr}_1(\rho) = \rho_2, \operatorname{tr}_2(\rho) = \rho_1\}$$

is non-empty. One may determine $\rho \in \mathcal{S}(\rho_1, \rho_2)$ with the maximum / minimum entropy $H(\rho)$, say, using gradient method, i.e., find the steepest descent direction $\nabla H(\rho)$ and change ρ to

$$\rho - t\nabla H(\rho)$$

for some suitable step size $t > 0$.