

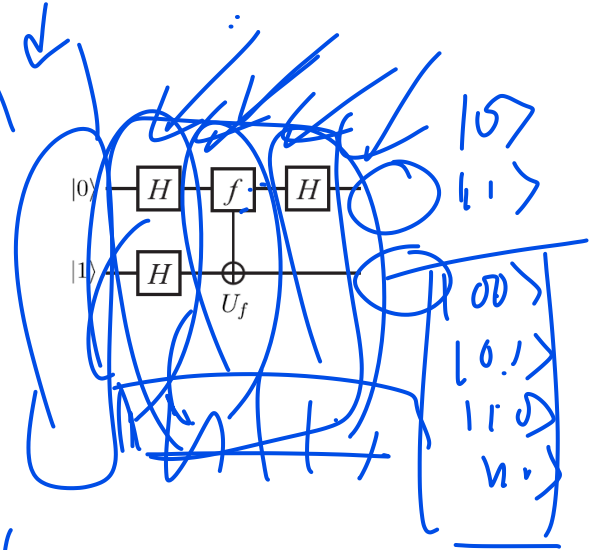
In this chapter, we introduce some simple algorithms. This demonstrate how one can use the quantum properties to solve certain problems efficiently. It should be emphasized that formulating the “right” questions to use quantum properties are important.

Simple quantum algorithms

5.1 Deutsch Algorithm

Let $f : \{0, 1\} \rightarrow \{0, 1\}$.

Decide whether $f(0) = f(1)$ or $f(0) \neq f(1)$ using one U_f evaluation.



Step 1 $|\psi_0\rangle = (H \otimes H)|01\rangle = (1/2)(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$.

Step 2 Let $U_f : |x\rangle|y\rangle \mapsto |x\rangle|y \oplus f(x)\rangle$. Then

$$\begin{aligned}
 |\psi_1\rangle &= U_f|\psi_0\rangle \\
 &= (1/2)(|0, f(0)\rangle - |0, 1 \oplus f(0)\rangle + |1, f(1)\rangle - |1, 1 \oplus f(1)\rangle) \\
 &= (1/2)(|0, f(0)\rangle - |0, \neg f(0)\rangle + |1, f(1)\rangle - |1, \neg f(1)\rangle).
 \end{aligned}$$

Step 3 $|\psi_2\rangle = (H \otimes I)|\psi_1\rangle$

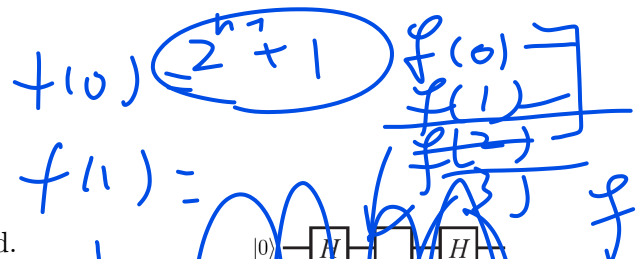
$$= \frac{1}{\sqrt{2}} [(|0\rangle + |1\rangle)(|f(0)\rangle - |\neg f(0)\rangle) + (|0\rangle - |1\rangle)(|f(1)\rangle - |\neg f(1)\rangle)]$$

Step 4 Measure the first qubit of $|\psi_2\rangle$:

Case 1. If $f(0) = f(1)$, then $|\psi_2\rangle = |0\rangle(|f(0)\rangle - |\neg f(0)\rangle)$ and we get the measurement ...

Case 2. If $f(0) \neq f(1)$, then $|\psi_2\rangle = |1\rangle(|f(0)\rangle - |\neg f(0)\rangle)$ and we get the measurement ...

Handwritten notes in blue ink include: (00) , (01) , (10) , (11) , $|0,1\rangle$, $|1,0\rangle$, $|1,1\rangle$, $|1,0\rangle$, $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, $|c\rangle$, and various arrows and scribbles.



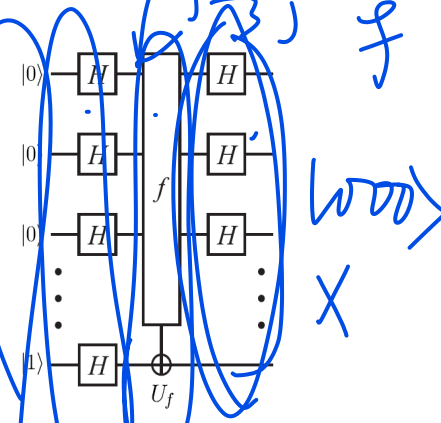
5.2.1 Deutsch-Jozsa Algorithm

Let $S_n = \{0, 1, \dots, 2^n - 1\}$ and $f : S_n \rightarrow \{0, 1\}$.
 We want to decide whether f is constant or balanced.

Step 0 $|\psi_0\rangle = |0\rangle^{\otimes n} |1\rangle$

Step 1 $|\psi_1\rangle = W_{n+1}|\psi_0\rangle = \frac{1}{\sqrt{2}}(\sum_x |x\rangle)(|0\rangle - |1\rangle)$

Step 2 Let $U_f : |x\rangle|c\rangle \mapsto |x\rangle|c \oplus f(x)\rangle$ and set



$$|\psi_2\rangle = U_f|\psi_1\rangle$$

$$= \frac{1}{\sqrt{2}} \sum_x (|x\rangle(|0\rangle - |1\rangle) \oplus f(x))$$

$$= \frac{1}{\sqrt{2}} \sum_x |x\rangle (-1)^{f(x)} (|0\rangle - |1\rangle)$$

(as $|c\rangle = |0\rangle - |1\rangle$ changes to $\pm|c\rangle$ depending on $f(x) = |0\rangle$ or $|1\rangle$)

$$= \frac{1}{\sqrt{2}} \sum_x (-1)^{f(x)} |x\rangle (|0\rangle - |1\rangle)$$

Handwritten notes and diagrams illustrating the algorithm's logic and state evolution:

- Vertical list of states: $|x\rangle$, $|1\rangle$, $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$.
- Matrix representations of W_2 and U_f gates.
- Diagram showing the decomposition of W_2 into two H gates.
- Diagram showing the decomposition of U_f into a $CNOT$ gate and H gates.
- Final state decomposition: $(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle$.
- Measurement results: $|0\rangle$ for constant functions and $|1\rangle$ for balanced functions.

Step 3 $|\psi_3\rangle = (W_n \otimes I_2)|\psi_2\rangle = \gamma \left(\sum_{x,y} (-1)^{f(x)} (-1)^{x \cdot y} |y\rangle \right) (|0\rangle - |1\rangle)$.

Note that $W_1|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $W_1|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

so that $\gamma W_1(|0\rangle + |1\rangle) = \sum_{x,y \in \{0,1\}} (-1)^{x \cdot y} |y\rangle$. Then

$$\begin{aligned} & W_2(\sum_{x_1, x_2} |x_1 x_2\rangle) \\ &= (\sum_{x_1, y_1} (-1)^{x_1 y_1} |y_1\rangle) (\sum_{x_2, y_2} (-1)^{x_2 y_2} |y_2\rangle) \\ &= \sum_{(x_1, x_2), (y_1, y_2)} (-1)^{(x_1, x_2) \cdot (y_1, y_2)} |y_1 y_2\rangle \\ &= \sum_{x, y} (-1)^{x \cdot y} |y\rangle. \end{aligned}$$

Here we are summing up the entries in each row.

Handwritten notes and diagrams:

At the top, there is a circled expression $\sum_{x,y} (-1)^{x \cdot y} |y\rangle$ with a checkmark to its right.

Below this, there are several diagrams and calculations:

- A box containing $H \otimes H$.
- A matrix with entries $1, -1, -1, 1$ in a 2×2 grid, with a circled 1.000 to its left and a 2 below it.
- A vertical column of entries $1, -1, -1, 1$.
- A vertical column of entries $0, 0, 0, 0$.
- Other scribbled-out notes and matrices.

Step 4 Measure the first n qubits.

Case 1. If f is constant, then $|\psi_3\rangle = \tilde{\gamma}|0\rangle^{\otimes n}(|0\rangle - |1\rangle)$.

Case 2. If f is balanced, then the probability of the measurement of the first n -qubits equal $|y\rangle = |0 \cdots 0\rangle$ is proportional to $\sum_x (-1)^{f(x)} (-1)^{x \cdot 0} = \sum_x (-1)^{f(x)} = 0$ because half of the $f(x)$ values are 0 and the rest are 1.

A closed look for a 2-qubit example

$$f(0,0) = 0, f(0,1) = 1, f(1,0) = 1, f(1,1) = 0.$$

$$\begin{aligned} U_f(W_2|00\rangle|1\rangle) &= \gamma U_f \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\ &= \gamma \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right). \end{aligned}$$

Applying $H \otimes H \otimes I_2$ to the vector, we can only get $|00\rangle$ when all $(-1)^{f(x)}$ are equal for all $|x\rangle$. Else, it has the form $\sum_y c_y |y\rangle$, where $c_y = \sum_x (-1)^{f(x)} (-1)^{x \cdot y}$.

Here note that:

$$W_2(\sum_{x_1 x_2} |x_1 x_2\rangle) = (\sum_{x_1, y_1} (-1)^{x_1 y_1} |y_1\rangle) (\sum_{x_2, y_2} (-1)^{x_2 y_2} |y_2\rangle) = \sum_{(x_1, x_2), (y_1, y_2)} (-1)^{(x_1, x_2) \cdot (y_1, y_2)} |y_1 y_2\rangle.$$

5.2.2 Bernstein-Vazirani algorithm

In the above example, $f(x_1, x_2) = (1, 1) \cdot (x_1, x_2)$.

Then the resulting measurement of the first two qubits of $|\psi_3\rangle$ yields $|11\rangle$.

In general, let $f(x) = c \cdot x = (c_{n-1}, \dots, c_0) \cdot (x_{n-1}, \dots, x_0)$. Apply the Deutsch-Jozsa algorithm to get:

$$|\psi_3\rangle = \gamma \left(\sum_{x, y} (-1)^{c \cdot x} (-1)^{x \cdot y} |y\rangle \right) (|0\rangle - |1\rangle) = \gamma |c\rangle (|0\rangle - |1\rangle).$$

Measuring the first n -qubits will give $c = (c_{n-1}, \dots, c_0)$.

5.3 Simon Algorithm

Let $f : \{0, 1\}^n \rightarrow \{0, 1\}^n$. Determine the nonzero $p \in \{0, 1\}^n$ if $f(x \oplus p) = f(x)$.

1. Set $|\psi_0\rangle = |0\rangle|0\rangle$ in $\mathbf{C}^N \otimes \mathbf{C}^N$ with $N = 2^n$.
Use the Walsh-Hadamard transformation W_n to get

$$|\psi_1\rangle = (W_n \otimes I)|\psi_0\rangle = \eta \sum_{x=0}^{2^n-1} |x\rangle|0\rangle,$$

$$\text{where } \eta = \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{2^n}}.$$

2. Use U_f and n controlled-NOT gates with control qubits $f_1(x), \dots, f_k(x)$ to get $|\psi_2\rangle = \eta \sum_x |x\rangle|f(x)\rangle$.
3. Apply measurement $f(x_0)$ to the second state to get

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|x_0\rangle + |x_0 + p\rangle)|f(x_0)\rangle.$$

4. Apply $W_n \otimes I$ again to get

$$\begin{aligned} |\psi_4\rangle &= \eta \sum_y (-1)^{x_0 \cdot y} [1 + (-1)^{p \cdot y}] |y\rangle |f(x_0)\rangle \\ &= \eta \sqrt{2} \sum_{p \cdot y = 0} (-1)^{x_0 \cdot y} |y\rangle |f(x_0)\rangle. \end{aligned}$$

5. Measure the first state to get $|y\rangle$ such that $p \cdot y = 0$.

The only states $|y\rangle$ with positive probability in the sum are those satisfying $p \cdot y = 0$.

Thus, a measurement will always yield such a vector $y_1 = (y_{11} \cdots y_{1n})$.

Repeat this to get linearly independent

$$y_1, \dots, y_{n-1} \text{ such that } p \cdot y_j = 0$$

for all j , i.e., we have a linear system

$$(y_{ij})(p_0, \dots, p_{n-1})^t = (0, \dots, 0)^t.$$

We need to do it in $O(n)$ attempts with a good probability. Then solve for p .

