Bipartite and multipartite systems

A system may have two components described by two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Then the bipartite system is represented by $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. A general state in $\mathcal{H}$ has the form

$$|x\rangle = \sum_{i,j} c_{ij}|e_{1,i}\rangle \otimes |e_{2,j}\rangle \quad \text{with} \quad \sum_{i,j} |c_{ij}|^2 = 1,$$

where $\{|e_{r,1}\rangle, |e_{r,2}\rangle, \dots\}$ is an orthonormal basis for $\mathcal{H}_r$ with $r \in \{1, 2\}$.

Then $\{|e_{1,i}e_{2,j}\rangle : i = 1, 2, \dots, j = 1, 2, \dots\}$ is an orthonrmal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Example For example, $\mathbb{C}^2$ has orthonormal basis $\{|0\rangle, |1\rangle\}$ with

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$a|0\rangle + b|1\rangle$

$\begin{pmatrix} a \\ b \end{pmatrix}$

Then $\mathbb{C}^2 \otimes \mathbb{C}^2$ has orthonormal basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ consisting of the 4 columns of the identity matrix I_4.

Similarly, $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ has orthonormal basis $\{|000\rangle, \dots, |111\rangle\}$ consisting of the columns of I_8.

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|a|^2 + |b|^2 = 1$

In general, if $U = [|u_1\rangle \cdots |u_m\rangle]$ such that the columns of U form an orthonormal basis for $\mathbb{C}^m$, and $V = [|v_1\rangle \cdots |v_n\rangle]$ such that the columns of V form an orthonormal basis for $\mathbb{C}^n$, then the columns of $U \otimes V = [|u_1v_1\rangle \cdots |u_mv_n\rangle]$ form an orthonormal basis for $\mathbb{C}^m \otimes \mathbb{C}^n$.

$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ $|a|^2 + |b|^2 + |c|^2 + |d|^2$

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$m \cdot n$

$\boxed{\mathbb{C}^m \otimes \mathbb{C}^n}$

$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \dots \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

$|000\rangle$ $|111\rangle$

8 $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Separable states, entangled states, Schmidt decomposition

A state of the form $|x\rangle = |x_1\rangle \otimes |x_2\rangle$ is a **separable state** or a **tensor product state**. Otherwise, it is an **entangled state**.

Example Let

$$|x\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle = \begin{pmatrix} c_{00} \\ c_{01} \\ c_{10} \\ c_{11} \end{pmatrix} \in \mathbb{C}^2 \otimes \mathbb{C}^2.$$

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} \right\|$$

Question How to detect that it is a tensor state?

Answer Check whether the rows of the matrix $C = \begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix}$ are multiples of each other. If yes, we can write $C = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} (b_1 \ b_2)^t$ for some unit vectors $|u\rangle = (a_1, a_2)^t, |v\rangle = (b_1, b_2)^t$. Then $|x\rangle = |u\rangle \otimes |v\rangle$. If not, $|x\rangle$ is entangled.

$$\langle u \otimes v | u \otimes v \rangle = \langle u|u\rangle \langle v|v\rangle$$

Remark Most states in $\mathcal{H}_1 \otimes \mathcal{H}_2$ are entangled states, which are most useful for quantum computing.

$$\mathbb{C}^4 \quad \| \quad \mathbb{C}^2 \otimes \mathbb{C}^2 \qquad \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \qquad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} (b_1, b_2) \qquad \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \begin{pmatrix} c \\ d \end{pmatrix} \\ b \begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix}$$

Theorem Suppose $\mathcal{H}_1, \mathcal{H}_2$ have finite dimensions, say, m and n. Every state $|x\rangle$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ admits a **Schmidt decomposition**

$r \le \min\{m, n\}$

$$\mathbb{C}^m \otimes \mathbb{C}^n \quad |x\rangle = \sum_{j=1}^{r} s_j |u_j\rangle \otimes |v_j\rangle, \qquad |x\rangle = \sum c_{ij} |e_{1,i}\rangle|e_{2,j}\rangle$$

$|x\rangle \in \mathbb{C}^{mn}$

where $s_j > 0$ are the Schmidt coefficients satisfying $\sum_{j=1}^{r} s_j^2 = 1$, r is the **Schmidt number** of $|x\rangle$, $\{|u_1\rangle, \ldots, |u_r\rangle\}$ is an orthonormal set of $\mathcal{H}_1$ and $\{|v_1\rangle, \ldots, |v_r\rangle\}$ is an orthonormal set of $\mathcal{H}_2$.

$\cong \mathbb{C}^m \otimes \mathbb{C}^n$

Proof. Assume $\mathcal{H}_1$ and $\mathcal{H}_2$ orthonormal bases $\{|e_{1,1}, \ldots, |e_{1,m}\rangle\}$ and $\{|e_{2,1}, \ldots, |e_{2,n}\rangle\}$. Every state has the form

$\mathbb{C}^m \otimes \mathbb{C}^n$

$$|x\rangle = \sum_{j=1} c_{rs}|e_{1,r}\rangle \otimes |e_{2,s}\rangle$$

$|e_i\rangle \otimes |f_j\rangle$

If C has rank one, then $C = (a_1, \ldots, a_m)^t(b_1, \ldots, b_n)$ so that $C = |u\rangle \otimes |v\rangle$ with $|u\rangle = \sum_{j=1}^{m} a_j|e_{1,j}\rangle$ and $|v\rangle = \sum_{j=1}^{n} b_j^*|e_{2,j}\rangle$. Because $\||x\rangle\| = 1$, we may assume that $(a_1, \ldots, a_m)^t$ and $(b_1, \ldots, b_n)^t$ are unit vectors and so are $|u\rangle, |v\rangle$.

In general, suppose $C = [c_{ij}]$ has singular decomposition

$$\sum_{j=1}^{r} s_j|\alpha_j\rangle\langle|\beta_j| = \sum_{j=1}^{r} s_j C_j,$$

$\sum s_j |\alpha_j\rangle\langle\beta_j|$

where $C_j = |\alpha_j\rangle\langle\beta_j|$ for $j = 1, \ldots, r$.

One can then use C_j as the coefficient matrix of $|x_j\rangle$ to construct tensor state $|x_j\rangle = |u_j\rangle \otimes |v_j\rangle$ so that

$$|x\rangle = \sum_{j=1}^{r} s_j|x_j\rangle = \sum_{j=1}^{r} s_j|u_j\rangle \otimes |v_j\rangle. \qquad \square$$

Example Suppose $|x\rangle = \sum_{i,j} c_{ij}|e_{1,i}e_{2,j}\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^3$ with

$$(c_{ij}) = UDV^t = d_1|u_1\rangle\langle v_1| + d_2|u_2\rangle\langle v_2|,$$

where

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad D = \frac{1}{5}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}, \quad V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \end{pmatrix}.$$

Then

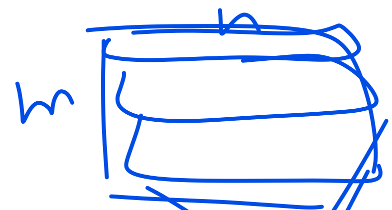
$$|x\rangle = \frac{4}{5}|u_1\rangle|v_1\rangle + \frac{3}{5}|u_2\rangle|v_2\rangle,$$

where

$$|u_1\rangle = (1, i)^t/\sqrt{2},\ |u_2\rangle = (1, -i)^t/\sqrt{2},$$

$$|v_1\rangle = (1, 1, 0)^t/\sqrt{2},\ |v_2\rangle = (0, 0, 1)^t.$$

Remark Extending the results to $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ for $k \geq 3$ is an open problem.



$|x\rangle = \sum s_j |x_j\rangle|y_j\rangle|z_j\rangle$

$\mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^p$

$\mathbb{R}^m \otimes \mathbb{R}^n \otimes \mathbb{R}^p$

No-cloning theorem

Theorem (Wootters and Zurek) An unknown quantum system cannot be cloned by unitary transformations.

Proof. Suppose there would exist a unitary transformation U that makes a clone of a quantum system. Namely, suppose U acts, for any state $|\varphi\rangle$, as

$$U : |\varphi 0\rangle \to |\varphi\varphi\rangle.$$

Let $|\varphi\rangle$ and $|\phi\rangle$ be two states that are linearly independent. Then we should have $U|\varphi 0\rangle = |\varphi\varphi\rangle$ and $U|\phi 0\rangle = |\phi\phi\rangle$ by definition. Then the action of U on $|\psi\rangle = \frac{1}{\sqrt{2}}(|\varphi\rangle + |\phi\rangle)$ yields

$$U|\psi 0\rangle = \frac{1}{\sqrt{2}}(U|\varphi 0\rangle + U|\phi 0\rangle) = \frac{1}{\sqrt{2}}(|\varphi\varphi\rangle + |\phi\phi\rangle).$$

If U were a cloning transformation, we must also have

$$U|\psi 0\rangle = |\psi\psi\rangle = \frac{1}{2}(|\varphi\varphi\rangle + |\varphi\phi\rangle + |\phi\varphi\rangle + |\phi\phi\rangle),$$

which contradicts the previous result. Therefore, there does not exist a unitary cloning transformation. $\square$

Remark There is proof using the fact that information cannot be transmitted faster than light speed. See the supplementary note.

Qubits

- Mathematically, qubit is a vector in $|x\rangle = a|0\rangle + b|1\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2$ with $|a|^2 + |b|^2$ realized by physical quantum states such as the vertically and horizontally polarized photons, or spin 1/2 in NMR system.

- Note that measurement will give $|0\rangle$ or $|1\rangle$ even a qubit can assume infinitely many states. The probability for the measurement on $|x\rangle$ yielding $|0\rangle$ is $\langle x|(|0\rangle\langle 0|)|x\rangle = |a|^2$.

- Even if we can get the information $|a|$ and $|b|$ by **measuring** many identical $|x\rangle$ if it is available, we cannot get complete information of $|x\rangle\langle x|$.

- Using the measurable states $P_1 = |0\rangle\langle 0|, P_2 = |1\rangle\langle 1|$ to get information of $\langle x|P_1 x\rangle, \langle x|P_2 x\rangle$, we have the "diagonal entries" of $\rho = |x\rangle\langle x|$, which are $|a|^2, |b|^2$.

- In order to obtain complete information of $|x\rangle\langle x|$, we may apply unitary $U_1, \dots, U_r$ and measure the diagonal $U_j|x\rangle\langle x|U_j^\dagger$ to access information of the off-diagonal entries. Such study is known as **state tomography** problem.

One may consider qutrits in $\mathbb{C}^3$ and qudits in $\mathbb{C}^n$.

Bloch sphere and Bloch ball

Since two unit vectors $|x\rangle$ and $e^{it}|x\rangle$ represent the same quantum state, it is convenient to use the rank one orthogonal projection $\rho = |x\rangle\langle x|$, which will be called a pure state, to represent the state.

$|x\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

$e^{i\theta}|x\rangle = e^{i\theta}\begin{pmatrix} a \\ b \end{pmatrix}$

More generally, one may consider the mixed state $\rho \in M_n$ of the form

$$\sum_{j=1}^{r} p_r |x_j\rangle\langle x_j|$$

$|x\rangle\langle x| = e^{i\theta}|x\rangle\langle x|e^{-i\theta}$

with probability vector $(p_1, \ldots, p_r)$ and pure states $|x_1\rangle\langle x_1|, \ldots, |x_r\rangle\langle x_r|$.

For qubits, a mixed state has the form

$$\rho = \frac{1}{2}(I_2 + u \cdot \sigma) = \frac{1}{2}(\sigma_0 + u_1\sigma_1 + u_2\sigma_2 + u_3\sigma_3)$$

$U \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} a^* & b^* \end{pmatrix} U^*$

$= \begin{pmatrix} |a|^2 & ab^* \\ ba^* & |b|^2 \end{pmatrix}$

with $|u| = \sqrt{u_1^2 + u_2^2 + u_3^2} \le 1$.

$R_x \; R_y \qquad S^2$

Here $(\sigma_1, \sigma_2, \sigma_3) = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices"

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Bloch

$= \begin{pmatrix} \frac{1}{2} + \frac{\alpha}{2} & \frac{\beta}{2} - i\frac{\gamma}{2} \\ \frac{\beta}{2} + i\frac{\gamma}{2} & \frac{1}{2} - \frac{\alpha}{2} \end{pmatrix}$

- The eigenvalues of ρ are $\frac{1}{2}(1 \pm |u|)$.
- ρ is a pure state if and only if $|u| = 1$.
- In such a case, we may let

$$u = (u_1, u_2, u_3) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

$\frac{1}{2}\begin{pmatrix} 1+\alpha & \beta - i\gamma \\ \beta + i\gamma & 1-\alpha \end{pmatrix}$

$1 - \alpha^2 - \beta^2 - \gamma^2$

$\frac{1}{2} I_2 + \alpha\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \beta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \gamma\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Multi-qubit systems and entangled states

Given n qubits $|x_1\rangle, \ldots, |x_n\rangle$, we can consider the tensor product $|x_1\rangle \otimes \cdots \otimes |x_n\rangle \in \mathbb{C}^N$ with $N = 2^n$. Most state vectors

$$\sum_{i_k=0,1} a_{i_1 \cdots i_n} |x_{i_1}\rangle \otimes \cdots \otimes |x_{i_n}\rangle \in \mathbb{C}^N$$

are entangled state vectors, which are not of the tensor form.

Notation We often assume $|x_j\rangle \in \{|0\rangle, |1\rangle\}$, and regard

$$|x\rangle = |x_{i_1} \cdots x_{i_n}\rangle = |q_{n-1} \cdots q_0\rangle$$

$|q_{n-1}\rangle \cdots \otimes |q_0\rangle$

as a binary number, and

$$|\psi\rangle = \sum_{i_k=0,1} a_{i_1 \cdots i_n} |x_{i_1} \cdots x_{i_n}\rangle.$$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Example

$$|x\rangle = \frac{1}{2} \boxed{\sum_{i,j\in\{0,1\}}} |ij\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \boxed{\frac{1}{2} \sum_{x=0}^{3} |x\rangle}.$$

$0 \quad 1 \quad 2 \quad 3$

$I_2 = \sigma_0$

$|x\rangle = \frac{1}{\sqrt{8}} \left(|000\rangle + \cdots + |111\rangle \right) \quad \frac{1}{2} \sum_{x=0} |x\rangle$

In quantum computing, we often implement quantum operation of the form:

$$\sum_x |x\rangle|0\rangle \to \sum_x |x\rangle|f(x)\rangle.$$

$f: \{0,1\} \to \{0,1\}$

For example, if $f(0) = f(1) = 1$, there is U such that

$$U|00\rangle = |01\rangle, \quad U|10\rangle = |11\rangle.$$

$f(0) = f(1) = 1$

There are many choices for U. For example, we may set $U|01\rangle = |00\rangle, U|11\rangle = |10\rangle$.