$\sum (c_{ij}) |e_i f_j\rangle$ $\qquad \mathbb{C}^m \otimes \mathbb{C}^n$

**Some important entangled states**

**Example** The Bell states

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

are entangled states and form an orthonormal basis for the two qubit systems.

$|000\rangle \qquad |111\rangle$

**Example** In the 3 qubit system, we have that GHZ state and W state:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \quad \text{and} \quad |W\rangle \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle).$$

$\in \mathbb{C}^8 \qquad \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \qquad |v_1\rangle \otimes |v_2\rangle$

$\sum (c_{ij}) |e_i\rangle |f_j\rangle$

$= \sum \ldots$

**Example** One can do measurement of the first qubit for a state vector in a $n$ qubit system. For instance,

$$|x\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle, \quad |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1.$$

We measure the first qubit with respect to the basis $\{|0\rangle, |1\rangle\}$. Set

$$|x\rangle = |0\rangle(a|0\rangle + b|1\rangle) + |1\rangle(c|0\rangle + d|1\rangle)$$

$$= u|0\rangle((a/u)|0\rangle + (b/u)|1\rangle) + v|1\rangle((c/v)|0\rangle + (d/v)|1\rangle),$$

where $u = \sqrt{|a|^2 + |b|^2}$ and $v = \sqrt{|c|^2 + |d|^2}$. We can measure the first qubit, say, by setting $A = (|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes I_2$ so that

$$M_0 = |0\rangle\langle 0| \otimes I_2, \quad M_1 = |1\rangle\langle 1| \otimes I_2.$$

Applying $M_0$ and $M_1$, we obtain 0 with probability $\langle x|M_0|x\rangle = u^2$ and 1 with probability $v^2$; the state $|x\rangle$ collapses to

$$|0\rangle \otimes ((a/u)|0\rangle + (b/u)|1\rangle) \text{ and } |1\rangle \otimes ((c/v)|0\rangle + (d/v)|1\rangle),$$

respectively, upon measurement.

$a|0\rangle + c|1\rangle$

$b|0\rangle + d|1\rangle$

**Einstein-Podolsky-Rosen (EPR) Phenomenon**

- Consider the EPR state

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

  Alice gets the first particle and Bob gets the second one.

- When Alice measures, Bob's particle will change instantaneously to $|1\rangle$ or $|0\rangle$ depending on the measured outcome of Alice being $|0\rangle$ or $|1\rangle$.

- For example, set up the apparatus for the observable

$$A = |0\rangle\langle 0| \otimes I_2 - |1\rangle\langle 1| \otimes I_2.$$

- If Alice sees the reading 1, then Bob's qubit is to $|1\rangle$; if Alice sees the reading $-1$, then Bob's qubit is $|0\rangle$.

- Alice cannot control her measurement and hence the reading of Bob! So, it does not violate the special theory of relativity. (It is impossible that information travels faster than light!)

- However, they can measure their individual states around the same time, and decide to make a move according to $|01\rangle$ or $|10\rangle$ occur.

- Bell proposed an experiment which confirmed that there cannot be a hidden rule governing the measurement of the entangled pair.

## 0.1 Bell inequality

About 30 years after the EPR paper was published, an experiment test was proposed to check whether the measurement of entangled pairs follow a certain predetermined rule imposed by Nature, or the postulate of quantum mechanics.

Here is the proposed experiments. Suppose Charlie prepares an entangled pair of qubits (photons or particles) and sends the first one to Alice and the second one to Bob. Alice will apply one of her two measurement schemes, say, $Q$ and $R$, each will produce a measured value in $\{1, -1\}$. Bob will also apply one of his two measurement schemes, say, $S$ and $T$, each will produce a measured value in $\{1, -1\}$.

Let us consider

$$QS + RS + RT - QT = (Q + R)S + (R - Q)T.$$

Because $R, Q \in \{1, -1\}$, it follows that either $(Q + R)S = 0$ or $(R - Q)T = 0$. As a result, $QS + RS + RT - QT \in \{2, -2\}$.

Suppose there is a hidden rule governing the measurement outcomes, and $\boxed{p(q, r, s, t)}$ is the probability that, *before* the measurements are performed, the system is in the state $(Q, R, S, T) = (q, r, s, t)$. Then the expectation value $E(QS + RS + RT - QT) = E(QS) + E(RS) + E(RT) - E(QT)$ satisfies

$$\begin{aligned} |E(QS + RS + RT - QT)| &= \sum_{(q,r,s,t)} p(q, r, s, t)|qs + rs + rt - qt| \\ &\leq \sum_{(q,r,s,t)} p(q, r, s, t) \cdot 2 = 2. \end{aligned}$$

So, we get the **Bell inequality**

$$|E(QS) + E(RS) + E(RT) - E(QT)| \leq 2. \qquad (0.1)$$

Suppose Charlie prepares an entangled state

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

and gives Alice the first qubit, and Bob the second one. Alice uses the measurement operators $Q = \sigma_z$ and $R = \sigma_x$, and Bob uses the measurement operators $S = \frac{-1}{\sqrt{2}}(\sigma_z + \sigma_x)$ and $T = \frac{1}{\sqrt{2}}(\sigma_z - \sigma_x)$. Then

$$E(QS) = \langle\Psi^-|Q \otimes S|\Psi^-\rangle = \frac{1}{\sqrt{2}}, \quad E(RS) = \langle\Psi^-|R \otimes S|\Psi^-\rangle = \frac{1}{\sqrt{2}},$$

$$E(RT) = \langle\Psi^-|R \otimes T|\Psi^-\rangle = \frac{1}{\sqrt{2}}, \quad E(QT) = \langle\Psi^-|Q \otimes T|\Psi^-\rangle = \frac{-1}{\sqrt{2}},$$

and hence

$$E(QS + RS + RT - QT) = 4/\sqrt{2} = 2\sqrt{2}. \tag{0.2}$$

This equality clearly violates the Bell inequality.

To determine whether (0.1) or (0.2) is valid, Alice and Bob can estimate $E(QS)$ by performing measurements on many copies of $|\Psi^-\rangle$, and record their results. After the experiments, they can multiply their measurements when they used the measurement schemes $Q$ and $S$, respectively. Similarly, they can estimate $E(RS), E(RT), E(QT)$, so as to obtain an estimate of $E(QS + RS + RT - QT)$.

Experimental results showed strong support to (0.2). Hence, the EPR proposal that there is a hidden rule governing the measurement results of entangled pair was ruled out.

**Measurements**

For each outcome $m$, construct a measurement operator $M_m$ so that the probability of obtaining outcome $m$ in the state $|x\rangle$ is computed by

$$p(m) = \langle x|M_m^\dagger M_m|x\rangle$$

and the state immediately after the measurement is

$$|m\rangle = \frac{M_m|x\rangle}{\sqrt{p(m)}}.$$

**Example** Let $M = \{M_0, M_1\}$ with $M_0 = |0\rangle\langle 0|$ and $M_1 = |1\rangle\langle 1|$. Then for $|x\rangle = a|0\rangle + b|1\rangle$ with $a \neq 0$, $p(0) = |a|^2$, $M_0|x\rangle = a|0\rangle/|a|$, which is the same as the vector state $|0\rangle$.

- In general, suppose an observable $M$ is given with measurement operators $M_m$. Then setting $P_i = M_i^\dagger M_i$, we require that $\sum_m P_m = I_n$.
- If there are many copy of a state $|x\rangle$, then the expected value of $M$ is

  $$E(M) = \langle M\rangle = \sum_m mp(m) = \sum_m m\langle x|P_m|x\rangle = \langle x|M|x\rangle.$$

  Here $M$ can be identified with $\sum_m mP_m$.
- The standard derivation is

  $$\Delta(M) = \sqrt{\langle (M - \langle M\rangle)^2} = \sqrt{\langle M^2\rangle - \langle M\rangle^2}.$$

- The variance (square of standard deviation) is

  $$\langle (M - \langle M\rangle)^2\rangle = \langle x|M^2|x\rangle - \langle x|M|x\rangle^2.$$

**Another proof of no-cloning theorem**

The no-cloning theorem may be proved by using the special theory of relativity, which assumes no information can propagate faster than the speed of light.

Suppose Alice and Bob share a Bell state

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle) = \frac{1}{\sqrt{2}}(|+\rangle|-\rangle - |-\rangle|+\rangle).$$

where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. Readers are encouraged to verify the second equality. Alice keeps the first qubit while Bob keeps the second. If Alice wants to send Bob a bit "0", she measures her qubit in $\{|0\rangle, |1\rangle\}$ basis while if she wants to send "1", she employs $\{|+\rangle, |-\rangle\}$ basis for her measurement. Bob always measures his qubit in $\{|0\rangle, |1\rangle\}$ basis.

After Alice's measurment and before Bob's measurment, Bob's qubit is $|0\rangle$ or $|1\rangle$ if Alice sent "0" while it is $|+\rangle$ or $|-\rangle$ if Alice sent "1".

Suppose Bob is able to clone his qubit. He makes many copies of his qubit and measures them in $\{|0\rangle, |1\rangle\}$ basis. If Alice sent "0", Bob will obtain $0, 0, 0, \ldots$ or $1, 1, 1, \ldots$ while if she sent "1", Bob will obtain approximately 50% of 0's and 50% of 1's. Suppose Bob received $|\pm\rangle$ and made $N$ clones, then the probability of obtaining the same outcome is $1/2^{N-1}$, which is negligible if $N$ is sufficiently large. Note that Bob obtains the bit Alice wanted to send immediately after Alice's measurement assuming it does not take long to clone his qubit. This could happen even if Alice and Bob are separated many light years apart, thus in contradiction with the special theory of relativity. $\square$

$p_1 |x_1\rangle\langle x_1| + p_2 |x_2\rangle\langle x_2|$

**Mixed States and Density Matrices**

- A system is in a mixed state if there is a (classical) probability $p_i$ that the system is in state $|x_i\rangle$ for $i = 1, \ldots, N$.
- If there is only one possible state, i.e., $p_1 = 1$, then the system is in pure state.
- The expectation value (mean) of the measurement of the system corresponding to the observable described by the Hermitian matrix $A$ is

$$\langle A \rangle = \sum_{j=1}^{N} p_j \langle x_j | A | x_j \rangle = \operatorname{tr}(A\rho),$$

where

$$\rho = \sum_{j=1}^{N} p_j |x_j\rangle\langle x_j|$$

is a density operator (matrix).

**Example** $\frac{1}{2}(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|) = \frac{1}{2} I_2$ is a maximally mixed state.

It is the mixed state of $\frac{1}{2}(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|)$ with

$$|e_1\rangle = (\cos\theta, \sin\theta)^t \quad \text{and} \quad |e_2\rangle = (\sin\theta, -\cos\theta)^t \qquad \theta \in [0, 2\pi).$$